Appendix 1

Detailed expression for the first term of $B_6\left(\mathbf{u}_6,\mathbf{v}\right)$ given by formula (72) :

$$\sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\int_0^L \mathbf{S}_l\left[v_k^{(q,o-q)}\right](s)\,A_{qop}^{kl\{u\}}(s)\,ds = \sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\int_0^L v_{k,s}^{(q,o-q)}A_{qop}^{k1\{u\}}(s)\,ds$$
$$+\sum_{o=1}^{5}\sum_{q=1}^{5}\sum_{u=1}^{2} q\int_0^L v_k^{(q,o-q)}A_{q-1o-1p}^{k2\{u\}}(s)\,ds + \sum_{o=1}^{5}\sum_{q=0}^{4}\sum_{u=1}^{2}\left(o-q\right)\int_0^L v_k^{(q,o-q)}A_{qo-1p}^{k3\{u\}}(s)\,ds.$$

Appendix 2

In this section, we give the justification of the estimate given by formula **Error! Reference source not found.** valid in linear elasticity (section *4.2*).
The method used here consist to compute the upper bound of formula **Error! Reference source not found.** for the exact expression of the displacement field and then to deduce that for the linear model of section *4.2* obtained by truncation of the potential energy at seventh order (linear model of section 3), the estimate **Error! Reference source not found.**.
For the expression of the exact expression of the potential energy obtained with no truncation of the displacement field **Error! Reference source not found.** (we assume that **u** has a convergent infinite series), by using integration by part , we can obtain:

$$E_{pot}(\mathbf{u}) = \frac{1}{2}B_\infty(\mathbf{u},\mathbf{u}) - F_\infty(\mathbf{u}), \qquad (1)$$

in which

$$B_\infty(\mathbf{u},\mathbf{v}) = \sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=1}^{2}\int_0^L v_k^{(q,o-q)}(s)A_{\infty qop}^{*k\{u\}}(s)\,ds + v_k^{(q,o-q)}(L)A_{\infty qop}^{k1\{u\}}(L) - v_k^{(q,o-q)}(0)A_{\infty qop}^{k1\{u\}}(0)$$
$$+\sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w=0}^{\infty}\sum_{u=1}^{2}\int_0^L v_i^{(q,o-q)}(s)B_{\infty qowp}^{i\{u\}}(s)\,ds, \qquad (2)$$

$$F_\infty(\mathbf{v}) = \sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=1}^{2}\int_0^L v_i^{(q,o-q)}(s)C_{\infty qop}^{i\{u\}}(s)\,ds + \sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty} v_i^{(q,o-q)}(s=0)\,D_{\infty qop}^{i}$$
$$+\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\sum_{o=0}^{\infty}\int_0^L v_i^{(q,o-q)}(s)F_{\infty qop}^{i}(s)\,ds$$

in which

$$A_{\infty qop}^{*k\{u\}}\left(\mathbf{U}^{(p-0+1)}(s)\right) = -A_{\infty qop,s}^{k1\{u\}}\left(\mathbf{U}^{(p-0)}(s)\right) + q\,A_{\infty q-1o-1p}^{k2\{u\}}\left(\mathbf{U}^{(p-0+1)}(s)\right) + \left(o-q\right)A_{\infty q\,o-1p}^{k3\{u\}}\left(\mathbf{U}^{(p-0+1)}(s)\right),$$

with if $(q,o,p)\in \mathbf{I}_\infty = \left\{(q^*,o^*,p^*)\in \mathbb{N}^3 \text{ such that } 0\le q^*\le o^*\le p^*\right\}$ then

$$A_{\infty qop}^{kl\{1\}}\left(\mathbf{U}^{(p-0)}(s)\right) = \int_S y_2^q\, y_3^{o-q}\, \sigma^{kl\,(p-o)}\left(\mathbf{u}(\mathbf{y})\right)dy_2\,dy_3,$$

if $(q,o,p)\notin \mathbf{I}_\infty$ then $A_{\infty qop}^{kl\{1\}} = 0$,

if $(q,o,p)\in \mathbf{J}_\infty = \left\{(q^*,o^*,p^*)\in \mathbb{N}^3 \text{ such that } 0\le q^*\le o^*\le p^*\right\}$ then

