Supplementary Material: Evolutionary Dynamics of Coordinated Cooperation

Hisashi Ohtsuki *
*Correspondence:
Hisashi Ohtsuki:
ohtsuki_hisashi@soken.ac.jp

A A PROOF OF THE EXISTENCE OF A STATIONARY STATE

Consider the n-player public goods game, and consider a group of n players where the number of C_k players is given by I_k $(0 \leq k \leq n)$. By definition $I_0 + \cdots + I_n = n$ holds. The aim of this section is to prove the existence of at least one stationary state.

For that purpose, I recursively define the sequence of integers, $\{J_m\}_{m=0,1,\cdots}$, by

$$\begin{cases} J_0 = 0 \\ J_m = \displaystyle\sum_{k=0}^{J_{m-1}} I_k \quad (m \geq 1). \end{cases} \tag{S1}$$

It is easy to prove (by using mathematical induction) that each integer J_m is well-defined and upper-bounded by n.

Next I will prove by mathematical induction that the sequence $\{J_m\}_{m=0,1,\cdots}$ is non-decreasing. First, $J_1 - J_0 = I_0 \geq 0$ holds. Second, assume that $J_m - J_{m-1} \geq 0$ holds for $m \geq 1$. Then I obtain

$$J_{m+1} - J_m = \sum_{k=0}^{J_m} I_k - \sum_{k=0}^{J_{m-1}} I_k \geq 0 \tag{S2}$$

because I's are non-negative integers. This completes the proof.

Because the sequence $\{J_m\}_{m=0,1,\cdots}$ is a non-decreasing sequence of integers upper-bounded by n, there exists the smallest $m^* \geq 1$ satisfying $J_{m^*} = J_{m^*-1}$. Set $k^* = J_{m^*}$. Then I have

$$k^* = \sum_{k=0}^{k^*} I_k. \tag{S3}$$

Next I will prove that $I_{k^*} = 0$ holds. I consider two separate cases. First, suppose that m^*=1. From the definition of m^*, $J_1 = J_0$ holds. But from the definition $J_1 = I_0$ and $J_0 = 0$ hold. Therefore $I_0 = 0$ follows. From the definition of k^*, I have $k^* = J_1 = 0$. Therefore $I_{k^*} = I_0 = 0$ follows. Second, suppose that $m^* \geq 2$ holds. From the definition of m^* I have

$$0 = J_{m^*} - J_{m^*-1} = \sum_{k=0}^{J_{m^*-1}} I_k - \sum_{k=0}^{J_{m^*-2}} I_k. \tag{S4}$$

However, from the minimality of m^*, I have $J_{m^*-1} > J_{m^*-2}$. Therefore $I_{J_{m^*-1}} = I_{k^*} = 0$ holds. End of the proof.

Now I am ready for proving the original claim. Consider a state where players with strategies from C_0 to C_{k^*} adopt thought C and all the others adopt thought D. The total number of players adopting thought C at this state is calculated as $\sum_{k=0}^{k^*} I_k$, but from eq.(S3) it is equal to k^*. Because thresholds of those who currently have thought D are strictly greater than k^*, they do not want to change their current thought. For those who currently adopt C, there are $k^* - 1$ others players with thought C. Therefore players from C_0 to C_{k^*-1} do not want to change their thought. Because $I_{k^*} = 0$ holds, C_{k^*} strategists are absent from the group. Therefore the proposed state is a stationary state. This ends the proof.

B STOCHASTIC EVOLUTIONARY DYNAMICS

I consider the Fermi process described in the main text for a finite population of size M. Generally speaking, it is a Markov process, the state space of which is all possible partition of M, that is

$$\{(M_0, \cdots, M_n) \mid M_0 + \cdots + M_n = M, \quad M_k \in \{0, 1, \cdots, M\} \text{ (for all } k)\}, \tag{S5}$$

where M_k corresponds to the number of C_k players in the population

I first review some known results for $n = 2$. Suppose $M \geq 2$, and consider a single mutant strategy τ invading a population of resident strategy σ, where $\sigma, \tau \in \{0, 1, 2\}$. In what follows I will calculate the fixation probability of the mutant, which is denoted by $\pi_{\sigma \to \tau}$.

For that purpose, imagine that there are i mutants and $M - i$ residents in the population. Their average payoffs are calculated, respectively, as

$$\begin{aligned} w_\tau(i) &= \frac{1}{M-1}\left\{(i-1)a_{\tau,\tau} + (M-i)a_{\tau,\sigma}\right\} \\ w_\sigma(i) &= \frac{1}{M-1}\left\{ia_{\sigma,\tau} + (M-i-1)a_{\sigma,\sigma}\right\}. \end{aligned} \tag{S6}$$

According to a general argument about the fixation probability of a single mutant (Nowak et al. 2004), the fixation probability is calculated as

$$\pi_{\sigma \to \tau} = 1 \Bigg/ \sum_{j=0}^{M-1} \prod_{i=1}^{j} \exp\left[-s\{w_\tau(i) - w_\sigma(i)\}\right], \tag{S7}$$

where I employ the convention, $\prod_{i=1}^{0} \cdot = 1$. Applying eq.(S7) to the payoffs (S6) yields

$$
\begin{aligned}
\pi_{\sigma\to\tau} &= 1 \Bigg/ \sum_{j=0}^{M-1} \prod_{i=1}^{j} \exp\left[-\frac{s}{M-1}(u_1 i + u_0)\right] \\
&= 1 \Bigg/ \sum_{j=0}^{M-1} \exp\left[-\frac{s}{2(M-1)}\{u_1 j(j+1) + 2u_0 j\}\right],
\end{aligned}
\tag{S8}
$$

where

$$
\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & M & 0 & -M+1 \end{pmatrix} \begin{pmatrix} a_{\tau,\tau} \\ a_{\tau,\sigma} \\ a_{\sigma,\tau} \\ a_{\sigma,\sigma} \end{pmatrix}.
\tag{S9}
$$

