Supplementary Materials

A.1 Observable Implications of Assumption 3

In this section, we derive the two observable implications of Assumption 3 described in Section 3.3. First, Assumption 3 implies,

$$\mathbb{E}[Y_i(a)|C_i = a] \;=\; \mathbb{E}[Y_i(a)|S_i = a], \tag{10}$$

for all $a \in \mathcal{A}$. This relationship directly implies equation (2) under Assumptions 1 and 2. Second, note that equation (10) also implies,

$$\begin{aligned}
\mathbb{E}[Y_i(a)|C_i = a] \;&=\; \mathbb{E}[Y_i(a)|C_i = a, S_i = a]\Pr(C_i = a|S_i = a) \\
&\quad + \mathbb{E}[Y_i(a)|C_i \neq a, S_i = a]\Pr(C_i \neq a|S_i = a) \\
\Leftrightarrow \quad \mathbb{E}[Y_i(a)|C_i \neq a, S_i = a] \;&=\; \frac{\mathbb{E}[Y_i|C_i = a, D_i = 0] - \mathbb{E}[Y_i|C_i = S_i = a, D_i = 0]\Pr(C_i = a|S_i = a, D_i = 0)}{1 - \Pr(C_i = a|S_i = a, D_i = 0)}
\end{aligned}$$

for all $a \in \mathcal{A}$. Setting the unobserved term in the left-hand side to its theoretical maximum and minimum yields equation (3).

A.2 Derivation of Equation (4)

First, consider $\mathbb{E}[Y_i(a)|C_i = c]$. Assumptions 1 and 2 imply $\Pr(C_i = c, S_i = s) = \Pr(C_i = c, S_i = s|D_i = 0)$, $\mathbb{E}[Y_i(c)|C_i = c, S_i = s] \;=\; \mathbb{E}[Y_i|C_i = c, S_i = s, D_i = 0]$, $\mathbb{E}[Y_i(a)] = \mathbb{E}[Y_i|A_i = a, D_i = 1]$, and $\mathbb{E}[Y_i(a)|S_i = s] = \mathbb{E}[Y_i|S_i = s, A_i = a, D_i = 1]$. Now, note that

$$\mathbb{E}[Y_i|A_i = a, D_i = 1] \;=\; \mathbb{E}[Y_i(a)] \;=\; \sum_{c'=0}^{J-1}\mathbb{E}[Y_i(a)|C_i = c']\Pr(C_i = c'),$$

by Assumptions 1, 2 and the law of total expectation. Substituting observed outcomes from the free-choice group and rearranging terms, we have

$$\mathbb{E}[Y_i(a)|C_i = c] \;=\; \frac{1}{\Pr(C_i = c|D_i = 0)}\left\{\begin{array}{l}
\mathbb{E}[Y_i|A_i = a, D_i = 1] \\
-\mathbb{E}[Y_i|C_i = a, D_i = 0]\Pr(C_i = a|D_i = 0) \\
-\sum_{c' \notin \{a,c\}}\mathbb{E}[Y_i(a)|C_i = c']\Pr(C_i = c'|D_i = 0)
\end{array}\right\}$$

because of Assumptions 1 and 2. By the same token,

$$\mathbb{E}[Y_i(a')|C_i = c] = \frac{1}{\Pr(C_i = c|D_i = 0)} \left\{ \begin{array}{l} \mathbb{E}[Y_i|A_i = a', D_i = 1] \\ -\mathbb{E}[Y_i|C_i = a', D_i = 0] \Pr(C_i = a|D_i = 0) \\ -\sum_{c' \notin \{a', c\}} \mathbb{E}[Y_i(a')|C_i = c'] \Pr(C_i = c'|D_i = 0) \end{array} \right\}$$

The quantity of interest is therefore

$$\begin{aligned} \tau(a, a'|c) &= \frac{1}{\Pr(C_i = c|D_i = 0)} \left\{ \begin{array}{l} \mathbb{E}[Y_i|A_i = a, D_i = 1] \\ -\mathbb{E}[Y_i|C_i = a, D_i = 0] \Pr(C_i = a|D_i = 0) \\ -\sum_{c' \notin \{a, c\}} \mathbb{E}[Y_i(a)|C_i = c'] \Pr(C_i = c'|D_i = 0) \end{array} \right\} \\ &\quad - \frac{1}{\Pr(C_i = c|D_i = 0)} \left\{ \begin{array}{l} \mathbb{E}[Y_i|A_i = a', D_i = 1] \\ -\mathbb{E}[Y_i|C_i = a', D_i = 0] \Pr(C_i = a'|D_i = 0) \\ -\sum_{c' \notin \{a', c\}} \mathbb{E}[Y_i(a')|C_i = c'] \Pr(C_i = c'|D_i = 0) \end{array} \right\} \end{aligned}$$

for any a, a' and c. Thus, under Assumptions 1 and 2, we have $2(J-2)$ terms that remain unidentified when $a \neq a' \neq c$. When $a' = c$, the above simplifies to

$$\begin{aligned} \tau(a, c|c) &= \mathbb{E}[Y_i(a)|C_i = c] - \mathbb{E}[Y_i|C_i = c, D_i = 0] \\ &= \frac{1}{\Pr(C_i = c|D_i = 0)} \left\{ \begin{array}{l} \mathbb{E}[Y_i|A_i = a, D_i = 1] \\ -\mathbb{E}[Y_i|C_i = a, D_i = 0] \Pr(C_i = a|D_i = 0) \\ -\sum_{c' \notin \{a, c\}} \mathbb{E}[Y_i(a)|C_i = c'] \Pr(C_i = c'|D_i = 0) \end{array} \right\} \\ &\quad - \mathbb{E}[Y_i|C_i = c, D_i = 0] \end{aligned}$$

and $J - 2$ terms remain unidentified.

A.3 Proof of Proposition 1

We begin by establishing several lemmas.

Lemma .1 *Let $\Gamma_a(y, c|s, a) = \Pr(Y_i(a) \leq y, C_i \leq c|S_i = s, C_i \neq a)$. Under Assumptions 1 and 2, the sharp upper and lower bounds on $\Gamma_a(y, c|s, a)$, denoted by $\Gamma_a^+(y, c|s, a)$ and $\Gamma_a^-(y, c|s, a)$ respectively, are identified as follows.*

$$\begin{aligned} \Gamma_a^+(y, c|s, a) &= \min\left\{ H(c|s, a, 0),\ \frac{F(y|s, a, 1) - F(y|s, a, 0)P(a|s, 0)}{1 - P(a|s, 0)} \right\}, \\ \Gamma_a^-(y, c|s, a) &= \max\left\{ 0,\ H(c|s, a, 0) + \frac{F(y|s, a, 1) - F(y|s, a, 0)P(a|s, 0)}{1 - P(a|s, 0)} - 1 \right\}, \end{aligned}$$

for $y \in \mathcal{Y}$, $a, c, s \in \mathcal{A}$ and $d \in \{0,1\}$, where $H(c|s,a,0) = \Pr(A_i \leq c | S_i = s, A_i \neq a, D_i = 0)$ and $F(y|s,a,d)$ and $P(a|s,0)$ are as defined in Proposition 1.

Proof. By the Fréchet-Hoeffding theorem, the sharp upper and lower bounds of the bivariate joint distribution function $\Gamma_a(y,c|s,a)$ are given by,

$$\Gamma_a^{+}(y,c|s,a) = \min\left\{\Gamma_a(\infty,c|s,a), \Gamma_a(y,\infty|s,a)\right\}, \tag{11}$$

$$\Gamma_a^{-}(y,c|s,a) = \max\left\{0, \Gamma_a(\infty,c|s,a) + \Gamma_a(y,\infty|s,a) - 1\right\}. \tag{12}$$

Under Assumption 1, $\Gamma_a(\infty,c|s,a) = \Pr(C_i \leq c|S_i = s, C_i \neq a) = \Pr(A_i \leq c|S_i = s, A_i \neq a, D_i = 0) = H(c|s,a,0)$ for any c, s and $a \in \mathcal{A}$. Under Assumptions 1 and 2, we have

$$\begin{aligned}
\Gamma_a(y,\infty|s,a) &= \Pr(Y_i(a) \leq y|S_i = s, C_i \neq a) \\
&= \frac{\Pr(Y_i(a) \leq y|S_i = s) - \Pr(Y_i(a) \leq y, C_i = a|S_i = s)}{\Pr(C_i \neq a|S_i = s)} \\
&= \frac{\Pr(Y_i(a) \leq y|S_i = s) - \Pr(Y_i(a) \leq y|C_i = a, S_i = s)\Pr(C_i = a|S_i = s)}{1 - \Pr(C_i = a|S_i = s)} \\
&= \frac{F(y|s,a,1) - F(y|s,a,0)P(a|s,0)}{1 - P(a|s,0)},
\end{aligned}$$

for any a and $s \in \mathcal{A}$. Substituting these to equations (11) and (12) yields the results in Lemma .1. ∎

Lemma .2 *Let A_i^*, C_i^* and S_i^* be reordered versions of A_i, C_i and S_i, respectively, such that $C_i^* = 0$ iff $C_i = c$ (and likewise for A_i^* and S_i^*). Then, the resulting sharp bounds on $\Gamma_a(y, c \mid s,a) - \Gamma_a(y, c-1 \mid s,a)$ are also the sharp bounds on $\Gamma_a^*(y, 0 \mid s,a) - \Gamma_a^*(y, -1 \mid s,a)$ for any y and $c \in \mathcal{A}$, where $\Gamma_a^*(y, c \mid s,a) = \Pr(Y_i(a) \leq y, C_i^* \leq c|S_i = s, C_i \neq a)$.*

