Supplement to “Optimal exercise of American options under stock pinning”

Abstract

This supplement is structured as follows. Section A contains all the proofs omitted in the paper. Section B states technical lemmas required by these proofs.

Keywords: American option; Brownian bridge; free-boundary problem; optimal stopping; option pricing; put-call parity; stock pinning.

A Main proofs

Proof of Proposition 1. Take an admissible pair (t,x) satisfying $x \geq S$ and $t < T$, and consider the stopping time $\tau_\varepsilon := \inf\{0 \leq s \leq T-t : X_{t+s} \leq S-\varepsilon \mid X_t = x\}$ (assume for convenience that $\inf\{\emptyset\} = T-t$), for $\varepsilon > 0$. Notice that $\mathbb{P}_{t,x}[\tau_\varepsilon < T-t] > 0$, which implies that $V(t,x) \geq \mathbb{E}_{t,x}\left[e^{-\lambda\tau_\varepsilon}G(X_{t+\tau_\varepsilon})\right] > 0 = G(x)$, from where it comes that $(t,x) \in C$.

Define $b(t) := \sup\{x \in \mathbb{R} : (t,x) \in D\}$. The above arguments guarantee that $b(t) < S$ for all $t \in [0,T)$, and we get from (1) that $b(T) = S$. Furthermore, from (1) it can be easily noticed that as λ increases $V(t,x)$ decreases and therefore $b(t)$ increases, and since $b(t)$ is known to be finite for all t when $\lambda = 0$ (see Subsection 3.2), then we can guarantee that $b(t) > -\infty$ for all values of λ.

Notice that, since D is a closed set, $b(t) \in D$ for all $t \in [0,T]$. In order to prove that D has the form claimed in Proposition 1, let us take $x < b(t)$ and consider the OST $\tau^* = \tau^*(t,x)$. Then, relying on (1), (3) and (5), we get

$$\begin{aligned}
V(t,x) - V(t,b(t)) &\leq \mathbb{E}_{t,x}\left[e^{-\lambda\tau^*}G(X_{t+\tau^*})\right] - \mathbb{E}_{t,b(t)}\left[e^{-\lambda\tau^*}G(X_{t+\tau^*})\right] \\
&\leq \mathbb{E}_{t,0}\left[\left(X_{t+\tau^*} + b(t)\frac{T-t-\tau^*}{T-t} - X_{t+\tau^*} - x\frac{T-t-\tau^*}{T-t}\right)^+\right] \\
&= (b(t)-x)\mathbb{E}\left[\frac{T-t-\tau^*}{T-t}\right] \\
&\leq b(t) - x,
\end{aligned} \tag{27}$$

where we used the relation

$$G(a) - G(b) \leq (b-a)^+, \tag{28}$$

for all $a,b \in \mathbb{R}$, for the second inequality. Since $V(t,b(t)) = S - b(t)$, we get from the above relation that $V(t,x) \leq S - x = G(x)$, which means that $(t,x) \in D$ and therefore $\{(t,x) \in [0,T] \times \mathbb{R} : x \leq b(t)\} \subset D$. On the other hand, if $(t,x) \in D$, then $x \geq b(t)$, which proves the reverse inclusion.

Take now $t, t' \in [0,T]$ and $x \in \mathbb{R}$ such that $t' < t$ and $(t,x) \in C$, then, since the function $t \mapsto V(t,x)$ is non-increasing for all $x \in \mathbb{R}$ (see (iv) from Proposition 2), $V(t',x) \geq V(t,x) > G(x)$, i.e., $(t',x) \in C$. Hence, b is non-decreasing.

Finally, in order to prove the right-continuity of b, let us fix $t \in (0,T)$ and notice that, since b is non-decreasing, then $b(t^+) \geq b(t)$. On the other hand, as D is a closed set and $(t+h, b(t+h)) \in D$ for all $0 < h \leq T-t$, then $(t^+, b(t^+)) \in D$ or, equivalently, $b(t^+) \leq b(t)$. ■

Proof of Proposition 2. *(i)* Half of the statement relies on the result obtained in Peskir and Shiryaev (2006, Section 7.1) relative to the Dirichlet problem. Specifically, it states that V is $\mathcal{C}^{1,2}$ on C and $\partial_t V + \mathbb{L}_X V = \lambda V$ on C. On the other hand, since $V(t,x) = G(x) = S - x$ for all $(t,x) \in D$, V is $\mathcal{C}^{1,2}$ also on D.

(ii) We easily get the convexity of $x \mapsto V(t,x)$ by plugging-in (3) into (1). To prove (13) let us fix an arbitrary point $(t,x) \in [0,T] \times \mathbb{R}$, consider $\tau^* = \tau^*(t,x)$ and $\tau_\varepsilon = \tau^*(t, x+\varepsilon)$ for some $\varepsilon \in \mathbb{R}$. Since $\tau_\varepsilon \to \tau$ a.s., by arguing similarly to (27), we get

$$\begin{aligned} \varepsilon^{-1}(V(t,x+\varepsilon) - V(t,x)) &\leq -\mathbb{E}\left[e^{-\lambda\tau_\varepsilon}\frac{T-t-\tau_\varepsilon}{T-t}\right] \\ &\xrightarrow{\varepsilon\to 0} -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right], \end{aligned} \tag{29}$$

where the limit is valid due to the dominated convergence theorem. For $\varepsilon < 0$ the reverse inequality emerges, giving us, after making $\varepsilon \to 0$, the relation $\partial_x^- V(t,x) \leq -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right] \leq \partial_x^+ V(t,x)$, which, due to the continuity of $x \mapsto \partial_x V(t,x)$ on $(-\infty, b(t))$ and on $(b(t), \infty)$ for all $t \in [0,T]$ (V is $\mathcal{C}^{1,2}$ on C and on D), turns into $\partial_x V(t,x) = -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right]$ for all (t,x) where $t \in [0,T]$ and $x \neq b(t)$. For $x = b(t)$ the equation (13) also holds true and it turns into the smooth fit condition *(iii)* proved later on.