$A_{\infty\, qop}^{kl\{2\}}\left(\mathbf{U}^{(p-0)}(s)\right) = -k(s)\int_S y_2^{q+1} y_3^{o-q} \sigma^{kl\,(p-o)}\left(\mathbf{u}(\mathbf{y})\right) dy_2\, dy_3,$

if $(q,o,p) \notin \mathbf{J}_\infty$ then

$A_{\infty\, qop}^{kl\{2\}}(s) = 0$

and if $(q,o,w,p) \in \mathbf{K}_\infty = \left\{ \left(q^*, o^*, w^*, p^*\right) \in \mathbb{N}^4 \text{ such that } 0 \le q^* \le o^* \le w^* \le p^* \right\}$ then

$B_{\infty\, qowp}^{i\{1\}}\left(\mathbf{U}^{(p-w)}(s)\right) = -\int_S \Gamma_{kl}^{i\,(w-o)}(\mathbf{y}) y_2^q y_3^{o-q} \sigma^{kl\,(p-w)}\left(\mathbf{u}(\mathbf{y})\right) dy_2\, dy_3,$

if $(q,o,w,p) \notin \mathbf{K}_\infty$ then

$B_{\infty\, qowp}^{i\{1\}} = 0,$

if $(q,o,w,p) \in \mathbf{L}_\infty = \left\{ \left(q^*, o^*, w^*, p^*\right) \in \mathbb{N}^4 \text{ such that } 0 \le q^* \le o^* \le w^* \le p^* \right\}$ then

$B_{\infty\, qowp}^{i\{2\}}\left(\mathbf{U}^{(p-w)}(s)\right) = k(s)\int_S \Gamma_{kl}^{i\,(w-o)}(\mathbf{y}) y_2^{q+1} y_3^{o-q} \sigma^{kl\,(p-w)}\left(\mathbf{u}(\mathbf{y})\right) dy_2\, dy_3\,,$

if $(q,o,w,p) \notin \mathbf{L}_\infty$ then

$B_{\infty\, qowp}^{i\{2\}} = 0,$

if $(q,o,p) \in \mathbf{I}_\infty$ then

$C_{\infty\, qop}^{i\{1\}}(s) = \int_S y_2^q y_3^{o-q}\, f^{i\,(p-o)}(\mathbf{y}) dy_2\, dy_3,$

if $(q,o,p) \notin \mathbf{I}_\infty$ then

$C_{\infty\, qop}^{i\{1\}} = 0,$

if $(q,o,p) \in \mathbf{J}_\infty$ then

$C_{\infty\, qop}^{i\{2\}}(s) = -k(s)\int_S y_2^{q+1} y_3^{o-q}\, f^{i\,(p-o)}(\mathbf{y}) dy_2\, dy_3,$

if $(q,o,p) \notin \mathbf{J}_\infty$ then

$C_{\infty\, qop}^{i\{2\}}(s) = 0,$

if $(q,o,p) \in \mathbf{I}_\infty$ then

$D_{\infty\, qop}^{i} = \int_S y_2^q y_3^{o-q}\, \overline{\tau}^{\,i(p-o)}\left(s=0, y_2, y_3\right) dy_2\, dy_3\,,$

if $(q,o,p) \notin \mathbf{I}_\infty$ then $D_{\infty\, qop}^{i} = 0,$

if $(q,o,p) \in \mathbf{M}_\infty = \left\{ \left(q^*, o^*, p^*\right) \in \mathbb{N}^3 \text{ such that } 0 \le q^* \le o^* \le p^* \right\}$ then

$F_{\infty\, qop}^{i}(s) = \int_{\partial S} D_{lat}^{(p-o)}(\mathbf{y}) y_2^q y_3^{o-q}\, \overline{t}_{lat}^{\,i}(\mathbf{y}) ds_{lat},$

with $D_{lat}^{(n)}(\mathbf{y})$ are the terms of the asymptotic expansion of $\left\| \boldsymbol{\tau}_{lat}\left(y_1, \boldsymbol{\theta}\right) \wedge \mathbf{g}_1\left(y_1, \boldsymbol{\theta}\right) \right\|$

$\left(\left\| \boldsymbol{\tau}_{lat}\left(y_1, \boldsymbol{\theta}\right) \wedge \mathbf{g}_1\left(y_1, \boldsymbol{\theta}\right) \right\| = \sum_{n=0}^{\infty} D_{lat}^{(n)}(\mathbf{y}) \right),$

if $(q,o,p) \notin \mathbf{M}_\infty$ then

$F_{\infty\, qop}^{i}(s) = 0.$

From (1), we obtain the exact Euler-Lagrange equations associated to the complete expression of the potential energy :