Similarly, I consider the case of $n = 3$. Suppose $M \geq 3$ and consider mutants τ and residents σ, where $\sigma, \tau \in \{0, 1, 2, 3\}$. When there are i mutants and $(M - i)$ residents, their average payoffs are

$$
\begin{aligned}
w_\tau(i) &= \frac{1}{(M-1)(M-2)}\{(i-1)(i-2)a_{\tau,\tau\tau} + 2(i-1)(M-i)a_{\tau,\tau\sigma} + (M-i)(M-i-1)a_{\tau,\sigma\sigma}\} \\
w_\sigma(i) &= \frac{1}{(M-1)(M-2)}\{i(i-1)a_{\sigma,\tau\tau} + 2i(M-i-1)a_{\sigma,\tau\sigma} + (M-i-1)(M-i-2)a_{\sigma,\sigma\sigma}\},
\end{aligned}
\tag{S10}
$$

respectively. Applying eq.(S7) to the payoffs (S10) yields

$$
\begin{aligned}
\pi_{\sigma\to\tau} &= 1 \Bigg/ \sum_{j=0}^{M-1} \prod_{i=1}^{j} \exp\left[-\frac{s}{(M-1)(M-2)}(v_2 i^2 + v_1 i + v_0)\right] \\
&= 1 \Bigg/ \sum_{j=0}^{M-1} \exp\left[-\frac{s}{6(M-1)(M-2)}\{v_2 j(j+1)(2j+1) + 3v_1 j(j+1) + 6v_0 j\}\right],
\end{aligned}
\tag{S11}
$$

where

$$
\begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & -1 & 2 & -1 \\ -3 & 2M+2 & -2M+1 & 1 & -2M+2 & 2M-3 \\ 2 & -2M & M^2-M & 0 & 0 & -M^2+3M-2 \end{pmatrix} \begin{pmatrix} a_{\tau,\tau\tau} \\ a_{\tau,\tau\sigma} \\ a_{\tau,\sigma\sigma} \\ a_{\sigma,\tau\tau} \\ a_{\sigma,\tau\sigma} \\ a_{\sigma,\sigma\sigma} \end{pmatrix}.
\tag{S12}
$$

C ADIABATIC LIMIT AND STRONG SELECTION FOR THE TWO-PERSON GAME

When I consider the adiabatic limit, $\mu \to 0$, I need to calculate fixation probabilities for all combinations of resident and mutant strategies. According to Eq. (S8), when selection is strong ($s \to \infty$) the fixation probability is determined solely by the sign of $f_{\sigma\to\tau}(j) \equiv u_1 j(j+1) + 2u_0 j$ as follows;

$$\pi_{\sigma\to\tau} = \begin{cases} 0 & \text{if at least one of } f_{\sigma\to\tau}(0), \cdots, f_{\sigma\to\tau}(M-1) \text{ is negative} \\ 1/K & \text{otherwise; } K \text{ is the number of zeros among } \{f_{\sigma\to\tau}(0), \cdots, f_{\sigma\to\tau}(M-1)\} \end{cases} \quad \text{(S13)}$$

(see eq.(S9) for the definitions of u_1 and u_0). Remember that $f_{\sigma\to\tau}(0) = 0$ always holds.

I will consider six separate cases below. Remember that I assume $1 < r < 2$.

When $(\sigma, \tau) = (1, 0)$

A calculation shows

$$f_{1\to 0}(j) = c(r-1)(1-p)j\{2M - (j+3)\}. \quad \text{(S14)}$$

Therefore I have

$$\pi_{1\to 0} = \begin{cases} 1/M & \text{if } p = 1 \\ 1/2 & \text{if } p \neq 1, M = 2 \\ 1 & \text{if } p \neq 1, M \geq 3. \end{cases} \quad \text{(S15)}$$

When $(\sigma, \tau) = (2, 0)$

A calculation shows

$$f_{2\to 0}(j) = -c\{2(r-1) + (2-r)M\}j, \quad \text{(S16)}$$

which is negative for all $j \geq 1$. Therefore $\pi_{2\to 0} = 0$.

When $(\sigma, \tau) = (0, 1)$

A calculation shows

$$f_{0\to 1}(j) = -c(r-1)(1-p)j(j-1). \quad \text{(S17)}$$

Therefore I have

$$\pi_{0\to 1} = \begin{cases} 1/M & \text{if } p = 1 \\ 1/2 & \text{if } p \neq 1, M = 2 \\ 0 & \text{if } p \neq 1, M \geq 3. \end{cases} \quad \text{(S18)}$$

When $(\sigma, \tau) = (2, 1)$

A calculation shows

$$f_{2\to 1}(j) = c(r-1)pj(j-1), \quad \text{(S19)}$$

Therefore I have

$$\pi_{2\to 1} = \begin{cases} 1/M & \text{if } p = 0 \\ 1/2 & \text{if } p \neq 0. \end{cases} \quad \text{(S20)}$$

When $(\sigma, \tau) = (0, 2)$
A calculation shows

$$f_{0\to 2}(j) = c\{2(r-1) + (2-r)M\}j, \tag{S21}$$

which is always positive for $j \geq 1$. Therefore I have $\pi_{0\to 2} = 1$.