Furthermore, since $\mathbb{P}_{t,x}[\tau^* < T-t] > 0$ (see Lemma 1), the expression (13) shows that $\partial_x V < 0$ and therefore $x \mapsto V(t,x)$ is strictly decreasing for all $t \in [0,T]$.

(iii) Take a pair $(t,x) \in [0,T) \times \mathbb{R}$ lying on the OSB, i.e., $x = b(t)$, and consider $\varepsilon > 0$. Since $(t,x) \in D$ and $(t, x+\varepsilon) \in C$, we have that $V(t,x) = G(x)$ and $V(t,x+\varepsilon) > G(x+\varepsilon)$. Thus, taking into account the inequality (28), we get $\varepsilon^{-1}(V(t,x+\varepsilon) - V(t,x)) > \varepsilon^{-1}(G(x+\varepsilon) - G(x)) \geq -1$, which, after making $\varepsilon \to 0$ turns into $\partial_x^+ V(t,x) \geq -1$. On the other hand, by considering the optimal stopping time $\tau_\varepsilon := \tau^*(t, x+\varepsilon)$ and following the same

arguments showed at (29), we get that $\partial_x^+ V(t,x) \le -1$. Therefore $\partial_x^+ V(t,b(t)) = -1$ for all $t \in [0,T)$. Since $V = G$ in D, it follows straightforwardly that $\partial_x^- V(t,b(t)) = -1$ and hence the smooth fit condition holds.

(iv) Notice that, due to (i), alongside to (4), (3), (13), and recalling that $x \mapsto V(t,x)$ is convex (and therefore $\partial_{x^2} V \ge 0$), we get that

$$\begin{aligned}\partial_t V(t,x) &\le \lambda V(t,x) - \frac{T-t-x}{T-t}\partial_x V(t,x)\\ &= -\mathbb{E}\left[e^{-\lambda\tau^*}(x-S)\frac{1-t-\tau^*}{1-t}\left(\lambda + \frac{x-S}{1-t}\right)\right].\end{aligned}$$

Therefore $\partial_t V \le 0$ on the set $C_S := [0,T) \times [S,\infty) \subset C$.

For some small $\varepsilon > 0$, denote by $(X_s^{[t,T]})_{s\ge 0}^{T-t+\varepsilon}$ a process such that, for $s \in [0, T-t]$ it behaves as the Brownian bridge $X^{[t,T]}$, and the remaining part stays constant at the value S, *i.e.*, $X_s^{[t,T]} = S$ for $s \in [T-t, T-t+\varepsilon]$. Let $\mu^{[t,T]}$ be its drift, and define the process $(X_s^{[t-\varepsilon,T]})_{s\ge 0}^{T-t+\varepsilon} = X^{[t-\varepsilon,T]}$ with drift $\mu^{[t-\varepsilon,T]}$. Since $\mu^{[t,T]}(t,x) \ge \mu^{[t-\varepsilon,T]}$ whenever $x \le S$, Theorem 1.1 from Ikeda and Watanabe (1977) guarantees that $X_s^{[t,T]} \ge X_s^{[t-\varepsilon,T]}$ $\mathbb{P}_{t,x}$-a.s., for all (t,x) and for all $s \le \tau^S$, where $\tau^S := \inf\{s \in [0,T-t] : X_s^{[t,T]} > S\}$.

Assume now that both processes start at $x \le S$, and consider the stopping time $\tau^* = \tau^*(t,x)$. Therefore, since $V(t + s\wedge\tau^*, X_{s\wedge\tau^*}^{[t,T]})$ and $V(t+s, X_s^{[t-\varepsilon,T]})$ are a martingale and a supermartingale (see Section 2.2 from Peskir and Shiryaev (2006)), respectively, we have

$$\begin{aligned}V(t,x) - V(t-\varepsilon,x) &\le \mathbb{E}\left[V(t+\tau^S\wedge\tau^*, X_{\tau^S\wedge\tau^*}^{[t,T]}) - V(t-\varepsilon+\tau^S\wedge\tau^*, X_{\tau^S\wedge\tau^*}^{[t-\varepsilon,T]})\right]\\ &\le \mathbb{E}\left[\mathbb{I}(\tau^* \le \tau^S, \tau^* < T-t)e^{-\lambda\tau^*}(X_{\tau^*}^{[t-\varepsilon,T]} - X_{\tau^*}^{[t,T]})^+\right]\\ &+ \mathbb{E}\left[\mathbb{I}(\tau^*\wedge\tau^S = T-t)e^{-\lambda\tau^*}(X_{T-t}^{[t-\varepsilon,T]} - X_{T-t}^{[t,T]})^+\right]\\ &+ \mathbb{E}\left[\mathbb{I}(\tau^S \le \tau^*, \tau^S < T-t)V(t+\tau^S, X_{\tau^S}^{[t,T]}) - V(t-\varepsilon+\tau^S, X_{\tau^S}^{[t-\varepsilon,T]})\right]\\ &\le \mathbb{E}\left[\mathbb{I}(\tau^S \le \tau^*, \tau^S < T-t)V(t-\varepsilon+\tau^S, S) - V(t-\varepsilon+\tau^S, X_{\tau^S}^{[t-\varepsilon,T]})\right]\\ &\le 0,\end{aligned}$$

where the second inequality comes after noticing that τ^* is not optimal for $X^{[t-\varepsilon,T]}$ and using (28); the third inequality holds since $X_{\tau^*}^{[t-\varepsilon,T]} \le X_{\tau^*}^{[t,T]}$ for $\tau^* \le \tau^S$, $X_{T-t}^{[t-\varepsilon,T]} \le X_{T-t}^{[t,T]}$ whenever $\tau^*\wedge\tau^S = T-t$, and the fact that $\partial_t V \le 0$ on the set C_S; and the last inequality relies on the increasing behavior of $x \mapsto V(t,x)$. Finally, after dividing by ε and taking $\varepsilon \to 0$, we get the claimed result. Note that a similar argument for a different gain function has recently appeared in Tiziano and Milazzo (2019).