For k = 1, 2, 3

$$\sum_{u=1}^{2}\sum_{p=0}^{\infty}\left(A_{\infty qop}^{*k\{u\}}\left(s\right)-C_{\infty qop}^{k\{u\}}\left(s\right)+\sum_{w=0}^{\infty}B_{\infty qowp}^{k\{u\}}\left(s\right)\right)-\sum_{p=0}^{\infty}F_{\infty qop}^{k}\left(s\right)=0, \tag{3}$$

and Neumann type boundary conditions at s = 0,

$$\sum_{p=0}^{\infty}(\sum_{u=1}^{2}A_{\infty qop}^{k1\{u\}}\left(0\right)+D_{\infty qop}^{k})=0, \tag{4}$$

Now we have to express the dual energy (25) ($E_d\left(\boldsymbol{\sigma}\right)=\widetilde{E}_d\left(\tilde{\boldsymbol{\sigma}}\right)$) in terms of curvilinear coordinates:

$$E_d\left(\boldsymbol{\sigma}\right)=-\frac{1}{2}\int_{\mathbf{B}}D_{ijkl}\,\boldsymbol{\sigma}^{ij}\boldsymbol{\sigma}^{kl}\sqrt{g}\,d\mathbf{y}+\int_{\Sigma_L}\boldsymbol{\sigma}^{i1}\,\overline{u}_i\sqrt{g}\,d\Sigma_L\,, \tag{5}$$

in which D_{ijkl} are the covariant components of the compliance tensor.

Let **u** be a displacement field that is associared with $\boldsymbol{\sigma}(\sigma^{ij}\left(\mathbf{u}\right)=E^{ijkl}u_{k\|l}$), if we insert the series expansion for **u** in **Error! Reference source not found.** and by using integration by part, expression (5) of the dual energy becomes:

$$E_d\left(\boldsymbol{\sigma}\right)=-\frac{1}{2}B_{\infty}\left(\mathbf{u},\mathbf{u}\right)+F_{d\infty}\left(\mathbf{u}\ \right), \tag{6}$$

in which $F_{d\infty}(\mathbf{u})=\sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=1}^{2}\overline{u}_k^{(q,\,o-q)}\left(L\right)A_{\infty qop}^{k1\{u\}}\left(L\right)$.

Now we want to vanish the first variation of the dual energy (variational formulation). We note that the variation of the stress tensor have to fulfill the condition associated with the three-dimensional equilibrium condition with vanishing body force and the three dimensional stress boundary condition with free traction. The displacement field **v** associated to this stress tensor has a complete expansion of the type **Error! Reference source not found.** with no truncation:

For k = 1, 2, 3

$$\sum_{u=1}^{2}\sum_{p=0}^{\infty}\left(A_{\infty qop}^{*k\{u\}}\left(\mathbf{V}^{(p-o)}(s)\right)+\sum_{w=0}^{\infty}B_{\infty qowp}^{k\{u\}}\left(\mathbf{V}^{(p-w)}(s)\right)\right)=0, \tag{7}$$

and Neumann type boundary conditions at s = 0,

$$\sum_{p=0}^{\infty}\sum_{u=1}^{2}A_{\infty qop}^{k1\{u\}}\left(\mathbf{V}^{(p-o)}(0)\right)=0. \tag{8}$$

Then (6)-(8) show that the variational formulation of the dual energy is :

$$\sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=1}^{2}\left(\overline{u}_k^{(q,\,o-q)}\left(L\right)-u_k^{(q,\,o-q)}\left(L\right)\right)A_{\infty qop}^{k1\{u\}}\left(\mathbf{V}^{(p-o)}(L)\right)=0. \tag{9}$$

In other words this variational formulation gives the Dirichlet boundary condition at $s = L$:

$$\overline{u}_k^{(q,\,o-q)}\left(L\right)-u_k^{(q,\,o-q)}\left(L\right)=0. \tag{10}$$