When $(\sigma, \tau) = (1, 2)$
A calculation shows

$$f_{1\to 2}(j) = -c(r-1)pj\{2M - (j+3)\}. \tag{S22}$$

Therefore I have

$$\pi_{1\to 2} = \begin{cases} 1/M & \text{if } p = 0 \\ 1/2 & \text{if } p \neq 0, M = 2 \\ 0 & \text{if } p \neq 0, M \geq 3. \end{cases} \tag{S23}$$

Given these fixation probabilities, I can calculate the stationary distribution of the Fermi process over monomorphic states, namely the relative fraction of time the process spends at all-C_0, all-C_1 and all-C_2 states, which I denote by (q_0, q_1, q_2). Because $M = 2$ is somewhat a degenerate case, I will consider $M \geq 3$ in the following. Because a rare mutation produces one of the two strategies that are not present in the currently monomorphic population, the transition matrix between these three states is given by

$$\begin{aligned} &\begin{pmatrix} \rho_{0\to 0} & \rho_{1\to 0} & \rho_{2\to 0} \\ \rho_{0\to 1} & \rho_{1\to 1} & \rho_{2\to 1} \\ \rho_{0\to 2} & \rho_{1\to 2} & \rho_{2\to 2} \end{pmatrix} \\ &= \begin{pmatrix} 1 - \left(\frac{1}{2}\pi_{0\to 1} + \frac{1}{2}\pi_{0\to 2}\right) & \frac{1}{2}\pi_{1\to 0} & \frac{1}{2}\pi_{2\to 0} \\ \frac{1}{2}\pi_{0\to 1} & 1 - \left(\frac{1}{2}\pi_{1\to 0} + \frac{1}{2}\pi_{1\to 2}\right) & \frac{1}{2}\pi_{2\to 1} \\ \frac{1}{2}\pi_{0\to 2} & \frac{1}{2}\pi_{1\to 2} & 1 - \left(\frac{1}{2}\pi_{2\to 0} + \frac{1}{2}\pi_{2\to 1}\right) \end{pmatrix}, \end{aligned} \tag{S24}$$

where $\rho_{\sigma\to\tau}$ represents the transition probability that a resident population of strategy C_σ is taken over strategy C_τ when one random mutant of unknown identity arises in the resident population. The stationary distribution is given as a right eigenvector of this transition matrix;

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \rho_{0\to 0} & \rho_{1\to 0} & \rho_{2\to 0} \\ \rho_{0\to 1} & \rho_{1\to 1} & \rho_{2\to 1} \\ \rho_{0\to 2} & \rho_{1\to 2} & \rho_{2\to 2} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix}. \tag{S25}$$

I consider three separate cases below.

When $p = 0$
The transition matrix is

$$\begin{pmatrix} \rho_{0\to 0} & \rho_{1\to 0} & \rho_{2\to 0} \\ \rho_{0\to 1} & \rho_{1\to 1} & \rho_{2\to 1} \\ \rho_{0\to 2} & \rho_{1\to 2} & \rho_{2\to 2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2M} & \frac{1}{2M} \\ \frac{1}{2} & \frac{1}{2M} & 1 - \frac{1}{2M} \end{pmatrix}, \tag{S26}$$

and the stationary distribution is $(q_0, q_1, q_2) = \frac{1}{M+3}(1, 1, M+1)$.

When $0 < p < 1$
The transition matrix is

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{3}{4} \end{pmatrix}, \tag{S27}$$

and the stationary distribution is $(q_0, q_1, q_2) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

When $p = 1$
The transition matrix is

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2M} & \frac{1}{2M} & 0 \\ \frac{1}{2M} & 1 - \frac{1}{2M} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{3}{4} \end{pmatrix}, \tag{S28}$$

and the stationary distribution is $(q_0, q_1, q_2) = \frac{1}{M+4}(1, M+1, 2)$.

D OUTCOMES OF NEGOTIATION IN THE THREE-PERSON GAME

Here I investigate outcomes of the three-person public goods games played by C_0, C_1, C_2, C_3 players. In Table 2 in the main text I list up all the possible compositions of players. There are 20 different cases. In 14 cases, there is only one stationary state and it is easy to confirm that the negotiation process always leads to that state irrespective of players' initial thought.

All that remains is to study the other 6 cases. As an example, here I describe my detailed analysis of the case of (C_0, C_2, C_2), that is, when one C_0 player and two C_2 players are matched.

Because three players are involved and because each player has either thought C or D, there are $2^3 = 8$ possible states. Thanks to the symmetry between the two C_2 players, however, I do not have to distinguish their identity, and therefore I should study only the number of C_2 players whose current thought is C. This reasoning reduces the number of states from 8 to 6. More specifically, by (u, v) $(u = \{0, 1\}, v = \{0, 1, 2\})$ I hereafter mean the state where the number of C_0 players whose current thought is C and the number of C_2 players whose current thought is C are u and v, respectively.

Let $\phi_{u,v}(t)$ be the probability that the three players is at state (u, v) after t steps of update. Because each player independently has C as his initial thought with probability p and D with probability $1-p$, I obtain

$$\begin{pmatrix} \phi_{0,0}(0) \\ \phi_{0,1}(0) \\ \phi_{0,2}(0) \\ \phi_{1,0}(0) \\ \phi_{1,1}(0) \\ \phi_{1,2}(0) \end{pmatrix} = \begin{pmatrix} (1-p)^3 \\ 2p(1-p)^2 \\ p^2(1-p) \\ p(1-p)^2 \\ 2p^2(1-p) \\ p^3 \end{pmatrix}. \tag{S29}$$

Let us consider transitions between states. For example, imagine state $(0, 1)$, where C_0 player's thought is D and one C_2 player has thought C and the other C_2 player has thought D. If the C_0 player is chosen for updating his thought, he changes his thought from D to C because he always wants to cooperate. If the C_2 player with currently C-thought is chosen for the update, on the other hand, he will change his thought to D because he finds no cooperators among the other two players. If the C_2 player with currently D-thought is chosen for the update, he stays with the same thought because he finds only one cooperator among the other two players. In sum, the transition from state $(0,1)$ to state $(1,1)$ occurs with probability $1/3$, to state $(0,0)$ occurs with probability $1/3$, and no transition occurs with probability $1/3$. A similar calculation leads to the following transition matrix between states;

$$\begin{pmatrix}\phi_{0,0}(t+1)\\ \phi_{0,1}(t+1)\\ \phi_{0,2}(t+1)\\ \phi_{1,0}(t+1)\\ \phi_{1,1}(t+1)\\ \phi_{1,2}(t+1)\end{pmatrix} = \begin{pmatrix}2/3 & 1/3 & 0 & 0 & 0 & 0\\ 0 & 1/3 & 2/3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1/3 & 0 & 0 & 1 & 1/3 & 0\\ 0 & 1/3 & 0 & 0 & 1/3 & 0\\ 0 & 0 & 1/3 & 0 & 1/3 & 1\end{pmatrix} \begin{pmatrix}\phi_{0,0}(t)\\ \phi_{0,1}(t)\\ \phi_{0,2}(t)\\ \phi_{1,0}(t)\\ \phi_{1,1}(t)\\ \phi_{1,2}(t)\end{pmatrix}. \tag{S30}$$