(v) Let $(X_{t_i+s}^{[t_i,T]})_{s\geq 0}^{[0,T-t_i]}$ be a Brownian bridge going from $X_{t_i} = x$ to $X_T = S$ for any $x \in \mathbb{R}$, with $i = 1, 2$. Notice that, according to (3), the following holds:

$$X_{t_2+s'}^{[t_2,T]} \stackrel{d}{=} r^{1/2} X_{t_1+s}^{[t_1,T]} + (1 - r^{1/2})(S - x)\frac{s}{T - t_1}, \tag{30}$$

where $r = \frac{T-t_2}{T-t_1}$, $s \in [0, T - t_1]$, and $s' = sr \in [0, T - t_2]$.

Take $0 \leq t_1 < t_2 < T$, consider $\tau_1 := \tau^*(t_1, x)$, and set $\tau_2 := \tau_1 r$. Since $t \mapsto V(t,x)$ is decreasing for every $x \in \mathbb{R}$, then

$$\begin{aligned}
0 &\leq V(t_1,x) - V(t_2,x) \\
&\leq \mathbb{E}_{t_1,x}\left[e^{-\lambda\tau_1} G\left(X_{t_1+\tau_1}^{[t_1,T]}\right)\right] - \mathbb{E}_{t_2,x}\left[e^{-\lambda\tau_2} G\left(X_{t_2+\tau_2}^{[t_2,T]}\right)\right] \\
&\leq \mathbb{E}\left[e^{-\lambda\tau_2}\left(G\left(X_{t_1+\tau_1}^{[t_1,T]}\right) - G\left(X_{t_2+\tau_2}^{[t_2,T]}\right)\right)\right] \\
&\leq \mathbb{E}\left[\left(X_{t_2+\tau_2}^{[t_2,T]} - X_{t_1+\tau_1}^{[t_1,T]}\right)^+\right]. \\
&= \mathbb{E}\left[\left(\left(r^{1/2} - 1\right)\left(X_{t_1+\tau_1}^{[t_1,T]} + (S - x)\frac{\tau_1}{T - t_1}\right)\right)^+\right] \\
&\leq \left(\left(r^{1/2} - 1\right)\left(S + \mathbb{I}(x \leq S)(S - x)\right)\right)^+,
\end{aligned}$$

where the first equality comes after applying (30) and the last inequality takes place since $r < 1$ and $X_{t_1+\tau_1}^{[t_1,T]} \leq S$.

Hence, $V(t_1,x) - V(t_2,x) \to 0$ as $t_1 \to t_2$, i.e., $t \mapsto V(t,x)$ is continuous for every $x \in \mathbb{R}$, and thus, to address the continuity of V is sufficient to prove that, for a fixed t, $x \mapsto V(t,x)$ is uniformly continuous within a neighborhood of t. The latter comes after the following inequality, which comes right after applying similar arguments to those used in (27):

$$\begin{aligned}
0 \leq V(t,x_1) - V(t,x_2) &\leq (x_2 - x_1)\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T - t - \tau^*}{T - t}\right] \\
&\leq x_2 - x_1,
\end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$ are such that $x_1 \le x_2$ and $\tau^* = \tau^*(t, x_1)$. ∎

Proof of Proposition 3. We already proved the right-continuity of b in Proposition 1, so this proof is devoted to prove its left-continuity.

Let us assume that b is not left-continuous. Therefore, as b is non-decreasing, we can ensure the existence of a point $t_* \in (0, T)$ such that $b(t_*^-) < b(t_*)$, which allows us to take x' in the interval $(b(t_*^-), b(t_*))$ and consider the right-open rectangle $\mathcal{R} = [t', t_*) \times [b(t_*^-), x'] \subset C$ (see illustration of Figure 10), with $t' \in (0, t_*)$.

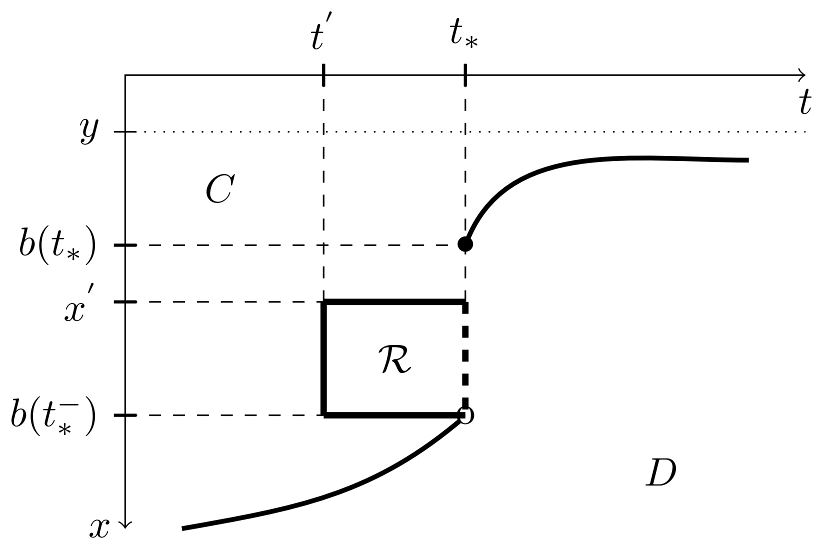

Figure 10: Graphical sketch of the proof of left continuity of b.

Applying twice the fundamental theorem of calculus, using that $(t, b(t)) \in D$ for all $t \in [0, T]$, the smooth fit condition (iii), and the fact that $x \mapsto V(t, x)$ is $\mathcal{C}^2$ on C, we obtain

$$V(t, x) - G(x) = \int_{b(t)}^{x} \int_{b(t)}^{u} (\partial_{x^2} V(t, v) - \partial_{x^2} G(v))\, \mathrm{d}v\, \mathrm{d}u, \tag{31}$$

for all $(t, x) \in \mathcal{R}$.