Proof. First, consider the sharp bounds on $\Gamma_a(y,c) - \Gamma_a(y,c-1)$. In addition to the Fréchet-Hoeffding constraints on its constituent parts,

$$\begin{aligned}
\Gamma_a(y,c|s,a) &\in \left[\Gamma_a^{-}(y,c|s,a), \Gamma_a^{+}(y,c|s,a)\right] \\
\Gamma_a(y,c-1|s,a) &\in \left[\Gamma_a^{-}(y,c-1|s,a), \Gamma_a^{+}(y,c-1|s,a)\right],
\end{aligned}$$

the increase in cumulative probability from $c-1$ to c is also subject to

$$\Gamma_a(y,c) - \Gamma_a(y,c-1) \quad \in \quad [0, \Gamma_a(\infty,c) - \Gamma_a(\infty,c-1)]\,.$$

The combination of these constraints yields

$$\begin{aligned}
&\Gamma_a(y,c|s,a)\Gamma_a(y,c-1|s,a) \\
&\quad \in [0, \Gamma_a(\infty,c) - \Gamma_a(\infty,c-1)] \bigcup \\
&\qquad \left(\left[\Gamma_a^{*-}(y,c|s,a), \Gamma_a^{*+}(y,c|s,a)\right] - \left[\Gamma_a^{*-}(y,c-1|s,a), \Gamma_a^{*+}(y,c-1|s,a)\right] \right) \\
&\quad \in \left[\max\left\{ \begin{array}{l} 0, \\ \max\left\{ \begin{array}{l} 0, \\ \Gamma_a(\infty,c|s,a) + \Gamma_a(y,\infty|s,a) - 1 \end{array} \right\} - \min\left\{ \begin{array}{l} \Gamma_a(\infty,c-1|s,a), \\ \Gamma_a(y,\infty|s,a) \end{array} \right\} \end{array} \right\}, \right. \\
&\qquad \left. \min\left\{ \begin{array}{l} \Gamma_a(\infty,c) - \Gamma_a(\infty,c-1), \\ \min\left\{ \begin{array}{l} \Gamma_a(\infty,c|s,a), \\ \Gamma_a(y,\infty|s,a) \end{array} \right\} - \max\left\{ \begin{array}{l} 0, \\ \Gamma_a(\infty,c-1|s,a) + \Gamma_a(y,\infty|s,a) - 1 \end{array} \right\} \end{array} \right\} \right]
\end{aligned}$$

Next, consider the sharp bounds on $\Gamma_a^*(y,0 \mid s,a) - \Gamma_a^*(y,-1 \mid s,a)$. Because -1 lies below the lowest possible value of C_i^*, $\Gamma_a^*(y,-1 \mid s,a)$ is necessarily zero, and bounds on the difference reduce to bounds on $\Gamma_a^*(y,0 \mid s,a)$,

$$\begin{aligned}
\Gamma_a^*(y,0|s,a) \;&\in\; \left[\Gamma_a^{*-}(y,0|s,a), \Gamma_a^{*+}(y,0|s,a)\right] \\
&\in\; \left[\max\left\{0, \Gamma_a^*(\infty,0|s,a) + \Gamma_a^*(y,\infty|s,a) - 1\right\}, \min\left\{\Gamma_a^*(\infty,0|s,a), \Gamma_a^*(y,\infty|s,a)\right\}\right] \\
&\in\; \Big[\max\left\{0, \Pr(A_i = c \mid S_i = s, A_i \neq a, D_i = 0) + \Gamma_a(y,\infty|s,a) - 1\right\}, \\
&\qquad \min\left\{\Pr(A_i = c \mid S_i = s, A_i \neq a, D_i = 0), \Gamma_a(y,\infty|s,a)\right\}\Big] \\
&\in\; \Big[\max\left\{0, \Gamma_a(\infty,c|s,a) - \Gamma_a(\infty,c-1|s,a) + \Gamma_a(y,\infty|s,a) - 1\right\}, \\
&\qquad \min\left\{\Gamma_a(\infty,c|s,a) - \Gamma_a(\infty,c-1|s,a), \Gamma_a(y,\infty|s,a)\right\}\Big].
\end{aligned}$$

We now show that the upper bound on $\Gamma_a(y,c) - \Gamma_a(y,c-1)$ is identical to the upper bound on

$\Gamma_a^*(y,0 \mid s,a) - \Gamma_a^*(y,-1 \mid s,a)$ in each of the following four possible cases. (1) $\Gamma_a(\infty,c|s,a) \leq \Gamma_a(y,\infty|s,a)$ and $0 \geq \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$. The upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces to $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), \Gamma_a(\infty,c|s,a)-\max\{0,\Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1\}\}$. This implies $\Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a) \leq \Gamma_a(y,\infty|s,a)$, and so the upper bound on $\Gamma_a^*(y,0 \mid s,a)$ becomes $\Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a)$. Since $0 \geq \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$, the upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ further reduces to $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), \Gamma_a(\infty,c|s,a)\} = \Gamma_a(\infty,c)-\Gamma_a(\infty,c-1)$, which is identical to the upper bound on $\Gamma_a^*(y,0 \mid s,a)$. (2) $\Gamma_a(\infty,c|s,a) \leq \Gamma_a(y,\infty|s,a)$ and $0 < \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$. The upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ becomes $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), \Gamma_a(\infty,c|s,a)-(\Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1)\} = \Gamma_a(\infty,c)-\Gamma_a(\infty,c-1)$, since $1-\Gamma_a(y,\infty|s,a) > 0$. This is again identical to the upper bound on $\Gamma_a^*(y,0 \mid s,a)$. (3) $\Gamma_a(\infty,c|s,a) > \Gamma_a(y,\infty|s,a)$ and $0 \geq \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$. The upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces to $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), \Gamma_a(y,\infty|s,a) - \max\{0,\Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1\}\}$. Since $0 \geq \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$, the upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ further reduces to $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), \Gamma_a(y,\infty|s,a)\}$, which is the original upper bound given for $\Gamma_a^*(y,0 \mid s,a)$. (4) $\Gamma_a(\infty,c|s,a) > \Gamma_a(y,\infty|s,a)$ and $0 < \Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1$. The upper bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ further reduces to $\min\{\Gamma_a(\infty,c)-\Gamma_a(\infty,c-1), 1-\Gamma_a(\infty,c-1|s,a)\} = \Gamma_a(\infty,c)-\Gamma_a(\infty,c-1)$. This implies that $\Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a) < \Gamma_a(y,\infty|s,a)$. The upper bound on $\Gamma_a^*(y,0 \mid s,a)$ then also becomes $\Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a)$.

Finally, we show that the lower bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ is identical to the upper bound on $\Gamma_a^*(y,0 \mid s,a) - \Gamma_a^*(y,-1 \mid s,a)$ in each of the following three possible cases. (1) $0 \geq \Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1$. The lower bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces to $\max\{0, -\min\{\Gamma_a(\infty,c-1|s,a), \Gamma_a(y,\infty|s,a)\}\} = 0$. Because $\Gamma_a(\infty,c|s,a) \geq \Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a)$, the lower bound on $\Gamma_a^*(y,0 \mid s,a)$ also becomes 0. (2) $0 < \Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1$ and $\Gamma_a(\infty,c-1|s,a) \leq \Gamma_a(y,\infty|s,a)$. The lower bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces to $\max\{0,\Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1-\min\{\Gamma_a(\infty,c-1|s,a),\Gamma_a(y,\infty|s,a)\}\}$. Since $\Gamma_a(\infty,c-1|s,a) \leq \Gamma_a(y,\infty|s,a)$, the lower bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces further to $\max\{0, \Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1-\Gamma_a(\infty,c-1|s,a)\}$, which is the original lower bound given for $\Gamma^*_a(y,0 \mid s,a)$. (3) $0<\Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1$ and $\Gamma_a(\infty,c-1|s,a)>\Gamma_a(y,\infty|s,a)$. The lower bound on $\Gamma_a(y,c)-\Gamma_a(y,c-1)$ reduces further to $\max\{0,\Gamma_a(\infty,c|s,a)+\Gamma_a(y,\infty|s,a)-1-\Gamma_a(y,\infty|s,a)\}=0$. Since $\Gamma_a(\infty,c|s,a)-\Gamma_a(\infty,c-1|s,a)+\Gamma_a(y,\infty|s,a)-1<\Gamma_a(\infty,c|s,a)-\Gamma_a(y,\infty|s,a)+\Gamma_a(y,\infty|s,a)-1<0$ and $\Gamma_a(\infty,c|s,a)-1\leq 0$, the lower bound for $\Gamma^*_a(y,0 \mid s,a)$ is also zero. ∎

Lemma .3 *Let* $\Phi_a(y|s,c)=\Pr(Y_i(a)\leq y|S_i=s,C_i=c)$. *Under Assumptions 1 and 2, the sharp upper and lower bounds on* $\Phi_a(y|s,0)$, *denoted by* $\Phi^+_a(y|s,0)$ *and* $\Phi^-_a(y|s,0)$ *respectively, are identified as*

$$\begin{aligned}
\Phi^+_a(y|s,0) &= \min\left\{1,\ \frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)}{P(0|s,0)}\right\} \text{ and}\\
\Phi^-_a(y|s,0) &= \max\left\{0,\ 1+\frac{P(a|s,0)+F(y|s,a,1)-F(y|s,a,0)P(a|s,0)-1}{P(0|s,0)}\right\}
\end{aligned}$$

for $y\in\mathcal{Y}$ *and* $a,s\in\mathcal{A}$.