Note that states $(1,0)$ and $(1,2)$ are stationary states. Solving this recursion with the initial condition, eq.(S29), gives

$$\begin{pmatrix}\phi_{0,0}(\infty)\\ \phi_{0,1}(\infty)\\ \phi_{0,2}(\infty)\\ \phi_{1,0}(\infty)\\ \phi_{1,1}(\infty)\\ \phi_{1,2}(\infty)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\\ -\frac{1}{2}p^2-\frac{1}{2}p+1\\ 0\\ \frac{1}{2}p^2+\frac{1}{2}p\end{pmatrix}. \tag{S31}$$

At state $(1,0)$, C_0 players cooperates and two C_2 players do not. At state $(1,2)$ all the three players cooperate. Therefore, the expected payoff of the C_0 player is given by

$$a_{0,22} = \left(-\frac{1}{2}p^2-\frac{1}{2}p+1\right)\left(-c+\frac{rc}{3}\right)+\left(\frac{1}{2}p^2+\frac{1}{2}p\right)(-c+rc) = -c+\frac{p^2+p+1}{3}rc, \tag{S32}$$

and the expected payoff of a C_2 player is given by

$$a_{2,02} = \left(-\frac{1}{2}p^2-\frac{1}{2}p+1\right)\frac{rc}{3}+\left(\frac{1}{2}p^2+\frac{1}{2}p\right)(-c+rc) = -\frac{p^2+p}{2}c+\frac{p^2+p+1}{3}rc. \tag{S33}$$

The other 5 cases can be studied in a similar manner.

Another, and a little simpler derivation of these expected payoffs is to rely on the argument of ultimate probabilities of absorption. Let $\psi_{a,b}^{(a^*,b^*)}$ be the probability that the negotiation process starting from state (a,b) ultimately ends up at state (a^*,b^*). It is not difficult to see that they satisfy the following relation;

$$\begin{aligned}&\begin{pmatrix}\psi_{0,0} & \psi_{0,1} & \psi_{0,2} & \psi_{1,0} & \psi_{1,1} & \psi_{1,2}\end{pmatrix}\\ &\quad = \begin{pmatrix}\psi_{0,0} & \psi_{0,1} & \psi_{0,2} & \psi_{1,0} & \psi_{1,1} & \psi_{1,2}\end{pmatrix}\begin{pmatrix}2/3 & 1/3 & 0 & 0 & 0 & 0\\ 0 & 1/3 & 2/3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1/3 & 0 & 0 & 1 & 1/3 & 0\\ 0 & 1/3 & 0 & 0 & 1/3 & 0\\ 0 & 0 & 1/3 & 0 & 1/3 & 1\end{pmatrix},\end{aligned} \tag{S34}$$

where I omitted the superscript (a^*, b^*). Taking into account the fact that there are two stationary states, $(1,0)$ and $(1,2)$, the ultimate probability that the negotiation process arrives at $(a^*, b^*) = (1,0)$ should satisfy

$$\psi_{1,0}^{(1,0)} = 1, \quad \psi_{1,2}^{(1,0)} = 0. \tag{S35}$$

Solving eqs.(S34, S35) gives

$$\begin{pmatrix}\psi_{0,0}^{(1,0)} & \psi_{0,1}^{(1,0)} & \psi_{0,2}^{(1,0)} & \psi_{1,0}^{(1,0)} & \psi_{1,1}^{(1,0)} & \psi_{1,2}^{(1,0)}\end{pmatrix} = \begin{pmatrix}1 & \frac{3}{4} & \frac{1}{2} & 1 & \frac{1}{2} & 0\end{pmatrix}, \tag{S36}$$

and therefore the probability that the negotiation process arrives at state $(1,0)$ is given by

$$\begin{aligned}&\begin{pmatrix}\psi_{0,0}^{(1,0)} & \psi_{0,1}^{(1,0)} & \psi_{0,2}^{(1,0)} & \psi_{1,0}^{(1,0)} & \psi_{1,1}^{(1,0)} & \psi_{1,2}^{(1,0)}\end{pmatrix}\begin{pmatrix}\phi_{0,0}(0)\\ \phi_{0,1}(0)\\ \phi_{0,2}(0)\\ \phi_{1,0}(0)\\ \phi_{1,1}(0)\\ \phi_{1,2}(0)\end{pmatrix}\\ &= \begin{pmatrix}1 & \frac{3}{4} & \frac{1}{2} & 1 & \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}(1-p)^3\\ 2p(1-p)^2\\ p^2(1-p)\\ p(1-p)^2\\ 2p^2(1-p)\\ p^3\end{pmatrix} = -\frac{1}{2}p^2 - \frac{1}{2}p + 1.\end{aligned} \tag{S37}$$

A similar argument leads to the absorption probability to state $(1,2)$, too.

E REPLICATOR DYNAMICS FOR THE THREE-PERSON GAME

In this section I will pay attention to the replicator dynamics on the edges of the simplex, S_4. Remember that x_k $(k = 0,1,2,3)$ represents the frequency of strategy C_k in the population, and that w_k $(k = 0,1,2,3)$ represents the average payoff of strategy C_k. The following analysis assumes $0 < p < 1$ and $1 < r < 3$.

<u>On the C_0-C_1 edge</u>
A calculation shows

$$w_0 - w_1 = c(1-p)^2(r-1)(1-x_0)^2, \tag{S38}$$

which is positive in the interior of the edge. Hence C_0 dominates C_1.