On the other hand, if we set $m := -\sup_{(t,x)\in\mathcal{R}} \partial_x V(t, x)$, then we readily obtain from (13) that $m > 0$ (see Lemma 1), which, combined with $\partial_t V + \mathbb{L}_X V = \lambda V$ on C and $\partial_t V \le 0$ on C ((i) and (iv) from Proposition 2), along with the fact that $V(t, x) \ge 0$ for all admissible pairs (t,x), gives

$$\partial_{x^2} V(t,x) = \frac{2}{\sigma^2}\left(\lambda V(t,x) - \frac{S-x}{T-t}\partial_x V(t,x) - \partial_t V(t,x)\right)$$
$$\geq \frac{2m}{\sigma^2}\frac{S-x}{T-t} > 0, \tag{32}$$

for all $(t,x) \in \mathcal{R}$. Therefore, by noticing that $\partial_{x^2} G(x) = 0$ for all $x \in (b(t_*^-), x')$ and plugging-in (32) into (31), we get

$$\begin{aligned} V(t,x) - G(x) &\geq \int_{b(t)}^{x}\int_{b(t)}^{u} \frac{2m}{\sigma^2}\frac{S-x}{T-t}\,\mathrm{d}v\,\mathrm{d}u \\ &\geq \frac{2m}{\sigma^2}\frac{S-x}{T-t}\int_{b(t_*^-)}^{x}\int_{b(t_*^-)}^{u}\mathrm{d}v\,\mathrm{d}u \\ &= \frac{2m}{\sigma^2}\frac{S-x}{T-t}(x - b(t_*^-))^2. \end{aligned}$$

Finally, after taking $t \to t_*$ on both sides of the above equation, we obtain $V(t_*,x) - G(x) > 0$ for all $x \in (b(t_*^-), b(t_*))$, which contradicts the fact that $(t_*,x) \in D$. ■

Proof of Theorem 1.

Assume we have a function $c : [0,T] \to \mathbb{R}$ that solves the integral equation (20) and define the function

$$\begin{aligned} V^c(t,x) &= \int_t^T e^{-\lambda(u-t)}\left(\frac{1}{T-u} + \lambda\right)\mathbb{E}_{t,x}\left[(S - X_u)\,\mathbb{1}\left(X_u \leq c(u)\right)\right]\mathrm{d}u \qquad (33) \\ &= \int_t^T K_{\sigma,\lambda}(t,x,u,c(u))\,\mathrm{d}u, \end{aligned}$$

where $X = (X_s)_{s=0}^T$ is a Brownian bridge with σ volatility that ends at $X_T = S$, and $K_{\sigma,\lambda}$ is defined at (19). It turns out that $x \mapsto K_{\sigma,\lambda}(t,x,u,c(u))$ is twice continuously differentiable and therefore differentiating inside the integral symbol at (33) yields $\partial_x V^c(t,x)$ and $\partial_{x^2} V^c(t,x)$, and furthermore ensures their continuity on $[0,T) \times \mathbb{R}$.

Let us compute the operator $\partial_t + \mathbb{L}_X$ acting on the function V^c,

$$\partial_t V^c + \mathbb{L}_X V^c(t,x) = \lim_{h\downarrow 0}\frac{\mathbb{E}_{t,x}[V^c(t+h, X_{t+h})] - V^c(t,x)}{h}.$$

Define the function

$$I(t,u,x_1,x_2) := e^{-\lambda(u-t)}\left(\frac{1}{T-u} + \lambda\right)(S - x_1)\,\mathbb{1}\left(x_1 \leq x_2\right) \tag{34}$$

and notice that

$$
\begin{aligned}
\mathbb{E}_{t,x}[V^c(t+h, X_{t+h})] &= \mathbb{E}_{t,x}\left[\mathbb{E}_{t+h,X_{t+h}}\left[\int_{t+h}^{T} I(t+h,u,X_u,c(u))\,\mathrm{d}u\right]\right] \\
&= \mathbb{E}_{t,x}\left[\mathbb{E}_{t,x}\left[\int_{t+h}^{T} I(t+h,u,X_u,c(u))\,\mathrm{d}u \;\middle|\; \mathcal{F}_{t+h}\right]\right] \\
&= \mathbb{E}_{t,x}\left[\int_{t+h}^{T} I(t+h,u,X_u,c(u))\,\mathrm{d}u\right],
\end{aligned}
$$

where $(\mathcal{F}_s)_{s=0}^T$ is the natural filtration of X. Therefore,

$$
\begin{aligned}
&\partial_t V^c + \mathbb{L}_X V^c(t,x) \\
&\quad= \lim_{h\downarrow 0} \frac{\mathbb{E}_{t,x}\left[\int_{t+h}^{T} I(t+h,u,X_u,c(u))\,\mathrm{d}u\right] - \mathbb{E}_{t,x}\left[\int_{t}^{T} I(t,u,X_u,c(u))\,\mathrm{d}u\right]}{h} \\
&\quad= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}_{t,x}\left[\int_{t+h}^{T}\left(e^{\lambda h}-1\right) I(t,u,X_u,c(u))\,\mathrm{d}u\right] - \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}_{t,x}\left[\int_{t}^{t+h} I(t,u,X_u,c(u))\,\mathrm{d}u\right] \\
&\quad= \lambda V(t,x) - (S-x)\left(\frac{1}{T-t}+\lambda\right)\mathbb{1}(x\le c(t)).
\end{aligned}
$$

From this result, alongside with (4) and the fact that V^c, $\partial_x V^c$, and $\partial_{x^2} V^c$ are continuous on $[0,T)\times\mathbb{R}$, we get the continuity of $\partial_t V^c$ on $C_1 \cup C_2$, where

$$
\begin{aligned}
C_1 &:= \{(t,x)\in[0,T)\times\mathbb{R} : x > c(t)\}, \\
C_2 &:= \{(t,x)\in[0,T)\times\mathbb{R} : x < c(t)\}.
\end{aligned}
$$

Now define the function $F^{(t)}(s,x) := e^{-\lambda s}V^c(t+s,x)$ with $s\in[0,T-t)$, $x\in\mathbb{R}$, and consider the sets

$$
\begin{aligned}
C_1^t &:= \{(s,x)\in C_1 : t\le s<T\}, \\
C_2^t &:= \{(s,x)\in C_2 : t\le s<T\}.
\end{aligned}
$$

We claim that $F^{(t)}$ satisfies the (*iii-b*) version of the hypothesis of Lemma 2 taking $C = C_1^t$ and $D^\circ = C_2^t$. Indeed: $F^{(t)}$, $\partial_x F^{(t)}$, and $\partial_{x^2}F^{(t)}$ are continuous on $[0,T)\times\mathbb{R}$; it has been proved that $F^{(t)}$ is $\mathcal{C}^{1,2}$ on C_1^t and C_2^t; we are assuming that c is a continuous function of bounded variation; and $(\partial_t F^{(t)} + \mathbb{L}_X F^{(t)})(s,x) = -e^{-\lambda s}(S-x)\left(\frac{1}{T-t-s}+\lambda\right)\mathbb{1}(x\le c(t+s))$ is locally bounded on $C_1^t\cup C_2^t$.