Proof. First, note that

$$\begin{aligned}
\Phi_a(y|s,c) &= \Pr(Y_i(a)\leq y|S_i=s,C_i=c,C_i\neq a)\\
&= \frac{\Pr(Y_i(a)\leq y,C_i\leq c|S_i=s,C_i\neq a)-\Pr(Y_i(a)\leq y,C_i\leq c-1|S_i=s,C_i=\neq a)}{\Pr(C_i=c|S_i=s,C_i\neq a)}\\
&= \frac{\Gamma_a(y,c|s,a)-\Gamma_a(y,c-1|s,a)}{\Pr(C_i=c|S_i=s,C_i\neq a)},
\end{aligned}$$

for $c\neq a$. By Lemma .1, the sharp upper and lower bounds on $\Phi_a(y|s,c)$ are given by

$$\begin{aligned}
\Phi^+_a(y|s,c) &= \min\left\{1,\ \frac{\Gamma^+_a(y,c|s,a)-\Gamma^-_a(y,c-1|s,a)}{\Pr(C_i=c|S_i=s,C_i\neq a)}\right\},\\
\Phi^-_a(y|s,c) &= \max\left\{0,\ \frac{\Gamma^-_a(y,c|s,a)-\Gamma^+_a(y,c-1|s,a)}{\Pr(C_i=c|S_i=s,C_i\neq a)}\right\}.
\end{aligned}$$

Because $\Gamma^+_a(y,-1|s,a)=\Gamma^-_a(y,-1|s,a)=0$ and by Lemma .1, these bounds simplify when $c=0$ to

$$\Phi^+_a(y|s,0) = \frac{\Gamma^+_a(y,0|s,a)}{\Pr(C_i=0|S_i=s,C_i\neq a)}$$

$$
\begin{aligned}
&= \min\left\{\frac{H(0|s,a,0)}{\Pr(C_i=0|S_i=s,C_i\neq a)},\ \frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)}{\Pr(C_i=0|S_i=s,C_i\neq a)\{1-P(a|s,0)\}}\right\} \\
&= \min\left\{\frac{H(0|s,a,0)}{\Pr(A_i=0|S_i=s,A_i\neq a,D_i=0)},\right. \\
&\qquad\qquad \left.\frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)}{\Pr(A_i=0|S_i=s,A_i\neq a,D_i=0)\Pr(A_i\neq a|S_i=s,D_i=0)}\right\} \\
&= \min\left\{1, \frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)}{\Pr(A_i=0|S_i=s,D_i=0)}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_a^-(y|s,0) &= \frac{\Gamma_a^-(y,0|s,a)}{\Pr(C_i=0|S_i=s,C_i\neq a)} \\
&= \max\left\{0,\ \frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)-\{1-H(0|s,a,0)\}\{1-P(a|s,0)\}}{\Pr(C_i=0|S_i=s,C_i\neq a)\{1-P(a|s,0)\}}\right\} \\
&= \max\left\{0,\ \frac{F(y|s,a,1)-F(y|s,a,0)P(a|s,0)-1+P(a|s,0)}{\Pr(A_i=0|S_i=s,D_i=0)}+1\right\}. \qquad \blacksquare
\end{aligned}
$$

Now we provide a proof for the bounds in Proposition 1. We only consider the case of $c=0$. This can be done without loss of generality by Lemma .2. Now, note that $\tau(a,a'|0)$ can be written under Assumption 1 as,

$$
\tau(a,a'|0) \;=\; \sum_{s\in\mathcal{A}}\{\pi(a|s,0)-\pi(a'|s,0)\}\Pr(S_i=s|A_i=0,D_i=0), \tag{13}
$$

where $\pi(a|s,c)\equiv\mathbb{E}[Y_i(a)|S_i=s,C_i=c]$ for any a and $c\in\mathcal{A}$. Under Assumption 1, $\pi(a|s,0)$ can be point-identified when $a=0$ as

$$
\pi(0|s,0) \;=\; \mathbb{E}[Y_i|A_i=0,S_i=s,D_i=0], \tag{14}
$$

for any $s\in\mathcal{A}$, but not when $a\neq 0$. To find the sharp bounds on $\pi(a|s,0)$ when $a\neq 0$, note that

$$
\pi(a|s,0) \;=\; \lim_{y^*\to-\infty}\left\{\int_{y^*}^{\infty} 1-\Phi_a(y|s,0)\,\mathrm{d}y+y^*\right\}.
$$

By Lemma .3, $\pi^-(a|s,0)\leq\pi(a|s,0)\leq\pi^+(a|s,0)$ where

$$
\begin{aligned}
\pi^-(a|s,0) &\equiv \lim_{y^*\to-\infty}\left\{\int_{y^*}^{\infty} 1-\Phi_a^+(y|s,0)\,\mathrm{d}y+y^*\right\}, && (15)\\
\pi^+(a|s,0) &\equiv \lim_{y^*\to-\infty}\left\{\int_{y^*}^{\infty} 1-\Phi_a^-(y|s,0)\,\mathrm{d}y+y^*\right\}. && (16)
\end{aligned}
$$

The bounds, $\pi^-(a|s,0)$ and $\pi^+(a|s,0)$, are the sharp lower and upper bounds on $\pi(a|s,0)$ because $\Phi_a^+(y|s,0)$ and $\Phi_a^-(y|s,0)$ are the sharp upper and lower bounds on $\Phi_a(y|s,0)$, respectively.

Substituting Equations (14), (15) and (16) into Equation (13) and simplifying the terms yield the sharp bounds on $\tau(a,0|0)$,

$$\begin{aligned}
\sum_{s\in\mathcal{A}} \big\{\pi^-(a|s,0)\Pr(S_i=s|A_i=0,D_i=0)\big\} - \mathbb{E}[Y_i|A_i=0,D_i=0] & \\
\leq\ \tau(a,0|0)\ \leq & \\
\sum_{s\in\mathcal{A}} \big\{\pi^+(a|s,0)\Pr(S_i=s|A_i=0,D_i=0)\big\} - \mathbb{E}[Y_i|A_i=0,D_i=0] &
\end{aligned} \tag{17}$$

for any $a \in \mathcal{A}$. For $\tau(a,a')$ where $a \neq a'$, we obtain the following bounds,

$$\begin{aligned}
\sum_{s\in\mathcal{A}} \big\{\pi^-(a|s,0) - \pi^+(a'|s,0)\big\}\Pr(S_i=s|A_i=0,D_i=0) & \\
\leq\ \tau(a,a'|0)\ \leq & \\
\sum_{s\in\mathcal{A}} \big\{\pi^+(a|s,0) - \pi^-(a'|s,0)\big\}\Pr(S_i=s|A_i=0,D_i=0) &
\end{aligned} \tag{18}$$

which are not necessarily sharp because $\pi^-(a|s,0)$ and $\pi^+(a'|s,0)$ may not be simultaneously attainable, and vice versa. Finally, Lemma .2 implies that (17) and (18) are both valid as bounds for $\tau(a,c|c)$ and $\tau(a,a'|c)$, respectively, for any c $\in \mathcal{A}$. This completes the proof of Proposition 1. ∎

A.4 Proof of Proposition 2

We begin by considering the joint distribution of all variables in the study population when $J=3$:

$$\begin{aligned}
&\Pr(S_i=s,D_i=d,C_i=c,A_i=a,Y_i=y,Y_i(0)=y_0,Y_i(1)=y_1,Y_i(2)=y_2) \\
&= \Pr(Y_i(d)=y|A_i=a,Y_i(0)=y_0,Y_i(1)=y_1,Y_i(2)=y_2) \\
&\quad \times \Pr(A_i=a|C_i=c,D_i=d) \\
&\quad \times \Pr(S_i=s,C_i=c,Y_i(0)=y_0,Y_i(1)=y_1,Y_i(2)=y_2)\Pr(D_i=d) \\
&= \Pr(Y_i(d)=y|A_i=a,Y_i(0)=y_0,Y_i(1)=y_1,Y_i(2)=y_2) \\
&\quad \times \{\Pr(A_i=a|C_i=c,D_i=0)(1-d)+\Pr(A_i=a|D_i=1)d\}
\end{aligned}$$

$$\times \Pr(S_i = s, C_i = c, Y_i(0) = y_0, Y_i(1) = y_1, Y_i(2) = y_2) \Pr(D_i = d), \tag{19}$$

where the first equality follows from Assumption 1 and the fact that $Y_i(0)$, $Y_i(1)$, $Y_i(2)$ and A_i are sufficient for Y_i and that C_i and D_i are sufficient for A_i. The second equality is by Assumption 2. Note that $\Pr(Y_i(d) = y|Y_i(0), Y_i(1), Y_i(2))$ and $\Pr(A_i = a|C_i, D_i = 0)$ are degenerate and that $\Pr(A_i = a|D_i = 1)$ and $\Pr(D_i = d)$ are fixed by the experimental design. Therefore, the remaining component of equation (19), $\Pr(S_i = s, C_i = c, Y_i(0) = y_0, Y_i(1) = y_1, Y_i(2) = y_2)$, completely specifies the data generating process, with $|\mathcal{A}|^2 \cdot |\mathcal{Y}|^{|\mathcal{A}|} - 1 = J^2 2^J - 1$ free parameters needed to describe it. Balke (1995, Section 3.5) shows that bounds on counterfactual probabilities found by optimizing over such a complete model are sharp; that is, they are guaranteed to be at least as tight as bounds calculated from any partial (marginalized) model.