<u>On the C_1-C_2 edge</u>
A calculation shows

$$w_1 - w_2 = \frac{c}{2}p(1-p)(r-1)\{2x_1(1-x_1)+1\} > 0. \tag{S39}$$

Therefore, C_1 dominates C_2.

On the C_2-C_3 edge
A calculation shows

$$w_2 - w_3 = \frac{cp^2(3-p)^2(2r-3)^2}{9(r-1)}, \tag{S40}$$

which is positive unless $r = 3/2$ (note that $r = 3/2$ is a 'knife-edge' case and I do not pursue in the following). Therefore C_2 dominates C_3.

On the C_3-C_0 edge
A calculation shows

$$w_3 - w_0 = \frac{c}{3}(3-r) > 0. \tag{S41}$$

Therefore, C_3 dominates C_0.

On the C_0-C_2 edge
A calculation shows

$$w_2 - w_0 = \frac{c}{3}(1-p)x_2\{3(r-1)x_2 - (2+p)(2r-3)\}. \tag{S42}$$

When $r < 3/2$, the expression inside the curly brackets is positive, so the whole expression above is positive in the interior of the edge. Hence, C_2 dominates C_0. When $3/2 < r < \frac{3+3p}{1+2p}$, there is an unstable equilibrium P_{02} at

$$x_2^* = \frac{(2+p)(2r-3)}{3(r-1)}, \tag{S43}$$

and the system shows bistability. When $r > \frac{3+3p}{1+2p}$, I have

$$3(r-1)x_2 - (2+p)(2r-3) < 3(r-1)\cdot 1 - (2+p)(2r-3) = -(1+2p)r + (3+3p) < 0 \tag{S44}$$

in the interior of the edge and hence $w_2 < w_0$ holds, suggesting that C_0 dominates C_2.

On the C_1-C_3 edge
A calculation shows

$$w_3 - w_1 = \frac{c}{3}p(1-x_3)\{-3(r-1)x_3 + (2r-3)p + (-3r+6)\}. \tag{S45}$$

When $r < 3/2$, I have

$$-3(r-1)x_3 + (2r-3)p + (-3r+6) > -3(r-1)\cdot 1 + (2r-3)\cdot 1 + (-3r+6) = -4r+6 > 0 \tag{S46}$$

in the interior of the edge, and hence C_3 dominates C_1. When $3/2 < r < \frac{6-3p}{3-2p}$ there exists a stable equilibrium Q_{13} at

$$x_3^* = \frac{(2r-3)p + (-3r+6)}{3(r-1)}, \tag{S47}$$

and the system allows coexistence of C_1 and C_3. When $r > \frac{6-3p}{3-2p}$ I have

$$-3(r-1)x_3+(2r-3)p+(-3r+6) < -3(r-1)\cdot 0+(2r-3)p+(-3r+6) = -(3-2p)r+(6-3p) < 0 \tag{S48}$$

in the interior of the edge, suggesting that C_1 dominates C_3.

F ADIABATIC LIMIT AND STRONG SELECTION FOR THE THREE-PERSON GAME

I consider the adiabatic limit, $\mu \to 0$, for $n = 3$. When selection is strong ($s \to \infty$) the fixation probability of strategy τ invading the population of strategy σ is determined by the sign of $g_{\sigma\to\tau}(j) \equiv v_2 j(j+1)(2j+1) + 3v_1 j(j+1) + 6v_0 j$ (see eq. (S11)), as follows;

$$\pi_{\sigma\to\tau} = \begin{cases} 0 & \text{if at least one of } g_{\sigma\to\tau}(0), \cdots, g_{\sigma\to\tau}(M-1) \text{ is negative} \\ 1/K & \text{otherwise; } K \text{ is the number of zeros among } \{g_{\sigma\to\tau}(0), \cdots, g_{\sigma\to\tau}(M-1)\} \end{cases} \tag{S49}$$

(see eq.(S12) for the definition of v_2, v_1 and v_0). Note that $g_{\sigma\to\tau}(0) = 0$ always holds.

The function $g_{\sigma\to\tau}(j)$ can be a quadratic function of j even after factoring out one j, and therefore it is generally difficult to determine the sign of $g_{\sigma\to\tau}$ at M discrete points, $j = 0, \cdots, M-1$. Therefore, I restrict my attention to a large M and treat $z \equiv j/M$ as a continuous variable as an approximation, where $0 \le z \le 1$. More precisely speaking, I set $j = Mz$ and consider the polynomial $g(Mz)$, of which leading term with respect to M is M^3. For a large M, therefore, the sign of the following polynomial

$$\begin{aligned} G_{\sigma\to\tau}(z) &= \lim_{M\to\infty} \frac{g(Mz)}{M^3} \\ &= 2z\Bigg[(a_{\tau,\tau\tau} - 2a_{\tau,\tau\sigma} + a_{\tau,\sigma\sigma} - a_{\sigma,\tau\tau} + 2a_{\sigma,\tau\sigma} - a_{\sigma,\sigma\sigma})z^2 \\ &\qquad + 3(a_{\tau,\tau\sigma} - a_{\tau,\sigma\sigma} - a_{\sigma,\tau\sigma} + a_{\sigma,\sigma\sigma})z + 3(a_{\tau,\sigma\sigma} - a_{\sigma,\sigma\sigma})\Bigg], \end{aligned} \tag{S50}$$

in $0 \le z \le 1$ determines the fixation probability. In particular, if there exists some $0 \le z^* \le 1$ such that $G_{\sigma\to\tau}(z^*) < 0$ holds, then the corresponding fixation probability is calculated as zero. Otherwise, I separately calculate $g_{\sigma\to\tau}(0)\ (=0), g_{\sigma\to\tau}(1), g_{\sigma\to\tau}(2) \cdots$ to find the number of consecutive zeros from $g_{\sigma\to\tau}(0)$, and calculate the fixation probability as $1/K$, where K is the number of consecutive zeros. Note that, because $g_{\sigma\to\tau}$ is at most cubic in j, the number of consecutive zeros can be either $1, 2, 3$ or M (in the last case $g_{\sigma\to\tau}$ is identical to zero).