Thereby, we can use the (*iii-b*) version of Lemma 2 to obtain the following change of variable formula, which is missing the local time term because of the continuity of F_x on $[0,T) \times \mathbb{R}$:

$$
\begin{aligned}
&e^{-\lambda s} V^c(t+s,\ X_{t+s}) \\
&\qquad = V^c(t,x) - \int_t^{t+s} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \le c(u))\,\mathrm{d}u + M_s^{(1)},
\end{aligned} \tag{35}
$$

with $M_s^{(1)} = \int_t^{t+s} e^{-\lambda(u-t)} \sigma \partial_x V^c(u, X_u)\,\mathrm{d}B_u$. Notice that $(M_s^{(1)})_{s=0}^{T-t}$ is a martingale under $\mathbb{P}_{t,x}$.

In the same way, we can apply the (*iii-b*) version of Lemma 2 this time using the function $F(s,x) = e^{-\lambda s} G(X_{t+s})$, and taking $C = \{(s,x) \in [0, T-t) \times \mathbb{R} : x > S\}$ and $D^\circ = \{(s,x) \in [0,T-t) \times \mathbb{R} : x < S\}$, thereby getting

$$
\begin{aligned}
e^{-\lambda s} G(X_{t+s}) = G(x) &- \int_t^{t+s} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u < S)\,\mathrm{d}u \\
&- M_s^{(2)} + \frac{1}{2}\int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u = S)\,\mathrm{d}l_s^S(X),
\end{aligned} \tag{36}
$$

where $M_s^{(2)} = \sigma \int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u < S)\,\mathrm{d}B_u$, with $0 \le s \le T-t$, is a martingale under $\mathbb{P}_{t,x}$.

Consider the following stopping time for (t,x) such that $x \le c(t)$:

$$
\rho_c := \inf\left\{0 \le s \le T - t : X_{t+s} \ge c(t+s) \mid X_t = x\right\}. \tag{37}
$$

In this way, along with assumption $c(t) < S$ for all $t \in (0,T)$, we can ensure that $\mathbb{1}(X_{t+s} \le c(t+s)) = \mathbb{1}(X_{t+s} \le S) = 1$ for all $s \in [0, \rho_c)$, as well as $\int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u = S)\,\mathrm{d}l_s^S(X) = 0$. Recall that $V^c(t, c(t)) = G(c(t))$ for all $t \in [c, T)$ since c solves the integral equation (20). Moreover, $V^c(T,S) = 0 = G(S)$. Hence, $V^c(t + \rho_c, X_{t+\rho_c}) = G(X_{t+\rho_c})$. Therefore, we are able now to derive the following relation from equations (35) and (36):

$$
\begin{aligned}
V^c(t,x) &= \mathbb{E}_{t,x}[e^{-\lambda\rho_c}V^c(t+\rho_c,\ X_{t+\rho_c})] \\
&\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \le c(u))\,\mathrm{d}u\right] \\
&= \mathbb{E}_{t,x}\left[e^{-\lambda\rho_c}G(X_{t+\rho_c})\right] \\
&\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \le S)\,\mathrm{d}u\right] \\
&= G(x).
\end{aligned}
$$

The vanishing of the martingales $M^{(1)}_{\rho_c}$ and $M^{(2)}_{\rho_c}$ comes after using the optional stopping theorem (see, e.g., Section 3.2 from Peskir and Shiryaev (2006)). Therefore, we have just proved that $V^c = G$ on C_2.

Now define the stopping time

$$\tau_c := \inf\{0 \le u \le T-t : X_{t+u} \le c(t+u) \mid X_t = x\}$$

and plug-in it into equation (35) to obtain the expression

$$
\begin{aligned}
V^c(t,x) &= e^{-\lambda\tau_c}V^c(t+\tau_c,\ X_{t+\tau_c}) \\
&\quad + \int_t^{t+\tau_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \le c(u))\,\mathrm{d}u - M^{(1)}_{\tau_c}.
\end{aligned}
$$

Notice that, due to the definition of τ_c, $\mathbb{1}(X_{t+u} \le c(t+u)) = 0$ for all $0 \le u < \tau_c$ whenever $\tau_c > 0$ (the case $\tau_c = 0$ is trivial). In addition, the optional sampling theorem ensures that $\mathbb{E}_{t,x}[M^{(1)}_{\tau_c}] = 0$. Therefore, the following formula comes after taking $\mathbb{P}_{t,x}$-expectation in the above equation and considering that $V^c = G$ on C_2:

$$V^c(t,x) = \mathbb{E}_{t,x}[e^{-\lambda\tau_c}V^c(t+\tau_c, X_{t+\tau_c})] = \mathbb{E}_{t,x}\left[e^{-\lambda\tau_c}G(X_{t+\tau_c})\right],$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Recalling the definition of V from (1), we realize that the above equality leads to

$$V^c(t,x) \le V(t,x), \tag{38}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

Take $(t,x) \in C_2$ satisfying $x < \min\{b(t), c(t)\}$, where b is the OSB for (1), and consider the stopping time ρ_c defined as

$$\rho_b := \inf\left\{0 \leq s \leq T-t : X_{t+s} \geq b(t+s) \mid X_t = x\right\}.$$