We express the complete model in terms of $\phi_{y_0,y_1,y_2,s,c} \in \Phi$. First, note that $\sum_{y_0 \in \{0,1\}} \sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \sum_{s' \in \mathcal{A}} \sum_{c' \in \mathcal{A}} \phi_{y_a, y_{a'}, y_{a''}, s', c'} = 1$. Next, from the free-choice condition, we observe $\Pr(S_i = s, C_i = c, Y_i = y \mid D_i = 0)$, which is completely specified by $|\mathcal{A}|^2 \cdot |\mathcal{Y}| - 1 = 2J^2 - 1$ free parameters. We use the following $2J^2$ marginals as constraints on $\phi_{y_0,y_1,y_2,s,c}$ (with one redundant):

$$\Pr(S_i = s, A_i = c \mid D_i = 0) \; = \; \Pr(S_i = s, C_i = c) \; = \sum_{a \in \mathcal{A}} \sum_{y_a \in \{0,1\}} \phi_{y_0,y_1,y_2,s,c}, \tag{20}$$

$$\Pr(S_i = s, A_i = c, Y_i = 1 \mid D_i = 0) \; = \; \Pr(S_i = s, C_i = c, Y_i(c) = 1) \; = \sum_{a \neq c} \sum_{y_a \in \{0,1\}} \phi_{y_0,y_1,y_2,s,c}, \tag{21}$$

for all s and $c \in \mathcal{A}$. Similarly, from the forced-choice condition, we observe

$$\begin{aligned} &\Pr(S_i = s, A_i = a, Y_i = y \mid D_i = 1) \\ &\quad = \; \Pr(Y_i = y \mid S_i = s, A_i = a, D_i = 1) \Pr(A_i = a \mid D_i = 1) \Pr(S_i = s \mid D_i = 1) \end{aligned}$$

where the equality holds by Assumption 2. Because $\Pr(A_i = a \mid D_i = 1)$ is fixed a priori by randomization, the observed distribution from the forced-choice arm can be fully characterized by $(|\mathcal{Y}| - 1)|\mathcal{A}|^2 + |\mathcal{A}| - 1 = J^2 + J - 1$ free parameters. We use the following $J^2 + J$ margins as constraints on $\phi_{y_0,y_1,y_2,s,c}$, noting that one of them is redundant:

$$\Pr(S_i = s \mid A_i = a, D_i = 1) \;=\; \Pr(S_i = s) \;=\; \sum_{a \in \mathcal{A}} \sum_{y_a \in \{0,1\}} \sum_{c \in \mathcal{A}} \phi_{y_0,y_1,y_2,s,c}, \tag{22}$$

$$\Pr(S_i = s, Y_i = 1 \mid A_i = a, D_i = 1) \;=\; \Pr(S_i = s, Y_i(a) = 1) \;=\; \sum_{a' \in \mathcal{A}} \sum_{y_{a'} \in \{0,1\}} \sum_{c \in \mathcal{A}} \phi_{y_0,y_1,y_2,s,c} \cdot \mathbf{1}\{y_a = 1\},$$

for all s and $a \in \mathcal{A}$. However, note that equation (22) are merely linear combinations of equation (20) and can therefore be omitted.

Finally, the quantity of interest can be written in terms of $\phi_{y_0,y_1,y_2,s,c}$ as,

$$\begin{aligned}
\tau(a, a' \mid c) &= \mathbb{E}[Y_i(a) \mid C_i = c] - \mathbb{E}[Y_i(a') \mid C_i = c] \\
&= \frac{\sum_{y_0 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \sum_s \phi_{1,y_1,y_2,s,c}}{\Pr(A_i = c | D_i = 0)} - \frac{\sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \sum_s \phi_{y_0,1,y_2,s,c}}{\Pr(A_i = c | D_i = 0)},
\end{aligned}$$

assuming $a' = 1$ and $a = 0$ without loss of generality. Solving for the extrema of $\tau(a, a' \mid c)$ under the above set of linear constraints, which incorporate the full information in the observed data as well as probability axioms, yields its sharp upper and lower bounds as displayed in Proposition 2. ∎

A.5 Statistical Inference for the Bounds

Let $\boldsymbol{p} = [p_s] = [\Pr(S_i = 0), \cdots, \Pr(S_i = J-1)]^\top$ be a stochastic vector of stated-preference probabilities. $\boldsymbol{q} = [q_{sc}] = [\Pr(C_i = c | S_i = s)]$ is a row-stochastic matrix, where row s, denoted $\boldsymbol{q}_s$, represents the distribution of true preferences (C_i) among those with the stated preference $S_i = s$. Also let $\boldsymbol{\pi}^+ = \{\pi^+(a|s,c) : a, s, c \in \mathcal{A}\}$ and $\boldsymbol{\pi}^- = \{\pi^-(a|s,c) : a, s, c \in \mathcal{A}\}$, where $\pi^+(a|s,c)$ and $\pi^-(a|s,c)$ are defined in Appendix A.3. Let $\boldsymbol{F}^1 = \{F(y|s,a,d) : s, a \in \mathcal{A}, d = 1\}$ and $\boldsymbol{F}^0 = \{F(y|s,a,d) : s, a \in \mathcal{A}, d = 0\}$, where $F(y|s,a,d)$ is defined in Proposition 1. Finally, we use $\boldsymbol{\tau}^+$ and $\boldsymbol{\tau}^-$ to denote the sets of the upper and lower bounds on $\tau(a, a'|c)$ for all $a, a', c \in \mathcal{A}$, respectively, and $\boldsymbol{X}$ to indicate all observed data.

Our goal is to approximate the posterior distribution of $(\boldsymbol{\tau}^-, \boldsymbol{\tau}^+)$ with Monte Carlo simulations. We begin by the general bounds in Proposition 1. Note that $\boldsymbol{\tau}^-$ and $\boldsymbol{\tau}^+$ are deterministic functions of $\boldsymbol{\pi}^-$,

$\boldsymbol{\pi}^+$, $\boldsymbol{p}$ and $\boldsymbol{q}$, such that

$$\begin{aligned}
\tau^-(a, a'|c) &= \sum_{s\in\mathcal{A}} \left(\pi^-(a|s,c) - \pi^+(a'|s,c)\right) \frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}} q_{s'c}p_{s'}}, \\
\tau^+(a, a'|c) &= \sum_{s\in\mathcal{A}} \left(\pi^+(a|s,c) - \pi^-(a'|s,c)\right) \frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}} q_{s'c}p_{s'}}
\end{aligned}$$

for all $a, a', c \in \mathcal{A}$. Therefore, we consider the problem of simulating samples from the joint posterior of $\boldsymbol{\pi}^-$, $\boldsymbol{\pi}^+$, $\boldsymbol{p}$ and $\boldsymbol{q}$, which can be written as,

$$f(\boldsymbol{\pi}^+, \boldsymbol{\pi}^-, \boldsymbol{p}, \boldsymbol{q}|\boldsymbol{X}) = f(\boldsymbol{\pi}^+, \boldsymbol{\pi}^-|\hat{\boldsymbol{F}}^1, \hat{\boldsymbol{F}}^0, \boldsymbol{q})\, f(\boldsymbol{q}|\boldsymbol{n}_s^0)\, f(\boldsymbol{p}|\boldsymbol{n})$$

under Assumptions 1 and 2, where $\hat{\boldsymbol{F}}^1$ and $\hat{\boldsymbol{F}}^0$ are empirical CDFs corresponding to $\boldsymbol{F}^1$ and $\boldsymbol{F}^0$, respectively. For $\boldsymbol{p}$ and $\boldsymbol{q}$, we use the noninformative improper priors $\boldsymbol{p} \sim$ Dirichlet($\mathbf{0}$) and $\boldsymbol{q}_s \sim$ Dirichlet($\mathbf{0}$) $\forall\, s \in \mathcal{A}$. Then, $\boldsymbol{q}_s \mid \boldsymbol{X} \sim$ Dirichlet($\boldsymbol{n}_s^0$) $\forall\, s$ and $\boldsymbol{p} \mid \boldsymbol{X} \sim$ Dirichlet($\boldsymbol{n}$).

