THE HAMILTONIAN FORMULATION

IN RELATIVITY

A thesis presented to the University of Leicester

for the degree of Doctor of Philosophy by

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Thesis

From its first introduction, action
has always been looked upon as something
whose sole   "raison dêtre"   is to be
varied - and, moreover, *varied in such a
way as to defy the laws of nature!*

A.S. Eddington,
"The mathematical theory
of relativity."

## ACKNOWLEDGEMENTS

## SYNOPSIS

Like any major breakthrough in thinking, the theory of relativity caused a great upheaval in our attitude to science. Seventy years after the advent of relativity we are still coming to terms with the changes it has brought in our outlook. Part of this process is simply the valid translation of pre-relativistic laws and concepts into the 4-dimensional language of relativity - a problem by no means as easy as would at first seem; the aim of this thesis is to survey the ways in which the methods of analytical mechanics may be translated into a relativistic setting.

Chapter 1 provides an introduction to the work in the form of a non-rigorous discussion of the historical and mathematical development of electromagnetism, analytical mechanics and relativity, and ends with a presentation of the basics of the functional calculus. This is needed in the presentation of field theory given in chapter 2. We see two possibilities for the relativistic formulation of analytical mechanics, and field theory represents the first of these possibilities. In the absence of any real grounds for continuing on this tack we then move on to the other possibility in chapter 3, where we review the attempts of a number of authors to formulate relativistic particle mechanics as a Hamiltonian system. This then leads in chapter 4 to our own such attempt, based mainly on the work of Synge, which we have named homogeneous mechanics. After the main exposition of the theory the work of the remaining chapters 5 and 6 is then to apply the above theory (not always successfully) to a number of cases where analytical mechanics has in the past proven itself an invaluable tool: namely, the areas of symmetries and quantum theory.

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CHAPTER 1

INTRODUCTION

## 1.1 Electromagnetism

The "golden age" of Greek philosophical thought, spanning the centuries from 600 BC to 200 AD, forms the basis for most of present-day science, and the supposedly modern fields of electromagnetism and relativity are by no means exceptions. Amongst the most prominent figures of this time was the philosopher Aristotle (born 384 BC), who devoted his whole life to the search for a conceptual model of the universe. His approach to the problem was along the Platonic lines of pure reasoning, although in fact his theories were to an extent "soiled" by some observational evidence.

Aristotle's universe comprised five basic elements: earth, air, fire, water and the supremely pure fifth element - the aether. Without going into the complexities of the Aristotelian description, it is relevant to emphasise the underlying philosophy behind the model. As far as Aristotle was concerned, it was self-evident that a sustaining force was required for the continuous motion of the heavens. Consequently his universe was permeated throughout by the aether in the form of a vast fluid continuum whose endless swirling and eddying carried the planets along their preordained courses. Yet this aesthetic picture became increasingly unmanageable as observational astronomy began to reveal the full complexity of the stellar motions. This struggle between Aristotelian theory and observational fact slowly became equated with the battle between ecclesiastical orthodoxy and the rising hereticism of the Renaissance, until matters reached a head with the findings of Galileo Galilei

(1564-1642). Galileo showed from his experiments that natural motion was not coming to rest, as was previously thought, but rather uniform motion in a straight line. This paved the way towards a whole new conception of the universe as a sort of receptacle into which objects could be placed, and within which they could move unhindered but for interactions between themselves. It was in the context of this conception that Sir Isaac Newton (1642-1727) formulated his system of mechanics.

One notable difference between the Newtonian and Aristotelian systems was that Greek astronomy was geometrical, not dynamical. With their obsession for "perfect" geometrical figures the Greeks thought of the motions of the heavenly bodies as uniform and circular, or compounded of circular motions; they had no conception of force. Instead the various heavenly bodies were fixed to celestial spheres which moved as a whole. Newton, with his universal law of gravitation, introduced a less geometrical point of view, yet it is interesting to note that there is a reversion to geometry in Einstein's general theory of relativity, from which the conception of force in the Newtonian sense has been excluded.

By the mid-nineteenth century the last remnants of Aristotle's description had been expunged from physics, and the Newtonian model held complete sway in all but one respect. Newtonian physics saw the universe as a container in which physical objects move, but the remaining space within that container was still occupied by the Greeks' fifth element, the aether. It had by now been established beyond all reasonable doubt that light is a wavelike phenomenon, and while the aether was no longer responsible for planetary motions it still performed the useful function of carrying light from source to

observer. The aether was supposed to be the medium through which light was 'propagated, just as sound waves travel through the air.