Since $V = G$ on D, the following equality holds true due to (14) and from noticing that $\mathbb{1}(X_{t+u} \leq b(t+u)) = 1$ for all $0 \leq u < \rho_b$:

$$\mathbb{E}_{t,x}[e^{-\lambda\rho_b} V(t+\rho_b, X_{t+\rho_b})] = G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right) \mathrm{d}u\right].$$

On the other hand, we get the next equation after substituting s for ρ_b at (35) and recalling that $V = G$ on C_2:

$$\begin{aligned}
&\mathbb{E}_{t,x}[e^{-\lambda\rho_b} V(t+\rho_b, X_{t+\rho_b})] \\
&\quad = G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u))\, \mathrm{d}u\right].
\end{aligned}$$

Therefore, we can use (38) to merge the two previous equalities into

$$\begin{aligned}
&\mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} - \lambda\right) \mathbb{1}(X_u \leq c(u))\, \mathrm{d}u\right] \\
&\quad \geq \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} - \lambda\right) \mathrm{d}u\right],
\end{aligned}$$

meaning that $b(t) \leq c(t)$ for all $t \in [0,T]$ since c is continuous.

Suppose there exists a point $t \in (0,T)$ such that $b(t) < c(t)$ and fix $x \in (b(t), c(t))$. Consider the stopping time

$$\tau_b := \inf\{0 \leq u \leq T-t : X_{t+u} \leq b(t+u) \mid X_t = x\}$$

and plug-in it both into (14) and (35) replacing s before taking the $\mathbb{P}_{t,x}$-expectation. We obtain

$$\begin{aligned}
&\mathbb{E}_{t,x}[e^{-\lambda\tau_b} V^c(t+\tau_b, X_{t+\tau_b})] \\
&\quad = \mathbb{E}_{t,x}[e^{-\lambda\tau_b} G(X_{t+\tau_b})] \\
&\quad = V^c(t,x) - \mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right) \mathbb{1}(X_u \leq c(u))\, \mathrm{d}u\right]
\end{aligned}$$

and

$$\mathbb{E}_{t,x}[e^{-\lambda\tau_b}V(t+\tau_b, X_{t+\tau_b})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_b}G(X_{t+\tau_b})] = V(t,x).$$

Thus, from (38) we get

$$\mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right] \leq 0.$$

Using the fact that $x > b(t)$ and the time-continuity of the process X, we can state that $\tau_b > 0$. Therefore, the previous inequality can only happen if $\mathbb{1}(X_s \leq c(s)) = 0$ for all $t \leq s \leq t+\tau_b$, meaning that $b(s) \geq c(s)$ for all $t \leq s \leq t+\tau_b$, which contradicts the assumption $b(t) < c(t)$. ■

Proof of Proposition 4. Since the OSP (21) satisfies the hypothesis stated in Corollary 2.9 from Peskir and Shiryaev (2006) (V_i lower semi-continuous and G_i upper semi-continuous), we can ensure the existence of the OSP $\tau_i^*(t,x)$ defined at (6) for the pair (t,x), where $i = 1,2$. Moreover, Theorem 2.4 from Peskir and Shiryaev (2006) guarantees that $\mathbb{P}_{t,x}^{(i)}[\tau_i^*(t,x) \leq \tau_*] = 1$ for any other OST τ_* of the OSP (21), where $\mathbb{P}_{t,x}^{(i)}$ denotes the law such that $\mathbb{P}_{t,x}^{(i)}\left[X_t^{(i)} = x\right] = 1$.

(i) Define the sets $D_i^{\alpha,A} := \{(t,x) \in [0,T]\times\mathbb{R} : (t, \alpha^{-1}(x-A)) \in D_i\}$ for $i = 1,2$, and notice that $\tau_1^*(t,x) \overset{\mathrm{d}}{=} \inf\{0 \leq s \leq T-t : X_{t+s}^{(2)} \in D_1^{\alpha,A} \mid X_t^{(2)} = \alpha x + A\}$ as well as $\tau_2^*(t,\alpha x + A) \overset{\mathrm{d}}{=} \inf\{0 \leq s \leq T-t : X_{t+s}^{(1)} \in D_2^{\alpha^{-1},-\alpha^{-1}A} \mid X_t^{(1)} = x\}$, for $(t,x) \in [0,T]\times\mathbb{R}$. Suppose that

$$\mathbb{E}_{t,\alpha x+A}\left[e^{-\lambda\tau_2^*(t,\alpha x+A)}G_2\left(X_{t+\tau_2^*(t,x+A)}^{(2)}\right)\right] > \mathbb{E}_{t,\alpha x+A}\left[e^{-\lambda\tau_1^*(t,x)}G_2\left(X_{t+\tau_1^*(t,x)}^{(2)}\right)\right].$$

Then,

$$\begin{aligned}
\mathbb{E}_{t,x}\left[e^{-\lambda\tau_2^*(t,\alpha x+A)}G_1\left(X_{t+\tau_2^*(t,\alpha x+A)}^{(1)}\right)\right] &= \mathbb{E}_{t,\alpha x+A}\left[e^{-\lambda\tau_2^*(t,\alpha x+A)}G_2\left(X_{t+\tau_2^*(t,\alpha x+A)}^{(2)}\right)\right] \\
&> \mathbb{E}_{t,\alpha x+A}\left[e^{-\lambda\tau_1^*(t,x)}G_2\left(X_{t+\tau_1^*(t,x)}^{(2)}\right)\right] \\
&= \mathbb{E}_{t,x}\left[e^{-\lambda\tau_1^*(t,x)}G_1\left(X_{t+\tau_1^*(t,x)}^{(1)}\right)\right]
\end{aligned}$$

which is a contradiction. Therefore, our original assumption has to be wrong, meaning that $\tau_2^*(t,\alpha x+A) \leq \tau_1^*(t,x)$ $\mathbb{P}_{t,x}^{(1)}$-a.s. as well as $\mathbb{P}_{t,\alpha x+A}^{(2)}$-a.s. (notice that $\mathbb{P}_{t,x}^{(1)} = \mathbb{P}_{t,\alpha x+A}^{(2)}$).