Now by combining (1)-(6), we see that the upper bound of **Error! Reference source not found.** becomes :

$$E_{pot}\left(\mathbf{u}\right)-E_d\left(\boldsymbol{\sigma}\left(\mathbf{u}\right)\right)=$$

$$\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\int_0^L u_k^{(q,o-q)}(s)\left(\sum_{u=1}^{2}\sum_{p=0}^{\infty}\left(A_{\infty qop}^{*k\{u\}}(s)-C_{\infty qop}^{k\{u\}}(s)+\sum_{w=0}^{\infty}B_{\infty qowp}^{k\{u\}}(s)\right)-\sum_{p=0}^{\infty}F_{\infty qop}^{k}(s)\right)ds$$

$$-\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}u_k^{(q,o-q)}(0)\left(\sum_{p=0}^{\infty}(\sum_{u=1}^{2}A_{\infty qop}^{k1\{u\}}(0)+D_{\infty qop}^{k})\right)$$

$$+\sum_{p=0}^{\infty}\sum_{o=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=1}^{2}\left(\overline{u}_k^{(q,o-q)}(L)-u_k^{(q,o-q)}(L)\right)A_{\infty qop}^{k1\{u\}}\left(\mathbf{U}^{(p-o)}(L)\right). \qquad (11)$$

In expression (11), we can observe in the first, second and third term of its right-hand side respectively appears the left hand side term of the Euler-Lagrange equation (3), (4) and (10) respectively. So for the exact solution $E_{pot}(\mathbf{u}_{sol})-E_d(\sigma(\mathbf{u}_{sol}))$ vanishes.

Now we consider the complete potential energy associated with the the truncated displacement field **Error! Reference source not found.**. Let $\mathbf{u}_{c5}$ be the solution of minimization of this complete potential energy, we see that Euler-Lagrange equations of this model can be written as follows :

$$\sum_{u=1}^{2}\sum_{p=0}^{4}\left(A_{qop}^{*k\{u\}}\left(\mathbf{U}_{c5}^{(p-0+1)}(s)\right)-C_{qop}^{k\{u\}}(s)+\sum_{w=0}^{4}B_{qowp}^{k\{u\}}\left(\mathbf{U}_{c5}^{(p-w)}(s)\right)\right)-\sum_{p=0}^{5}F_{qop}^{k}(s)+O(\rho^7)=0, \qquad (12)$$

in which

$$A_{qop}^{*k\{u\}}\left(\mathbf{U}_{c5}^{(p-0+1)}(s)\right)=-A_{qop,s}^{k1\{u\}}\left(\mathbf{U}_{c5}^{(p-0)}(s)\right)+q\,A_{q-1o-1p}^{k2\{u\}}\left(\mathbf{U}_{c5}^{(p-0+1)}(s)\right)+(o-q)A_{q\,o-1p}^{k3\{u\}}\left(\mathbf{U}_{c5}^{(p-0+1)}(s)\right),$$

with $\mathbf{U}_{c5}^{(m)}=\left\{\ \mathbf{u}_{c5}^{(i,i-j)}\ \text{ for }\ i=0,m+1 \text{ and } j=0,i\right\}$

and Neumann type boundary conditions at s = 0,

$$\sum_{p=0}^{4}(\sum_{u=1}^{2}A_{qop}^{k1\{u\}}\left(\mathbf{U}_{c5}^{(p-0)}(0)\right)+D_{qop}^{k})+O(\rho^7)=0 \ , \qquad (13)$$

We can observe that the difference between equations (12)-(13) and equations **Error! Reference source not found.-Error! Reference source not found.** are the terms $O(\rho^7)$ and represents the additional terms which appear when we consider the complete expression of the potential energy. For our purpose, it is not necessary to give their complete expression here.

Now we rewrite (11) by separating the terms of order of magnitude strictly greater than ρ^7 and the other ones for which obviously the order of magnitude is less than ρ^7 :

$$E_{pot}(\mathbf{u})-E_d(\sigma(\mathbf{u}))=-\sum_{p=0}^{4}\sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\int_0^L u_k^{(q,o-q)}(s)A_{qop,s}^{k1\{u\}}(s)ds+\sum_{p=0}^{4}\sum_{o=1}^{5}\sum_{q=1}^{5}\sum_{u=1}^{2}q\int_0^L u_k^{(q,o-q)}(s)A_{q-1o-1p}^{k2\{u\}}(s)ds$$

$$+\sum_{p=0}^{4}\sum_{o=1}^{5}\sum_{q=0}^{4}\sum_{u=1}^{2}(o-q)\int_0^L u_k^{(q,o-q)}A_{qo-1p}^{k3\{u\}}(s)ds+\sum_{o=0}^{4}\sum_{q=0}^{4}\int_0^L u_k^{(q,o-q)}(s)\sum_{u=1}^{2}\sum_{p=0}^{4}\left(-C_{qop}^{k\{u\}}(s)+\sum_{w=0}^{4}B_{qowp}^{k\{u\}}(s)\right)ds$$