The undulatory nature of light acquired increasing justification in the late nineteenth century through its connection with the work of James Clerk Maxwell (1831-1879) in electromagnetism. Maxwell was able to describe the entire electromagnetic interaction by a system of four differential equations:

$$\text{div } \underline{E} = \varrho/\varepsilon_o \; ; \quad \text{curl } \underline{E} = -\partial\underline{B}/\partial t \; ;$$
$$\text{div } \underline{B} = 0 \quad ; \quad \text{curl } \underline{B} = \mu_o(\underline{j} + \varepsilon_o \partial\underline{E}/\partial t), \qquad -- \;(1\text{-}1)$$

where $\underline{E}$, $\underline{B}$ are the electric and magnetic fields at a given point; $\varrho, \underline{j}$ the charge and current densities; and $\varepsilon_o$, $\mu_o$ the permittivity and permeability constants of free space. In a region of space containing no charge or current these equations admit a wave solution in $\underline{E}$ and $\underline{B}$, from which Maxwell concluded that it was possible for a system of electric and magnetic disturbances to propagate indefinitely through the aether with a speed $c = 1/\sqrt{(\varepsilon_o\mu_o)}$. This conclusion was later amply confirmed by the work of Hertz. From experimental values of $\varepsilon_o$ and $\mu_o$ this formula gives $c = 2.998\times10^8 \text{m/s}$ - precisely the measured speed of light, so the link between electromagnetism and light had now been established.

An overriding feature of Maxwell's theory was the concept of a field. The use of coordinates to label points in the absolute space of Newtonian mechanics established the mathematical concept of the field as a useful instrument in theoretical physics, but it was not until Maxwell's theory of electromagnetism that the field concept came into its own as a fully fledged dynamical variable. It is precisely this feature of the theory which enabled it to survive the aether theories and which provided a conceptual environment favourable to the development of relativity. The

overthrow of aether theories which relativity theory effected would not have been possible without some new concept to replace them - this was provided by the idea of a field.

The early studies of electromagnetic phenomena emphasised their mechanical context, however. Thus the concept of the charged particle came to play a central role in the development of electromagnetic theory. Coulomb's formulation of the law of force acting between charged bodies provided the means by which the quantity of charge carried by a body could be determined by mechanical measurements. Investigations by Faraday of the quant-itave laws governing the electrolytic decomposition of chemical substances led to the hypothesis of the atom of electricity. Then early in the twentieth century Millikan succeeded in verifying this hypothesis through his famous oil-drop experiment. The universal quantum of electricity was thus established as a physical fact: the total charge on any particle is always an integer multiple of the charge on the electron, defined arbitrarily to be negative - $e = -1.6 \times 10^{-19}$ C.

The usual mathematical representation of the electric field identifies the value of the field at a point with the mechanical response of a charged test body placed at that point. The ass-umption that this procedure actually measures the value of the field in the absence of the test body is sometimes justified by defining the field value to be the force on the test body divided by the charge in the limit of vanishing charge. Thus the elect-romagnetic force on a test particle of charge e at rest is $e\underline{E}$, and this defines the electric field vector $\underline{E}$. If the particle is moving with uniform velocity $\underline{v}$ then it is a matter of experience that the electromagnetic force can be written

$$\underline{F} = e(\underline{E} + \underline{v} \times \underline{B}). \qquad\qquad -- (1-2)$$

This is the Lorentz force law, which defines the magnetic field vector $\underline{B}$.

One way of escaping from this mechanical approach is to describe the charge present by a continuous field of charge density $\varrho$ and current density $\underline{j}$. If $\bar{\varrho}$ is the average charge density in an element of volume $\delta V$ which includes a fixed point P, then the value of $\varrho$ at P is defined as the limit of $\bar{\varrho}$ as $\delta V \to 0$ through the element shrinking to the point P. The current density $\underline{j}$ may then be defined by the relation

$$\underline{j} = \varrho\underline{v},$$

where $\underline{v}$ is the velocity field of the charge distribution. These two quantities obey a conservation law which is contained implicitly in Maxwell's equations (1-1):

$$\text{div } \underline{j} + \partial\varrho/\partial t = 0. \qquad -- (1-3)$$

In this continous description charged particles appear as point singularities in the field, which presents a problem in the description of the field in the neighbourhood of the singularity. In continuous charge distributions one is concerned only with the charge contained within a given finite volume - the amount of charge at a point has no meaning in this description. However, if point charges are introduced into the theory then we must demand the field at the point to be infinite. This so-called "self-energy" problem is still unresolved, since the various methods of eliminating the problem (eg, "renormalisation" techniques) all amount to statements on the internal structure of the charged particle. Rohrlich (1965) avoids this problem by considering a point particle to be one whose "radius" is too small to be observed. This does not mean that the radius vanishes - simply that it is undetermined; as a result the divergent terms do not occur in the theory. Nevertheless, particle and field theories

remain distinct from each other, despite the possible synthesis of the two provided by quantum theory. This duality has been the main motivation for this thesis: on the one hand the attempt to treat the field as the fundamental entity has led to the study of quantum field theory, while treating the particle as the fundamental entity has led to Dirac's (1928) relativistic theory of the electron. In the research summarised in this thesis we have tried to look at both the field (chapter 2) and the particle (chapter 3) approaches in the light of the canonical formulation of mechanics (see section 1.2) and to assess their relative merits as viable descriptions of nature.