Interchanging the roles of $t_1^*(t,x)$ and $t_2^*(t,\alpha x+A)$ along the argumentation given above, and making the corresponding rearrangements, we get the opposite inequality. Thus, since both D_1 and D_2 are closed sets, then $D_2 = D_1^{\alpha,A}$ or, reciprocally, $D_1 = D_2^{\alpha^{-1},-\alpha^{-1}A}$.

(ii) Fix $(t,x) \in [0,T] \times \mathbb{R}$ and let $\tau_1^* = \tau_1^*(t,x)$ as well as $\tau_2^* = \tau_2^*(t,x)$. Notice that $\tau_1^* \overset{\mathrm{d}}{=} \inf\{0 \le s \le T-t : X_{t+s}^{(2)} \in D_1 \mid X_t^{(2)} = x\}$ and $\tau_2^* \overset{\mathrm{d}}{=} \inf\{0 \le s \le T-t : X_{t+s}^{(1)} \in D_2 \mid X_t^{(1)} = x\}$. Suppose that

$$\mathbb{E}_{t,x}\left[e^{-\lambda\tau_2^*}G_2\left(X_{t+\tau_2^*}^{(2)}\right)\right] > \mathbb{E}_{t,x}\left[e^{-\lambda\tau_1^*}G_2\left(X_{t+\tau_1^*}^{(2)}\right)\right].$$

Since $G_1 = G_2$ on $D_1 \cup D_2$, then

$$\begin{aligned}\mathbb{E}_{t,x}\left[e^{-\lambda\tau_2^*}G_1\left(X_{t+\tau_2^*}^{(1)}\right)\right] &= \mathbb{E}_{t,x}\left[e^{-\lambda\tau_2^*}G_2\left(X_{t+\tau_2^*}^{(2)}\right)\right] \\ &> \mathbb{E}_{t,x}\left[e^{-\lambda\tau_1^*}G_2\left(X_{t+\tau_1^*}^{(2)}\right)\right] \\ &= \mathbb{E}_{t,x}\left[e^{-\lambda\tau_1^*}G_1\left(X_{t+\tau_1^*}^{(1)}\right)\right],\end{aligned}$$

which is an absurd and hence our assumption is wrong, this is, $\tau_2^* \le \tau_1^*$ $\mathbb{P}_{t,x}^{(1)}$-a.s. as well as $\mathbb{P}_{t,x}^{(2)}$-a.s. (notice that $\mathbb{P}_{t,x}^{(1)} = \mathbb{P}_{t,x}^{(2)}$).

Swapping the roles of t_1^* and t_2^* throughout the argumentation given above, and making the correspondent rearrangements, we get the opposite inequality. Therefore, since both D_1 and D_2 are closed sets, then $D_2 = D_1$. ■

Proof of Corollary 1. First, notice that in both scenarios, (i) and (ii), the conditions G_i being upper semi-continuous and V_i lower semi-continuous from Proposition 4 are fulfilled due to the continuity of G_i (see Remark 2.10 from Peskir and Shiryaev (2006)).

(i) Since $G_1(2S-x) = G_2(x)$ and $\left[2S - X_{t+s}^{(1)} \mid X_t^{(1)} = x\right] \overset{\mathrm{d}}{=} \left[X_{t+s}^{(2)} \mid X_t^{(2)} = 2S - x\right]$ for all $s \in [0, T-t]$, then we can apply (i) from Proposition 4 to show that $D_1 = \{(t,x) : (t, 2S-x) \in D_2\}$, and therefore $b_1 = 2S - b_2$.

(ii) Introduce the function $G(x) = S_2 - x$ and the Brownian bridge $(X_{t+s})_{s=0}^{T-t}$ such that $X_T = S_2$. Since $G(S_2 - x) = G_1(x)$ and $[X_{t+s} \mid X_t = S_2 - x] \overset{d}{=} [X_{t+s}^{(1)} \mid X_t^{(1)} = x]$

for $(t,x) \in [0,T] \times \mathbb{R}$, we get that $D_1 = \{(t,x) \in [0,T] \times \mathbb{R} : S_2 - x \in D\}$, and hence $b(t) = S_2 - b_1$, where D and b are, respectively, the stopping set and the OSB of the non-discounted OSP with gain function G and process $(X_{t+s})_{s=0}^{T-t}$.

Let us fix $t \in [0,T)$ and take x' satisfying $x' > S_2$. Consider $\varepsilon > 0$ such that $\varepsilon < x' - S_2$, as well as the stopping time $\tau_\varepsilon := \inf\{0 \leq s \leq T-t : X_{t+s} \leq S + \varepsilon \mid X_t = x'\}$. Since our underlying Brownian bridge process $X^{(1)}$ is continuous and it takes the value S_2 at the expiration date T, then $\mathbb{P}^{(1)}_{t,x'}[\tau_\varepsilon < T-t] = 1$ and thus $V(t,x') \geq \mathbb{E}_{t,x'}\left[G(X_{t+\tau_\varepsilon})\right] = -\varepsilon > S_2 - x^{'} = G(x')$, i.e., $(t,x') \notin D$. Therefore, $D \subset D_{S_2} := \{(t,x) \in [0,T] \times \mathbb{R} : x \leq S_2\}$.

On the other hand, recall from Proposition 1 that $D_2 \subset D_{S_2}$. Therefore, since $G(x) = G_2(x)$ for all x such that $(t,x) \in D_{S_2}$ for some $t \in [0,T]$, and $[X_{t+s} \mid X_t = x] \overset{\mathrm{d}}{=} \left[X^{(2)}_{t+s} \mid X^{(2)}_t = x\right]$ for all $s \in [0, T-t]$, then we can use (ii) from Proposition 4 in order to get the relation $b_2 = b = S_2 - b_1$. ■

B Auxiliary lemmas

Lemma 1 *Let $(X_{t+s})_{s=0}^{T-t}$ be a Brownian bridge from X_t to $X_T = S$ with volatility σ, where $t \in [0,T)$. Let b be the optimal stopping boundary associated to the OSP*

$$V(t,x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau} G(X_{t+\tau})\right],$$

with $G(x) = (G-x)^+$, and $\lambda \geq 0$. Then, $\sup_{(t,x)\in\mathcal{R}} \partial_x V(t,x) < 0$, where $\mathcal{R}$ is the set defined in the proof of Proposition 3.