$$+\sum_{p=0}^{5}\sum_{q=0}^{5}\sum_{o=0}^{5}\int_0^L v_i^{(q,o-q)}(s)F_{qop}^{i}(s)ds\ -\sum_{o=0}^{4}\sum_{q=0}^{4}u_k^{(q,o-q)}(0)\left(\sum_{p=0}^{\infty}(\sum_{u=1}^{2}A_{qop}^{k1\{u\}}(0)+D_{qop}^{k})\right)$$

$$+\sum_{p=0}^{4}\sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\left(\overline{u}_k^{(q,o-q)}(L)-u_k^{(q,o-q)}(L)\right)A_{qop}^{k1\{u\}}\left(\mathbf{U}^{(p-o)}(L)\right)+O(\rho^7). \qquad (14)$$

If we consider equation (14) for $\mathbf{u} = \mathbf{u}_{c5}$, equation (12) is satisfied and the summation of the five first terms of right hand side of equation (14) is of order of magnitude $O(\rho^7)$; equation (13) is satisfied and the sixth term of the right-hand side of equation (14) is also of order of magnitude $O(\rho^7)$ and finally $\mathbf{u}_{c5}^{(o-q,q)}(L) = \bar{\mathbf{u}}^{(o-q,q)}(L)$ is satisfied and the last term of the right-hand side of equation (14) vanishes and finally we obtain :

$$E_{pot}\left(\mathbf{u}_{c5}\right) - E_d\left(\boldsymbol{\sigma}\left(\mathbf{u}_{c5}\right)\right) = O(\rho^7). \tag{15}$$

Since the stress tensor associated with $\mathbf{u}_{c5}$ satisfies the equilibrium equation and the Neumann boundary condition, then **Error! Reference source not found.** is valid for $\mathbf{u}_{c5}$. Finally by combining **Error! Reference source not found.**, **Error! Reference source not found.** and (15), we obtain the estimate **Error! Reference source not found.**.
We subtract equation **Error! Reference source not found.** from equation (12) and equation **Error! Reference source not found.** from equation (13), we obtain :

$$\sum_{u=1}^{2}\sum_{p=0}^{4} A_{qop}^{*k\{u\}}\left(\mathbf{U}_{c5}^{(p-0+1)}(s)\right) - A_{qop}^{*k\{u\}}\left(\mathbf{U}_{6}^{(p-0+1)}(s)\right) + \sum_{u=1}^{2}\sum_{p=0}^{4}\sum_{w=0}^{4} B_{qowp}^{k\{u\}}\left(\mathbf{U}_{c5}^{(p-w)}(s)\right) - B_{qowp}^{k\{u\}}\left(\mathbf{U}_{6}^{(p-w)}(s)\right) + O\left(\rho^7\right) = 0, \tag{16}$$

$$\sum_{p=0}^{4}\sum_{u=1}^{2} A_{qop}^{k1\{u\}}\left(\mathbf{U}_{c5}^{(p-0)}(0)\right) - A_{qop}^{k1\{u\}}\left(\mathbf{U}_{6}^{(p-0)}(0)\right) + O\left(\rho^7\right) = 0. \tag{17}$$

Appendix 3

Detailed expression for the first term of $\delta\Psi_{el\,6}\left(\mathbf{x},\mathbf{x}^*\right)$ given by formula (111):

$$\sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\int_0^L \mathbf{S}_l\left[x_k^{*(q,o-q)}\right](s)A_{qo}^{+kl\{u\}}(s)ds = \sum_{o=0}^{4}\sum_{q=0}^{4}\sum_{u=1}^{2}\int_0^L x_{k,s}^{*(q,o-q)}A_{qo}^{+k1\{u\}}(s)ds$$
$$+\sum_{o=1}^{5}\sum_{q=1}^{5}\sum_{u=1}^{2} q\int_0^L x_k^{*(q,o-q)}A_{q-1o-1}^{+k2\{u\}}(s)ds + \sum_{o=1}^{5}\sum_{q=0}^{4}\sum_{u=1}^{2}(o-q)\int_0^L x_k^{*(q,o-q)}A_{qo-1}^{+k3\{u\}}(s)ds.$$