Below I will study twelve separate cases. Remember that I assume a large M and $1 < r < 3$ in the following.

When $(\sigma, \tau) = (1, 0)$

If $p = 1$ I have $g_{1\to 0}(j) \equiv 0$ for all j, and hence $\pi_{1\to 0} = 1/M$. If $p \ne 1$, $g_{1\to 0}(j)$ is quadratic in j even after factoring out one j, so I consider

$$G_{1\to 0}(z) = 2c(r-1)(1-p)^2 z(z^2 - 3z + 3), \tag{S51}$$

which is non-negative in $0 \le z \le 1$. Calculations show

$$\begin{aligned} g_{1\to 0}(0) &= 0 \\ g_{1\to 0}(1) &> 0, \end{aligned} \tag{S52}$$

and therefore $K = 1$ and $\pi_{1\to 0} = 1$. To summarize, I obtain

$$\pi_{1\to 0} = \begin{cases} 1/M & \text{if } p = 1 \\ 1 & \text{if } p \neq 1. \end{cases} \tag{S53}$$

When $(\sigma, \tau) = (2, 0)$

If $p = 1$ I have $g_{2\to 0}(j) \equiv 0$ for all j, and hence $\pi_{2\to 0} = 1/M$. If $p \neq 1$, $g_{2\to 0}(j)$ is quadratic in j even after factoring out one j, so I consider

$$G_{2\to 0}(z) = -c(1-p)z\{2(r-1)z^2 + (2pr - 2r - 3p)z - 2(2pr + r - 3p - 3)\}. \tag{S54}$$

For $G_{2\to 0}(z)$ to be non-negative in $0 \le z \le 1$,

$$\{2(r-1)z^2 + (2pr - 2r - 3p)z - 2(2pr + r - 3p - 3)\}|_{z=0,1} \le 0, \tag{S55}$$

is necessary and sufficient, which is equivalent to

$$r \ge \frac{3 + 3p}{1 + 2p}. \tag{S56}$$

For $r > (3 + 3p)/(1 + 2p)$, separate calculations show

$$\begin{aligned} g_{2\to 0}(0) &= 0 \\ g_{2\to 0}(1) &> 0, \end{aligned} \tag{S57}$$

and therefore $K = 1$ and $\pi_{2\to 0} = 1$. If $r < (3 + 3p)/(1 + 2p)$, on the other hand, I have $\pi_{2\to 0} = 0$. To summarize, I obtain

$$\pi_{2\to 0} = \begin{cases} 1/M & \text{if } p = 1 \\ 1 & \text{if } p \neq 1 \text{ and } r > \frac{3+3p}{1+2p} \\ 0 & \text{if } p \neq 1 \text{ and } r < \frac{3+3p}{1+2p}, \end{cases} \tag{S58}$$

where I avoided the evaluation of the knife-edge case, $r = (3 + 3p)/(1 + 2p)$.

When $(\sigma, \tau) = (3, 0)$

A straightforward calculation shows

$$g_{3\to 0}(j) = -2cj(M - 2)\{(3 - r)M + 3(r - 1)\}, \tag{S59}$$

which is negative for $M \geq 3$ and $j \geq 1$. Therefore I obtain

$$\pi_{3\to 0} = 0. \tag{S60}$$

When $(\sigma, \tau) = (0, 1)$

A straightforward calculation shows

$$g_{0\to 1}(j) = -2c(r-1)(1-p)^2 j(j-1)(j-2). \tag{S61}$$

If $p = 1$, I have $g_{0\to 1}(j) \equiv 0$ and therefore $\pi_{0\to 1} = 1/M$. If $p \neq 1$, I have $g_{0\to 1}(j) < 0$ for $j \geq 3$, and therefore $\pi_{0\to 1} = 0$ holds for $M \geq 4$. To summarize, I obtain

$$\pi_{0\to 1} = \begin{cases} 1/M & \text{if } p = 1 \\ 0 & \text{if } p \neq 1. \end{cases} \tag{S62}$$

When $(\sigma, \tau) = (2, 1)$

If $p = 0$ or 1, I have $g_{2\to 1}(j) \equiv 0$ for all j, and hence $\pi_{2\to 1} = 1/M$. If $p \neq 0, 1$, $g_{2\to 1}(j)$ is quadratic in j even after factoring out one j, so I consider

$$G_{2\to 1}(z) = -c(r-1)p(1-p)z(2z^2 - 3z - 3), \tag{S63}$$

which is non-negative in $0 \leq z \leq 1$. Calculations show

$$\begin{aligned} g_{2\to 1}(0) &= 0 \\ g_{2\to 1}(1) &> 0, \end{aligned} \tag{S64}$$

and therefore $K = 1$ and $\pi_{2\to 1} = 1$. To summarize, I obtain

$$\pi_{2\to 1} = \begin{cases} 1/M & \text{if } p = 0, 1 \\ 1 & \text{if } p \neq 0, 1. \end{cases} \tag{S65}$$

When $(\sigma, \tau) = (3, 1)$

A straightforward calculation leads to

$$g_{3\to 1}(j) = -cpj(j-1)\{2(r-1)j + M(2r-3)(p-3) - 2(r-1)(3p-7)\}. \tag{S66}$$

If $p = 0$, I have $g_{3\to 1}(j) \equiv 0$ for all j, and hence $\pi_{3\to 1} = 1/M$. If $p \neq 0$, $g_{3\to 1}(j) \leq 0$ holds for all $j = 0, \cdots, M-1$ if and only if $g_{3\to 1}(M-1) \leq 0$, because the expression inside the curly brackets in eq.(S66) is an increasing function of j. Solving $g_{3\to 1}(M-1) \leq 0$ for a large M gives the condition,

$$r \geq \frac{7-3p}{4-2p}. \tag{S67}$$

For $r > (7-3p)(4-2p)$,

$$\begin{aligned} g_{3\to1}(0) &= 0 \\ g_{3\to1}(1) &= 0 \\ g_{3\to1}(2) &> 0 \end{aligned} \tag{S68}$$

hold, so I have $K = 2$ and therefore $\pi_{3\to1} = 1/2$. If $r < (7-3p)/(4-2p)$, on the other hand, I have $\pi_{3\to1} = 0$. To summarize, I obtain

$$\pi_{3\to1} = \begin{cases} 1/M & \text{if } p = 0 \\ 1/2 & \text{if } p \neq 0 \text{ and } r > \frac{7-3p}{4-2p} \\ 0 & \text{if } p \neq 0 \text{ and } r < \frac{7-3p}{4-2p}, \end{cases} \tag{S69}$$

where I avoided the evaluation of the knife-edge case, $r = (7-3p)/(4-2p)$.