Proof. Take $0 < \varepsilon < 1$, let $\tau^* = \tau^*(t,x)$, and define

$$p(t,x) := \mathbb{P}\left[\tau^* \leq (T-t)(1-\varepsilon)\right].$$

Notice that

$$\begin{aligned}
p(t,x) &= \mathbb{P}_{t,x}\left[\min_{0\le s\le (T-t)(1-\varepsilon)}\{X_{t+s}-b(s)\}<0\right] \\
&\ge \mathbb{P}_{t,x}\left[\min_{0\le s\le (T-t)(1-\varepsilon)} X_{t+s}<b(t)\right] \\
&= \mathbb{P}\left[\min_{0\le s\le (T-t)(1-\varepsilon)}\left\{(S-x)\frac{s}{T-t}+\sigma\sqrt{\frac{T-t-s}{T-t}}W_s\right\}<b(t)-x\right] \\
&\ge \mathbb{P}\left[\min_{0\le s\le (T-t)(1-\varepsilon)}\left\{\sqrt{\frac{T-t-s}{T-t}}W_s\right\}<\sigma^{-1}(b(t)-\max\{x,S\})\right] \\
&= \mathbb{P}\left[\min_{0\le s\le (T-t)(1-\varepsilon)}\{W_s\}<\varepsilon^{-1/2}\sigma^{-1}(b(t)-\max\{x,S\})\right] \\
&= 2\mathbb{P}\left[W_{(T-t)(1-\varepsilon)}<\varepsilon^{-1/2}\sigma^{-1}(b(t)-\max\{x,S\}\right],
\end{aligned}$$

where the first inequality is justified since b is non-decreasing (see Proposition 1), while the last equality comes after applying the *reflection principle.* Therefore,

$$M := \inf_{(t,x)\in\mathcal{R}} p(t,x) > 0.$$

Finally, by using (13) we obtain the following relation for all $(t,x)\in\mathcal{R}$:

$$\begin{aligned}
\partial_x V(t,x) &\le -e^{-\lambda(T-t)}\mathbb{E}\left[\frac{T-t-\tau^*}{T-t}\mathbb{1}\left(\tau^*\le (T-t)(1-\varepsilon)\right)\right] \\
&\le -e^{-\lambda(T-t)}\varepsilon p(t,x) \\
&\le -e^{-\lambda(T-t)}\varepsilon M < 0.
\end{aligned}$$

■

For the sake of completeness, we formulate the following change-of-variable result by taking Theorem 3.1 from Peskir (2005a) and changing some of its hypothesis according to Remark 3.2 from Peskir (2005a). Specifically, the (*iii-a*) version of Lemma 2 comes after changing, in Peskir (2005a), (3.27) and (3.28) for the joint action of (3.26), (3.35), and (3.36). The (*iii-b*) version relaxes condition (3.35) into (3.37) in *ibid.*

Lemma 2 *Let $X=(X_t)_{t=0}^T$ be a diffusion process solving the SDE*

$$\mathrm{d}X_t = \mu(t,X_t)\,\mathrm{d}t+\sigma(t,X_t)\,\mathrm{d}B_t, \qquad 0\le t\le T,$$

in the Itô's sense. Let $b : [0, T] \to \mathbb{R}$ *be a continuous function of bounded variation, and let* $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying*

$$F \text{ is } \mathcal{C}^{1,2} \text{ on } C,$$
$$F \text{ is } \mathcal{C}^{1,2} \text{ on } D^\circ,$$

where $C = \{(t, x) \in [0, T] \times \mathbb{R} : x > b(t)\}$ *and* $D^\circ = \{(t, x) \in [0, T] \times \mathbb{R} : x < b(t)\}$.

Assume there exists $t \in [0, T]$ *such that the following conditions are satisfied:*

(i) $\partial_t F + \mu \partial_x F + (\sigma^2/2)\partial_{x^2} F$ *is locally bounded on* $C \cup D^\circ$;

(ii) the functions $s \mapsto \partial_x F(s, b(s)^{\pm}) := \partial_x F(s, \lim_{h \to 0+} b(s) \pm h)$ *are continuous on* $[0, t]$;

(iii) and either

(iii-a) $x \mapsto F(s, x)$ *is convex on* $[b(s) - \delta, b(s)]$ *and convex on* $[b(s), b(s) + \delta]$ *for each* $s \in [0, t]$, *with some* $\delta > 0$, *or,*

(iii-b) $\partial_{x^2} F = G_1 + G_2$ *on* $C \cup D^\circ$, *where* G_1 *is non-negative (or non-positive) and* G_2 *is continuous on* $\bar{C}$ *and* $\bar{D}^\circ$.

Then, the following change-of-variable formula holds

$$\begin{aligned} F(t, X_t) = {} & F(0, X_0) + \int_0^t (\partial_t F + \mu \partial_x F + (\sigma^2/2)\partial_{x^2} F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, \mathrm{d}s \\ & + \int_0^t (\sigma \partial_x F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, \mathrm{d}B_s \\ & + \frac{1}{2} \int_0^t (\partial_x F(s, X_s^+) - \partial_x F(s, X_s^-)) \mathbb{1}(X_s = b(s)) \, \mathrm{d}l_s^b(X), \end{aligned}$$

where $\mathrm{d}l_s^b(X)$ *is the local time of* X *at the curve* b *up to time* t, *i.e.,*

$$l_s^b(X) = \lim_{\varepsilon \to 0} \int_0^t \mathbb{1}(b(s) - \varepsilon \le X_s \le b(s) + \varepsilon) \, \mathrm{d}\langle X, X \rangle_s, \tag{39}$$

where $\langle X, X \rangle$ *is the predictable quadratic variation of* X, *and the limit above is meant in probability.*

References

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