We are now left with $f(\boldsymbol{\pi}^+, \boldsymbol{\pi}^-|\hat{\boldsymbol{F}}^1, \hat{\boldsymbol{F}}^0, \boldsymbol{q})$. Because of the way these bounds are constructed (see Proposition 1),

$$\begin{aligned}
\pi^+(a|s,c), \pi^-(a|s,c) &\perp\!\!\!\perp \pi^+(a|s',c), \pi^-(a|s',c) \mid \hat{\boldsymbol{F}}^1, \hat{\boldsymbol{F}}^0, \boldsymbol{q} \quad \text{and} \\
\pi^+(a|s,c), \pi^-(a|s,c) &\perp\!\!\!\perp \pi^+(a'|s,c), \pi^-(a'|s,c) \mid \hat{\boldsymbol{F}}^1, \hat{\boldsymbol{F}}^0, \boldsymbol{q}
\end{aligned}$$

for $s \neq s'$ and $a \neq a'$. Therefore, to fully characterize the posterior of $[\tau^-(a', a''|c), \tau^+(a', a''|c)]$ for each a, a'' and $c \in \mathcal{A}$, it is sufficient to only consider the bivariate posterior distribution of $[\pi^+(a|s,c), \pi^-(a|s,c)]$ for $a \in \{a', a''\}$ and $s \in \mathcal{A}$. Note that, under mild assumptions and with a sufficiently large sample size, the posterior for each pair $[\pi^+(a|s,c), \pi^-(a|s,c)]$ can be approximated by a bivariate normal distribution due to the Bayesian central limit theorem. That is, we have:

$$\begin{bmatrix} \pi^-(a|s,c) \\ \pi^+(a|s,c) \end{bmatrix} \mid \boldsymbol{q}, \boldsymbol{X} \approx \text{Normal}\left(\begin{bmatrix} \bar{\pi}^-(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) \\ \bar{\pi}^+(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) \end{bmatrix}, \begin{bmatrix} V^-(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & C(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) \\ C(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & V^+(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) \end{bmatrix} \right) \tag{23}$$

when N is sufficiently large, and the means and covariances can be approximated by the asymptotic means and covariances of the frequentist sampling distributions of $[\pi^-(a|s,c), \pi^+(a|s,c)]$, respectively, as shown below. Note that priors on $\pi^-(a|s,c)$, $\pi^+(a|s,c)$ can be ignored and therefore left unspecified when N is large because of the Bernstein-von Mises theorem.

Let $\underline{y}$ be the natural lower bound of $Y_i(a)$ if it exists and $\min\{Y_i : S_i = s, A_i = a\}$, which is the lowest point at which the estimated conditional CDF, $\hat{\Gamma}_a(y, \infty|s, a)$, is nonzero, if it does not. Let $\Gamma_a^{-1}(\cdot)$ be the inverse of $\Gamma_a(y, \infty|s, a)$ (see Section A.3 for the definition) with respect to y, so that $\Gamma_a^{-1}(\Gamma_a(y, \infty|s, a)) = y$, and let $\hat{\Gamma}_a^{-1}(\cdot)$ be its sample analogue, such that $\hat{\Gamma}_a^{-1}(p) = \min\{y : p \leq \hat{\Gamma}_a(y, \infty|s, a)\}$. Let $b = \frac{q_{sc}}{1-q_{sa}}$. For the means, note that the $\pi^-(a|s, c)$ and $\pi^+(a|s, c)$ are functions of $F(y|s, a, 0)$, $F(y|s, a, 1)$ and $P(a|s, 0)$ (as shown in Appendix A.3), which can be consistently estimated by their nonparametric maximum likelihood estimates $\hat{F}(y|s, a, 0)$, $\hat{F}(y|s, a, 1)$ and q_{sa}, respectively. This implies the following plug-in estimators for $\bar{\pi}^-(a|s, c, \boldsymbol{q}_s, \boldsymbol{X})$ and $\bar{\pi}^+(a|s, c, \boldsymbol{q}_s, \boldsymbol{X})$:

$$\begin{aligned}
\hat{\bar{\pi}}^-(a|s, c, \boldsymbol{q}_s, \boldsymbol{X}) &= \hat{\Gamma}_a^{-1}(b) - \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \frac{\hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa}}{q_{sc}} \,\mathrm{d}y \\
\hat{\bar{\pi}}^+(a|s, c, \boldsymbol{q}_s, \boldsymbol{X}) &= \hat{\Gamma}_a^{-1}(1-b) - \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \frac{q_{sa} + \hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa} - 1}{q_{sc}} \,\mathrm{d}y,
\end{aligned}$$

where we used the fact that $\Phi_a^+(y|s, c) = 1$ for $y \geq \Gamma_a^{-1}(b)$ and $\Phi_a^-(y|s, c) = 0$ for $y \leq \Gamma_a^{-1}(1-b)$ (see Appendix A.3 for the definitions of $\Phi_a^+(y|s, c)$ and $\Phi_a^-(y|s, c)$).

For the variances and covariances, we use the fact that for any ECDF $\hat{F}(\cdot)$, $\mathrm{Cov}\left(\hat{F}(a), \hat{F}(b)\right) = \frac{F(a)-F(a)F(b)}{n}$ for $a \leq b$ where n is the number of steps in $\hat{F}(\cdot)$.

$$\begin{aligned}
&V^-(a|s, c, \boldsymbol{q}_s, \boldsymbol{X}) \\
&\quad = \mathrm{Var}\left(\hat{\Gamma}_a^{-1}(b) - \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \frac{\hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa}}{q_{sc}} \,\mathrm{d}y\right) \\
&\quad = \left(\frac{1}{q_{sc}}\right)^2 \mathrm{Var}\left(\int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa} \,\mathrm{d}y\right) \\
&\quad = \left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \mathrm{Cov}\begin{pmatrix} \hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa}, \\ \hat{F}(x|s, a, 1) - \hat{F}(x|s, a, 0)q_{sa} \end{pmatrix} \mathrm{d}x\mathrm{d}y \\
&\quad = 2\left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} \mathrm{Cov}\begin{pmatrix} \hat{F}(y|s, a, 1) - \hat{F}(y|s, a, 0)q_{sa}, \\ \hat{F}(x|s, a, 1) - \hat{F}(x|s, a, 0)q_{sa} \end{pmatrix} \mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= 2\left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} \mathrm{Cov}\left(\hat{F}(y|s,a,1), \hat{F}(x|s,a,1)\right) \,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} \mathrm{Cov}\left(\hat{F}(y|s,a,0), \hat{F}(x|s,a,0)\right) \,\mathrm{d}x\mathrm{d}y \\
&= \frac{2}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} F(y|s,a,1)\left(1 - F(x|s,a,1)\right) \,\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{2}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} F(y|s,a,0)\left(1 - F(x|s,a,0)\right) \,\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

where n_{sa}^0 is as defined in Section 6 and $n_{sa}^1 = \sum_{i=1}^{N} \mathbf{1}\{S_i = s, A_i = a, D_i = 1\}$. Similarly,

$$
\begin{aligned}
&V^+(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) \\
&\quad = \mathrm{Var}\left(\hat{\Gamma}_a^{-1}(1-b) - \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \frac{q_{sa} + \hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa} - 1}{q_{sc}} \,\mathrm{d}y\right) \\
&\quad = \frac{2}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \int_{y}^{\infty} F(y|s,a,1)\left(1 - F(x|s,a,1)\right) \,\mathrm{d}x\mathrm{d}y \\
&\quad\quad + \frac{2}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \int_{y}^{\infty} F(y|s,a,0)\left(1 - F(x|s,a,0)\right) \,\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

We estimate these quantities by substituting $F(\cdot|s,a,d)$ with $\hat{F}(\cdot|s,a,d)$ for $d = 0, 1$. A small sample correction can optionally be applied to these estimates by replacing n_{sa}^d with $n_{sa}^d - 1$ for $d = 0, 1$.

The covariance between $\pi^-(a|s,c)$ and $\pi^+(a|s,c)$ depends on whether $b < \frac{1}{2}$, in which case they are based on disjoint (but still correlated) portions of the same ECDFs, or whether $b \geq \frac{1}{2}$, in which case they are based on overlapping regions of the ECDFs and are therefore more correlated. If $b \geq \frac{1}{2}$,

$$
\begin{aligned}
&C\left(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}\right) \\
&\quad = \mathrm{Cov}\Bigg(\hat{\Gamma}_a^{-1}(b) - \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \frac{\hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa}}{q_{sc}} \,\mathrm{d}y, \\
&\qquad\qquad \hat{\Gamma}_a^{-1}(1-b) - \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \frac{q_{sa} + \hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa} - 1}{q_{sc}} \,\mathrm{d}y\Bigg) \\
&\quad = \mathrm{Cov}\left(\int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \frac{\hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa}}{q_{sc}} \,\mathrm{d}y, \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \frac{\hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa}}{q_{sc}} \,\mathrm{d}y\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(1-b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \mathrm{Cov}\begin{pmatrix} \hat{F}(y|s,a,1)-\hat{F}(y|s,a,0)q_{sa}, \\ \hat{F}(x|s,a,1)-\hat{F}(x|s,a,0)q_{sa} \end{pmatrix} \mathrm{d}x\mathrm{d}y \\
&\quad +2\left(\frac{1}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} \mathrm{Cov}\begin{pmatrix} \hat{F}(y|s,a,1)-\hat{F}(y|s,a,0)q_{sa}, \\ \hat{F}(x|s,a,1)-\hat{F}(x|s,a,0)q_{sa} \end{pmatrix} \mathrm{d}x\mathrm{d}y \\
&\quad +\left(\frac{1}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(b)}^{\infty} \mathrm{Cov}\begin{pmatrix} \hat{F}(y|s,a,1)-\hat{F}(y|s,a,0)q_{sa}, \\ \hat{F}(x|s,a,1)-\hat{F}(x|s,a,0)q_{sa} \end{pmatrix} \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(1-b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} F(y|s,a,1)\left(1-F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{1}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(1-b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} F(y|s,a,0)\left(1-F(x|s,a,0)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{2}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} F(y|s,a,1)\left(1-F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{2}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} F(y|s,a,0)\left(1-F(x|s,a,0)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{1}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(b)}^{\infty} F(y|s,a,1)\left(1-F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{1}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(b)}^{\infty} F(y|s,a,0)\left(1-F(x|s,a,0)\right)\,\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

and if $b < \frac{1}{2}$,

$$
\begin{aligned}
&C\left(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}\right) \\
&\quad = \frac{1}{n_{sa}^1}\left(\frac{1}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} F(y|s,a,1)\left(1-F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\qquad +\frac{1}{n_{sa}^0}\left(\frac{q_{sa}}{q_{sc}}\right)^2 \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} F(y|s,a,0)\left(1-F(x|s,a,0)\right)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Again, we estimate these by replacing $F(\cdot|s,a,d)$ with $\hat{F}(\cdot|s,a,d)$ for $d = 0,1$. The small sample correction can also be applied.