When $(\sigma, \tau) = (0, 2)$

A straightforward calculation shows

$$g_{0\to2}(j) = c(1-p)j(j-1)\{2(r-1)j - M(2r-3)(p+2) + 2(r-1)(3p+4)\}. \tag{S70}$$

If $p = 1$, I have $g_{0\to2}(j) \equiv 0$ for all j, and hence $\pi_{0\to2} = 1/M$. If $p \neq 1$, I find that $g_{0\to2}(j) \geq 0$ holds for all $j = 0, \cdots, M-1$ if and only if $g_{0\to2}(2) \geq 0$ holds, because the expression inside the curly brackets in eq.(S70) is an increasing function of j. Solving $g_{0\to2}(2) \geq 0$ for a large M gives the condition,

$$r \leq \frac{3}{2}. \tag{S71}$$

For $r < 3/2$,

$$\begin{aligned} g_{0\to2}(0) &= 0 \\ g_{0\to2}(1) &= 0 \\ g_{0\to2}(2) &> 0 \end{aligned} \tag{S72}$$

hold, so I have $K = 2$ and therefore $\pi_{0\to2} = 1/2$. If $r > 3/2$, on the other hand, I have $\pi_{0\to2} = 0$. To summarize, I obtain

$$\pi_{0\to2} = \begin{cases} 1/M & \text{if } p = 1 \\ 1/2 & \text{if } p \neq 1 \text{ and } r < \frac{3}{2} \\ 0 & \text{if } p \neq 1 \text{ and } r > \frac{3}{2}, \end{cases} \tag{S73}$$

where I avoided the evaluation of the knife-edge case, $r = 3/2$.

When $(\sigma, \tau) = (1, 2)$

If $p = 0$ or 1, I have $g_{1\to2}(j) \equiv 0$ for all j, and hence $\pi_{1\to2} = 1/M$. If $p \neq 0, 1$, $g_{1\to2}(j)$ is quadratic in j

even after factoring out one j, so I consider

$$G_{1\to 2}(z) = c(r-1)p(1-p)z(2z^2 - 3z - 3), \tag{S74}$$

which is negative for $0 < z \leq 1$. Therefore, $\pi_{1\to 2} = 0$. To summarize, I obtain

$$\pi_{1\to 2} = \begin{cases} 1/M & \text{if } p = 0, 1 \\ 0 & \text{if } p \neq 0, 1. \end{cases} \tag{S75}$$

When $(\sigma, \tau) = (3, 2)$
A straightforward calculation shows

$$g_{3\to 2}(j) = 2c(r-1)p^2 j(j-1)(j-2). \tag{S76}$$

If $p = 0$, I have $g_{3\to 2}(j) \equiv 0$ for all j, and hence $\pi_{3\to 2} = 1/M$. If $p \neq 0$, I find that $g_{3\to 2}(0) = g_{3\to 2}(1) = g_{3\to 2}(2) = 0$ and $g_{3\to 2}(j) > 0$ for all $j \geq 3$, and therefore $K = 3$ and $\pi_{3\to 2} = 1/3$. To summarize, I obtain

$$\pi_{3\to 2} = \begin{cases} 1/M & \text{if } p = 0 \\ 1/3 & \text{if } p \neq 0. \end{cases} \tag{S77}$$

When $(\sigma, \tau) = (0, 3)$
A calculation shows

$$g_{0\to 3}(j) = 2cj(M-2)\{(3-r)M + 3(r-1)\}, \tag{S78}$$

which is zero at $j = 0$ but positive for $M \geq 3$ and $j \geq 1$. Therefore I obtain

$$\pi_{0\to 3} = 1. \tag{S79}$$

When $(\sigma, \tau) = (1, 3)$
If $p = 0$, I have $g_{1\to 3}(j) \equiv 0$ for all j, and hence $\pi_{1\to 3} = 1/M$. If $p \neq 0$, $g_{1\to 3}(j)$ is quadratic in j even after factoring out one j, so I consider

$$G_{1\to 3}(z) = cpz\{2(r-1)z^2 - (2rp - 3p + 3)z + 2(2rp - 3p - 3r + 6)\}. \tag{S80}$$

Let $H_{1\to 3}(z)$ be the expression inside the curly brackets in eq.(S80). The axis of the parabola $y = H_{1\to 3}(z)$ lies at $\hat{z} = (2rp-3p+3)/4(r-1)$, which is positive. If $\hat{z} < 1$ (which is equivalent to $r > (7-3p)/(4-2p)$), for $G_{1\to 3}(z)$ to be non-negative in $0 \leq z \leq 1$

$$H_{1\to 3}(\hat{z}) = -\frac{\{(4-2p)r - (7-3p)\}\{(4-2p)r + (3p-15) + 8r\}}{8(r-1)} \geq 0. \tag{S81}$$

is sufficient and necessary. However, this is never satisfied because

$$\begin{aligned} (4-2p)r-(7-3p) &> (7-3p)-(7-3p)=0 \\ (4-2p)r+(3p-15)+8r &> (7-3p)+(3p-15)+8r=8(r-1)>0, \end{aligned} \tag{S82}$$

and therefore $\pi_{1\to3}=0$ is concluded. If the axis lies in $\hat{z}\geq 1$ (which is equivalent to $r\leq(7-3p)/(4-2p)$), for $G_{1\to3}(z)$ to be non-negative in $0\leq z\leq 1$