Finally, in the special case of $a = c$, the quantity $\pi(a|s,c) = \pi(c|s,c)$ is point-identified. Therefore, equation (23) reduces to a univariate normal distribution such that $\bar{\pi} \equiv \bar{\pi}^-(c|s,c,\boldsymbol{q}_s,\boldsymbol{X}) = \bar{\pi}^+(c|s,c,\boldsymbol{q}_s,\boldsymbol{X})$ and $V \equiv V^-(c|s,c,\boldsymbol{q}_s,\boldsymbol{X}) = V^+(c|s,c,\boldsymbol{q}_s,\boldsymbol{X}) = C(c|s,c,\boldsymbol{q}_s,\boldsymbol{X})$. In fact, the estimators of these parameters provided above reduce to the sample mean and the sampling variance for the mean, respectively, for the corresponding subgroup:

$$
\begin{aligned}
\hat{\bar{\pi}} &= \underline{y} + \int_{\underline{y}}^{\infty} 1 - \hat{F}(y|s,c,0)\mathrm{d}y \\
&= \underline{y} + \int_{\underline{y}}^{\infty} \sum_{i=1}^{N} \Big(1 - \mathbf{1}\{Y_i \leq y\}\Big) \cdot \frac{\mathbf{1}\{S_i = s, A_i = c, D_i = 0\}}{n_{sc}^0}\, \mathrm{d}y \\
&= \underline{y} + \frac{1}{n_{sc}^0} \sum_{i=1}^{N} \left(\int_{\underline{y}}^{Y_i} 1\, \mathrm{d}y + \int_{Y_i}^{\infty} 0\, \mathrm{d}y \right) \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\} \\
&= \frac{1}{n_{sc}^0} \sum_{i=1}^{N} Y_i \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{V} &= \frac{2}{n_{sc}^0} \int_{\underline{y}}^{\infty} \int_{y}^{\infty} \hat{F}(y|s,c,0) \Big(1 - \hat{F}(x|s,c,0)\Big)\, \mathrm{d}x\mathrm{d}y \\
&= \frac{2}{n_{sc}^0} \int_{\underline{y}}^{\infty} \int_{y}^{\infty} \left(\sum_{i=1}^{N} \mathbf{1}\{Y_i \leq y\} \cdot \frac{\mathbf{1}\{S_i = s, A_i = c, D_i = 0\}}{n_{sc}^0} \right) \\
&\qquad \times \left(\sum_{j=1}^{N} \Big(1 - \mathbf{1}\{Y_j \leq x\}\Big) \cdot \frac{\mathbf{1}\{S_j = s, A_j = c, D_j = 0\}}{n_{sc}^0} \right)\, \mathrm{d}x\mathrm{d}y \\
&= \frac{2}{(n_{sc}^0)^3} \int_{\underline{y}}^{\infty} \left(\sum_{i=1}^{N} \mathbf{1}\{Y_i \leq y\} \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\} \right) \\
&\qquad \times \sum_{j=1}^{N} \left(\int_{y}^{\infty} \Big(1 - \mathbf{1}\{Y_j \leq x\}\Big) \cdot \mathbf{1}\{S_j = s, A_j = c, D_j = 0\}\, \mathrm{d}x \right)\, \mathrm{d}y \\
&= \frac{2}{(n_{sc}^0)^3} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\underline{y}}^{\infty} \mathbf{1}\{Y_i \leq y\} \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\} \\
&\qquad \times \Big(1 - \mathbf{1}\{Y_j \leq y\}\Big)(Y_j - y) \cdot \mathbf{1}\{S_j = s, A_j = c, D_j = 0\}\, \mathrm{d}y \\
&= \frac{2}{(n_{sc}^0)^3} \sum_{i=1}^{N} \sum_{j \in \mathcal{J}} \mathbf{1}\{S_i = s, A_i = c, D_i = 0\} \cdot \mathbf{1}\{S_j = s, A_j = c, D_j = 0\} \int_{Y_i}^{Y_j} (Y_j - y)\, \mathrm{d}y, \\
&\qquad \text{with } \mathcal{J} = \big\{ j \in 1, \cdots, N : Y_j \geq Y_i \big\} \\
&= \frac{1}{n_{sc}^0} \sum_{i=1}^{N} \sum_{j \in \mathcal{J}} \frac{(Y_j - Y_i)^2}{(n_{sc}^0)^2} \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\} \cdot \mathbf{1}\{S_j = s, A_j = c, D_j = 0\} \\
&= \frac{1}{(n_{sc}^0)^2} \sum_{i=1}^{N} (Y_i - \bar{\pi})^2 \cdot \mathbf{1}\{S_i = s, A_i = c, D_i = 0\},
\end{aligned}
$$

for any $c, s \in \mathcal{A}$. Again, a small sample correction can be applied for $\hat{V}$ by multiplying it by $n^0_{sc}/(n^0_{sc}-1)$.

For the binary-outcome bounds in Proposition 2, we employ a similar procedure. Let $\boldsymbol{H} = [H_{sa}] = [\Pr(Y_i = 1|S_i = s, A_i = a, D_i = 1)]$ and $\boldsymbol{G} = [G_{sa}] = [\Pr(Y_i = 1|S_i = s, A_i = a, D_i = 0)]$. In this case, $\boldsymbol{\tau}^-$ and $\boldsymbol{\tau}^+$ are completely determined by $\boldsymbol{H}$, $\boldsymbol{G}$, $\boldsymbol{p}$ and $\boldsymbol{q}$. The endpoints of the ACTE bounds $\tau^-(a, a'|c)$ and $\tau^+(a, a'|c)$ are respectively given by the solutions to the linear problem described in Proposition 2:

$$\min_{\Phi} \quad \text{and} \quad \max_{\Phi} \quad \frac{1}{\Pr(A_i = c|D_i = 0)} \left\{ \sum_{a'' \in \{0,1\}} \sum_{s \in \mathcal{A}} \left(\phi_{1,0,y_{a''},s,c} - \phi_{0,1,y_{a''},s,c} \right) \right\}, \tag{24}$$

s.t. $\phi_{y_0,y_1,y_2,s,c'} \geq 0 \ \forall \ y_0, y_1, y_2, s, c'$, $\sum_{y_0 \in \{0,1\}} \sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \sum_{s \in \mathcal{A}} \sum_{c' \in \mathcal{A}} \phi_{y_0,y_1,y_2,s,c'} = 1$, $\sum_{y_0 \in \{0,1\}} \sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \phi_{y_0,y_1,y_2,s,c'} \cdot \mathbf{1}\{y_{c'} = 1\} = q_{sc'} p_s G_{sc'} \ \forall \ s, c'$, $\sum_{y_0 \in \{0,1\}} \sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \phi_{y_0,y_1,y_2,s,c'} = q_{sc'} p_s \ \forall \ s, c'$, and $\sum_{y_0 \in \{0,1\}} \sum_{y_1 \in \{0,1\}} \sum_{y_2 \in \{0,1\}} \sum_{c' \in \mathcal{A}} \phi_{y_0,y_1,y_2,s,c'} \cdot \mathbf{1}\{y_{a''} = 1\} = p_s H_{sa''}$ $\forall s, a''$, where $\Phi \equiv \{\phi_{y_0,y_1,y_2,s,c} : y_0 \in \{0,1\}, y_1 \in \{0,1\}, y_2 \in \{0,1\}, s \in \mathcal{A}, c \in \mathcal{A}\}$.

The joint posterior of these parameters can be factorized as $f(\boldsymbol{H}, \boldsymbol{G}, \boldsymbol{p}, \boldsymbol{q}|\boldsymbol{X}) = f(\boldsymbol{H}|\hat{\boldsymbol{F}}^1)\, f(\boldsymbol{G}|\hat{\boldsymbol{F}}^0)$ $f(\boldsymbol{q}|\boldsymbol{n}^0_s) f(\boldsymbol{p}|\boldsymbol{n})$ under Assumptions 1 and 2. We use the improper priors $H_{sa} \sim$ Beta$(0,0)$ and $G_{sa} \sim$ Beta$(0,0)$. The posteriors are then given by $H_{sa} \sim$ Beta$(\sum_{i=1}^N \mathbf{1}\{Y_i = 1, S_i = s, A_i = a, D_i = 1\}, \sum_{i=1}^N \mathbf{1}\{Y_i = 0, S_i = s, A_i = a, D_i = 1\})$ and $G_{sa} \sim$ Beta$(\sum_{i=1}^N \mathbf{1}\{Y_i = 1, S_i = s, A_i = a, D_i = 0\}, \sum_{i=1}^N \mathbf{1}\{Y_i = 0, S_i = s, A_i = a, D_i = 0\})$.

A.6 Statistical Inference for the Sensitivity Analysis

Our approach to statistical inference for the sensitivity analysis in Section 5 is similar to the procedure outlined in Section 6. In addition to the parameters defined there, we have the naïve estimates $\boldsymbol{\eta} = \{\eta(a|s) : a, s \in \mathcal{A}\}$, where $\eta(a|s) = \mathbb{E}[Y_i|S_i = s, A_i = a, D_i = 1]$. For a given value of the sensitivity parameter, $\rho = \rho_{ac} = \rho_{a'c}$, the sets of upper and lower bounds on $\tau(a, a'|c)$ are denoted $\boldsymbol{\tau}^-_\rho$ and $\boldsymbol{\tau}^+_\rho$ for $a, a', c \in \mathcal{A}$.

Given $\boldsymbol{\pi}^-$, $\boldsymbol{\pi}^+$, $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{\eta}$, we can deterministically find $\boldsymbol{\tau}^-_\rho$ and $\boldsymbol{\tau}^+_\rho$. Each pair of $\tau^-_\rho(a, a'|c)$ and

$\tau^+_\rho(a,a'|c)$ are equal to the endpoints of the following interval:

$$\begin{aligned}
&\tau(a,a'|c) \in \\
&\quad\left(\left[\eta(a|c)-\rho_{ac},\ \eta(a|c)+\rho_{ac}\right]\bigcap\left[\sum_{s\in\mathcal{A}}\pi^-(a|s,c)\frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}}q_{s'c}p_{s'}},\ \sum_{s\in\mathcal{A}}\pi^+(a|s,c)\frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}}q_{s'c}p_{s'}}\right]\right) \\
&\qquad-\left(\left[\eta(a|c)-\rho_{a'c},\ \eta(a|c)+\rho_{ac}\right]\bigcap\left[\sum_{s\in\mathcal{A}}\pi^-(a'|s,c)\frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}}q_{s'c}p_{s'}},\ \sum_{s\in\mathcal{A}}\pi^+(a'|s,c)\frac{q_{sc}p_s}{\sum_{s'\in\mathcal{A}}q_{s'c}p_{s'}}\right]\right).
\end{aligned}$$

We therefore simulate from the posterior of $(\boldsymbol{\tau}^-,\boldsymbol{\tau}^+)$ by drawing samples of $\boldsymbol{\pi}^-$, $\boldsymbol{\pi}^+$, $\boldsymbol{\eta}$, $\boldsymbol{p}$ and $\boldsymbol{q}$,

$$f(\boldsymbol{\pi}^+,\boldsymbol{\pi}^-,\boldsymbol{\eta},\boldsymbol{p},\boldsymbol{q}|\boldsymbol{X}) \;=\; f(\boldsymbol{\pi}^+,\boldsymbol{\pi}^-,\boldsymbol{\eta}|\hat{\boldsymbol{F}}^1,\hat{\boldsymbol{F}}^0,\boldsymbol{q})\, f(\boldsymbol{q}|\boldsymbol{n}^0_s)\, f(\boldsymbol{p}|\boldsymbol{n})$$

under Assumptions 1 and 2. Note that this differs from Appendix A.5 only in that the distributions of $\boldsymbol{\pi}^+$ and $\boldsymbol{\pi}^-$ are considered jointly with $\boldsymbol{\eta}$. These have the additional independence relations

$$\begin{aligned}
\eta(a|s) \;&\perp\!\!\!\perp\; \pi^+(a|s',c),\pi^-(a|s',c),\eta(a|s') \mid \hat{\boldsymbol{F}}^1,\hat{\boldsymbol{F}}^0,\boldsymbol{q} \quad \text{and} \\
\eta(a|s) \;&\perp\!\!\!\perp\; \pi^+(a'|s,c),\pi^-(a'|s,c),\eta(a'|s) \mid \hat{\boldsymbol{F}}^1,\hat{\boldsymbol{F}}^0,\boldsymbol{q}
\end{aligned}$$

for $s\neq s'$ and $a\neq a'$.

We can therefore approximate the posterior of sensitivity bounds by Monte Carlo simulation of $\boldsymbol{p}$, $\boldsymbol{q}$, and the trivariate distributions $[\pi^-(a|s,c),\ \pi^+(a|s,c),\ \eta(a|c)]$ for $a\in\{a',a''\}$ and $s\in\mathcal{A}$. By the Bayesian central limit theorem, the latter is given by

$$\begin{bmatrix}\pi^-(a|s,c)\\ \pi^+(a|s,c)\\ \eta(a|c)\end{bmatrix} \mid \boldsymbol{q},\boldsymbol{X} \;\approx\; \text{Normal}\left(\begin{bmatrix}\bar{\pi}^-(a|s,c,\boldsymbol{q}_s,\boldsymbol{X})\\ \bar{\pi}^+(a|s,c,\boldsymbol{q}_s,\boldsymbol{X})\\ \bar{\eta}(a|c,\boldsymbol{X})\end{bmatrix},\boldsymbol{\Sigma}(a|s,c)\right), \quad \text{where}$$

$$\boldsymbol{\Sigma}(a|s,c) = \begin{bmatrix} V^-(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & C(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & C^-_\eta(a|s,c,\boldsymbol{q}_s,\boldsymbol{X})\\ C(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & V^+(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & C^+_\eta(a|s,c,\boldsymbol{q}_s,\boldsymbol{X})\\ C^-_\eta(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & C^+_\eta(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}) & V_\eta(a|s,\boldsymbol{q}_s,\boldsymbol{X})\end{bmatrix}$$

when N is large, and the additional parameters $\bar{\eta}$, C^-_η, C^+_η, and V_η are defined below.

Note that naïve estimate $\eta(a|s)$ is point-identified, and its posterior mean and variance are equivalent to the sample mean and the sampling variance for the mean for the corresponding forced-choice units. These are given by:

$$\begin{aligned}
\bar{\eta}(a|s) &= \underline{y} + \int_{\underline{y}}^{\infty} 1 - F(y|s,a,1)\,\mathrm{d}y, \\
V_{\eta}(a|s) &= \frac{2}{n_{sa}^{1}} \int_{\underline{y}}^{\infty} \int_{y}^{\infty} F(y|s,a,1)\left(1 - F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Derivations closely follow Section A.5 and therefore are omitted here. Estimation can be done by plug-in with an optional small sample correction.

The posterior of $\eta(a|s)$ covaries with those of $\pi^{-}(a|s,c)$ and $\pi^{+}(a|s,c)$ because the latter parameters depend partially on the ECDF of the same forced-choice units.

$$\begin{aligned}
&C_{\eta}^{-}\left(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}\right) \\
&= \mathrm{Cov}\left(\hat{\Gamma}_a^{-1}(b) - \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \frac{\hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa}}{q_{sc}}\,\mathrm{d}y, \underline{y} + \int_{\underline{y}}^{\infty} 1 - \hat{F}(y|s,a,1)\,\mathrm{d}y\right) \\
&= \frac{2}{n_{sa}^{1}\cdot q_{sc}} \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{y}^{\hat{\Gamma}_a^{-1}(b)} F(y|s,a,1)\left(1 - F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{n_{sa}^{1}\cdot q_{sc}} \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(b)} \int_{\hat{\Gamma}_a^{-1}(b)}^{\infty} F(y|s,a,1)\left(1 - F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&C_{\eta}^{+}\left(a|s,c,\boldsymbol{q}_s,\boldsymbol{X}\right) \\
&= \mathrm{Cov}\left(\hat{\Gamma}_a^{-1}(1-b) - \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \frac{q_{sa} + \hat{F}(y|s,a,1) - \hat{F}(y|s,a,0)q_{sa} - 1}{q_{sc}}\,\mathrm{d}y,\right. \\
&\qquad \left. \underline{y} + \int_{\underline{y}}^{\infty} 1 - \hat{F}(y|s,a,1)\,\mathrm{d}y\right) \\
&= \frac{1}{n_{sa}^{1}\cdot q_{sc}} \int_{\underline{y}}^{\hat{\Gamma}_a^{-1}(1-b)} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} F(y|s,a,1)\left(1 - F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{2}{n_{sa}^{1}\cdot q_{sc}} \int_{\hat{\Gamma}_a^{-1}(1-b)}^{\infty} \int_{y}^{\infty} F(y|s,a,1)\left(1 - F(x|s,a,1)\right)\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

for any $s, c \neq a \in \mathcal{A}$.

Thus, each draw of the sensitivity results from their posterior is generated by the following procedure:

1. Draw $\boldsymbol{p} \equiv [p_s] \sim \text{Dirichlet}(\boldsymbol{n})$, where $\boldsymbol{n} \equiv [n_s] = \left[\sum_{i=1}^{N} \mathbf{1}\{S_i = 0\}, \cdots, \sum_{i=1}^{N} \mathbf{1}\{S_i = J-1\}\right]^{\top}$.

2. For each $s \in \mathcal{A}$:

   (a) Draw $\boldsymbol{q}_s \equiv [q_{sa}] \sim \text{Dirichlet}(\boldsymbol{n}_s^0)$, where $\boldsymbol{n}_s^0 \equiv [n_{sa}^0] = \Big[\sum_{i=1}^N \mathbf{1}\{S_i = s, A_i = 0, D_i = 0\}, \cdots, \sum_{i=1}^N \mathbf{1}\{S_i = s, A_i = J-1, D_i = 0\}\Big]^\top$;

   (b) For each a and $c \in \mathcal{A}$, draw a triplet $[\pi^-(a|s,c), \pi^+(a|s,c), \eta(a|s)]$ from the trivariate normal distribution defined above.