$$H_{1\to3}(1)=-\{(4-2p)r-(7-3p)\}\geq 0, \tag{S83}$$

is necessary and sufficient, which is indeed satisfied because

$$(4-2p)r-(7-3p)\leq(7-3p)-(7-3p)=0. \tag{S84}$$

For $r<(7-3p)/(4-2p)$, I find

$$\begin{aligned} g_{1\to3}(0) &= 0 \\ g_{1\to3}(1) &> 0, \end{aligned} \tag{S85}$$

and therefore $K=1$ and $\pi_{1\to3}=1$ holds. To summarize, I obtain

$$\pi_{1\to3}=\begin{cases} 1/M & \text{if } p=0 \\ 1 & \text{if } p\neq 0 \text{ and } r<\frac{7-3p}{4-2p} \\ 0 & \text{if } p\neq 0 \text{ and } r>\frac{7-3p}{4-2p}, \end{cases} \tag{S86}$$

where I avoided the evaluation of the knife-edge case, $r=(7-3p)/(4-2p)$.

When $(\sigma,\tau)=(2,3)$

If $p=0$, I have $g_{2\to3}(j)\equiv 0$ for all j, and hence $\pi_{2\to3}=1/M$. If $p\neq 0$, $g_{2\to3}(j)$ is quadratic in j even after factoring out one j, so I consider

$$G_{2\to3}(z)=-2cp^2(r-1)z(z^2-3z+3), \tag{S87}$$

which is negative in $0<z\leq 1$. Therefore, $\pi_{2\to3}=0$. To summarize, I obtain

$$\pi_{2\to3}=\begin{cases} 1/M & \text{if } p=0 \\ 0 & \text{if } p\neq 0. \end{cases} \tag{S88}$$

Given these twelve fixation probabilities, I will calculate the stationary distribution of the Fermi process. Let q_i be the fraction of time that the stochastic process under the adiabatic limit and strong selection stays at the all-C$_i$ state $(i = 0, 1, 2, 3)$. It is calculated as the solution of

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{S89}$$

where the matrix with ρ's is the transition matrix. I consider three separate cases. Note that the following analysis assumes a large M.

(i) When $p = 0$

(i-a) When $r < 3/2$

The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} - \frac{2}{3M} & \frac{1}{3M} & \frac{1}{3M} \\ \frac{1}{6} & \frac{1}{3M} & 1 - \frac{2}{3M} & \frac{1}{3M} \\ \frac{1}{3} & \frac{1}{3M} & \frac{1}{3M} & 1 - \frac{2}{3M} \end{pmatrix}. \tag{S90}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{33 + 9M}(6, 9, 9 + 4M, 9 + 5M). \tag{S91}$$

(i-b) When $r > 3/2$

The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} - \frac{2}{3M} & \frac{1}{3M} & \frac{1}{3M} \\ 0 & \frac{1}{3M} & 1 - \frac{2}{3M} & \frac{1}{3M} \\ \frac{1}{3} & \frac{1}{3M} & \frac{1}{3M} & 1 - \frac{2}{3M} \end{pmatrix}. \tag{S92}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{12 + 3M}(3, 3, 3 + M, 3 + 2M). \tag{S93}$$

(ii) When $0 < p < 1$

(ii-a) When $r < 3/2$

The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{8}{9} \end{pmatrix}. \tag{S94}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{26}(2, 3, 6, 15). \tag{S95}$$

(ii-b) When $3/2 < r < (7-3p)/(4-2p)$
The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{8}{9} \end{pmatrix}. \tag{S96}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{10}(1, 1, 2, 6). \tag{S97}$$

(ii-c) When $(7-3p)/(4-2p) < r < (3+3p)/(1+2p)$
The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{2}{3} & \frac{1}{9} \\ \frac{1}{3} & 0 & 0 & \frac{13}{18} \end{pmatrix}. \tag{S98}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{18}(5, 5, 2, 6). \tag{S99}$$

(ii-d) When $r > (3+3p)/(1+2p)$
The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{1}{9} \\ \frac{1}{3} & 0 & 0 & \frac{13}{18} \end{pmatrix}. \tag{S100}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{16}(5, 4, 1, 6). \tag{S101}$$

(iii) When $p = 1$
(iii-a) When $r < 2$
The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} - \frac{2}{3M} & \frac{1}{3M} & \frac{1}{3M} & 0 \\ \frac{1}{3M} & \frac{2}{3} - \frac{2}{3M} & \frac{1}{3M} & 0 \\ \frac{1}{3M} & \frac{1}{3M} & 1 - \frac{2}{3M} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{8}{9} \end{pmatrix}. \tag{S102}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{9+M}(1, 1, 1+M, 6). \tag{S103}$$

(iii-b) When $r > 2$
The transition matrix is given by

$$\begin{pmatrix} \rho_{0\to0} & \rho_{1\to0} & \rho_{2\to0} & \rho_{3\to0} \\ \rho_{0\to1} & \rho_{1\to1} & \rho_{2\to1} & \rho_{3\to1} \\ \rho_{0\to2} & \rho_{1\to2} & \rho_{2\to2} & \rho_{3\to2} \\ \rho_{0\to3} & \rho_{1\to3} & \rho_{2\to3} & \rho_{3\to3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} - \frac{2}{3M} & \frac{1}{3M} & \frac{1}{3M} & 0 \\ \frac{1}{3M} & 1 - \frac{2}{3M} & \frac{1}{3M} & \frac{1}{6} \\ \frac{1}{3M} & \frac{1}{3M} & 1 - \frac{2}{3M} & \frac{1}{9} \\ \frac{1}{3} & 0 & 0 & \frac{13}{18} \end{pmatrix}. \tag{S104}$$

The stationary distribution is

$$(q_0, q_1, q_2, q_3) = \frac{1}{63 + 15M}(15, 15 + 8M, 15 + 7M, 18). \tag{S105}$$