3. For a given ρ, calculate a simulated draw of $[\tau_\rho^-(a,a'|c), \tau_\rho^+(a,a'|c)]$ according to equation (9).

The sensitivity procedure for binary outcomes differs only in the last two steps:

2. (b) For each $a \in \mathcal{A}$, draw H_{sa} and G_{sa} from the posteriors discussed in Sections 6 and A.5.

3. Calculate a simulated draw of $[\tau^-(a,a'|c), \tau^+(a,a'|c)]$ by solving the linear programming problem in equation (24), with the additional sensitivity constraints $\sum_{y_0\in\{0,1\}}\sum_{y_1\in\{0,1\}}\sum_{y_2\in\{0,1\}}\sum_{s\in\mathcal{A}} \phi_{y_0,y_1,y_2,s,c}\mathbf{1}\{y_{a^*}=1\} \geq (H_{sa^*} - \rho_{a^*c})\sum_{s\in\mathcal{A}} q_{sc}p_s$ and $\sum_{y_0\in\{0,1\}}\sum_{y_1\in\{0,1\}}\sum_{y_2\in\{0,1\}}\sum_{s\in\mathcal{A}} \phi_{y_0,y_1,y_2,s,c} \mathbf{1}\{y_{a^*}=1\} \leq (H_{sa^*} + \rho_{a^*c})\sum_{s\in\mathcal{A}} q_{sc}p_s$ for given c and $a^* \in \{a, a'\}$.

A.7 Additional Simulation Results

In this section, we present additional results from the simulations described in Section 8. First, we explore the performance of the EM-algorithm-based parametric approach proposed by Long et al. (2008) (hereafter LLL) in a setting close to our empirical application. This necessitates extending LLL's original methodology, as it was developed for a binary treatment. We thus modify their parametric model to accommodate a categorical treatment by modeling the treatment choice with the multinomial logit model, as opposed to the binary logit model. (We have confirmed that our own R implementation of this extension replicates the simulation results reported by LLL in their original article almost exactly.) To make LLL's approach comparable to our proposed method in terms of observed information used, we set subjects' stated preferences as the covariate in their choice and outcome models (i.e., $X_{1i} = X_{2i} = S_i$ using their notation). We then apply the LLL estimator to the same 500 simulated datasets as in Section 8.

	CD=0.00	CD=0.33	CD=0.67	CD=1.00
LLL	0.053	0.028	0.019	-0.020
naïve	0.002	0.011	0.023	0.038
min	-0.001	-0.001	-0.001	0.000
max	-0.001	-0.001	-0.001	-0.001

Table A.1: LLL bias for various CD values, holding OD at zero. Naïve and bounds biases from Section 8.1 are reproduced here for convenience.

	OD=0.00	OD=0.33	OD=0.67	OD=1.00
LLL	0.053	0.062	0.072	0.080
naïve	0.002	0.011	0.020	0.030
min	-0.001	0.001	0.001	0.001
max	-0.001	-0.002	-0.002	-0.001

Table A.2: LLL bias for various OD values, holding CD at zero. Naïve and bounds biases from Section 8.2 are reproduced here for convenience.

Tables A.1 and A.2 show the results in terms of bias at the sample size of 3,000 (second row from the top), along with the comparable results for the naïve estimator and our proposed bounds estimator (third row and below), which are reproduced from Tables 2 and 3 in the main text. Somewhat surprisingly, and contrary to the original findings by LLL based on a much simpler simulation setup, the LLL estimator exhibits substantial bias even when both CD and OD are zero. This suggests that finite-sample performance of the LLL estimator is rather poor when applied to datasets like ours, rendering it an unattractive option for inference.

Next, we contrast the proposed Bayesian inferential approach described in Section A.5 to an alternative method based on the nonparametric bootstrap. We construct the 95% bootstrap confidence intervals by taking the 2.5th and 97.5th percentiles of parameter estimates in 1000 bootstrap draws. For the bounds, we take those percentiles from the lower and upper bound estimates, respectively, to construct confidence intervals that are purported to cover the nonparametric bounds 95% of the time.

Tables A.3 and A.4 show estimated coverage rates for the 95% bootstrap confidence intervals at various values of the CD and OD parameters. The comparable results for our proposed Bayesian intervals can be found in Tables A.3 and A.4 in the main text. In general, we find that the coverage of the bootstrap intervals is noticeably below that of our proposed method, and the bootstrap coverage rates are

n	CD=0.00	CD=0.33	CD=0.67	CD=1.00
500	0.944	0.930	0.895	0.891
1000	0.941	0.915	0.911	0.906
3000	0.952	0.914	0.908	0.924
10000	0.930	0.924	0.924	0.928
50000	0.936	0.940	0.940	0.942

Table A.3: Bootstrap coverage rates for various CD values, holding OD at zero.

n	OD=0.00	OD=0.33	OD=0.67	OD=1.00
500	0.944	0.954	0.949	0.950
1000	0.941	0.960	0.956	0.959
3000	0.952	0.942	0.952	0.944
10000	0.930	0.952	0.950	0.958
50000	0.936	0.948	0.944	0.946

Table A.4: Bootstrap coverage rates for various OD values, holding CD at zero.

substantially below nominal at lower sample sizes and for larger values of the CD parameter.

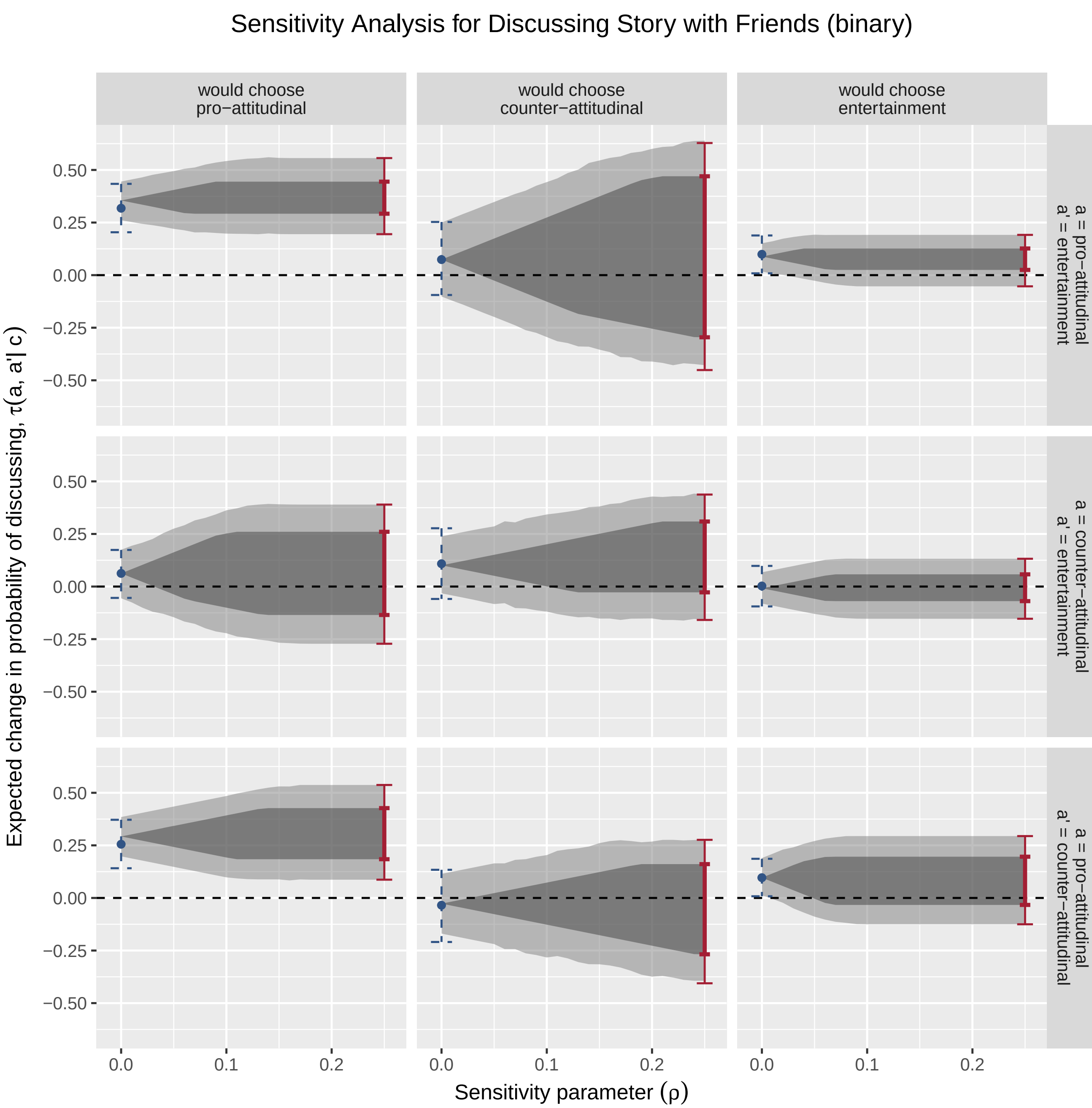

Figure A.1: Sensitivity Analysis for the ACTE of Partisan News Media (Binary Outcome). The plots correspond to the right panel of Figure 2. See caption for Figure 3 for the explanation of graph elements.