## 1.2 Analytical Mechanics

At the same time as Newton was writing his "Principia" the German philosopher Gottfried Leibnitz (1646-1716), together with Christiaan Huygens (1629-95), proposed an alternative form of mechanics in which the fundamental quantity was essentially the kinetic energy, rather than the force. This form of mechanics, based on the single scalar quantity of energy, then grew into what is now called _analytical mechanics_. D'Alembert's Principle may be considered the starting point of modern analytical mechanics. This appeared in d'Alembert's "Traite de Dynamique" of 1743, and commences from the fundamental Newtonian law of motion:

$$\underline{F} = d(m\underline{v})/dt,$$

whence
$$\underline{F} - d(m\underline{v})/dt = 0.$$

Now letting
$$\underline{I} = -d(m\underline{v})/dt,$$

we have
$$\underline{F} + \underline{I} = \underline{0}. \qquad\qquad -- (1-4)$$

We know that the vanishing of a force in Newtonian mechanics corresponds to a state of equilibrium, so (1-4) says that the

addition of the _force of inertia I_ to the other acting forces

produces equilibrium. At first sight this seems to be a mere

rephrasing of the law of motion, but its importance lies prec-

isely in this apparent simplicity. By this device the complex

problem of a dynamical situation is reduced to a more easily

soluble problem in statics. D'Alembert's Principle is now exp-

ressed as follows:

> The total virtual work done by the effective force $\underline{F}+\underline{I}$
> is zero for all variations of position $\delta\underline{r}$ which satisfy the
> given dynamical constraints of the problem, ie:

$$\delta W = (\underline{F}+\underline{I}).\delta\underline{r} = 0.$$

Lagrange's development of this principle consists of def-

ining the _Lagrangian_ L of the system as the kinetic energy minus

the potential energy:

$$L = T - V \; ; \qquad\qquad -- (1-5)$$

it then follows from d'Alembert's Principle that

$$\int_{t_1}^{t_2} \delta W \, dt = \delta\int_{t_1}^{t_2} L \, dt - [\sum_i m_i \underline{v}_i . \delta\underline{r}]_{t_1}^{t_2}$$

$$= 0.$$

If we now choose a specific variation for which $\delta r$ vanishes at

the endpoints of the path, then the boundary term vanishes and we

have

$$\delta I = \delta\int L \, dt = 0, \qquad\qquad -- (1-6)$$

where I is the _action_ of the system. (1-6) is called the _Action_

_Principle_, and states that the motion of an arbitrary mechanical

system occurs in such a way that the action I is stationary for

arbitrary variations of the configuration of the system, provided

the initial and final configurations are prescribed (ie,

$\delta r(t_1) = \delta r(t_2) = 0)$.

In the derivation of the action principle all scalar prod-

ucts have assumed implicitly that rectangular coordinates are

used.   However, the whole advantage of analytical mechanics is that  the coordinates are not restricted in this way: indeed it might  be  considered the first step from Newton's mechanics to relativity in that it commences the process of freeing mechanics from  a fixed frame of reference.  In general the coordinates of the  system  will  be  expressible in terms of N generalised coordinates $q^A$,  and  the potential and kinetic energies are functions  of the coordinates $q^A$ and the velocities $\dot{q}^A$.  In this case  the  necessary  and sufficient conditions for the action principle (1-6) to hold are

$$\delta I = \delta \int L dt$$

$$= \int \left\{ \frac{\partial L}{\partial q^A} \delta q^A + \frac{\partial L}{\partial \dot{q}^A} \delta \dot{q}^A \right\} dt$$

$$= \int \left\{ \frac{\partial L}{\partial q^A} \delta q^A + \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{q}^A} \delta q^A \right] - \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{q}^A} \right] \delta q^A \right\} dt$$

$$= \int \left\{ \frac{\partial L}{\partial q^A} - \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{q}^A} \right] \right\} \delta q^A dt + \left[ \frac{\partial L}{\partial \dot{q}^A} \delta q^A \right]_{t_1}^{t_2} .$$

On the assumption that the variation $\delta q$ vanishes at the endpoints of  the path, the boundary term disappears from this expression and we obtain the <u>Euler-Lagrange equations</u> of motion:

$$\frac{\partial L}{\partial q^A} - \frac{d}{dt}\left[ \frac{\partial L}{\partial \dot{q}^A} \right] = 0 .$$           -- (1-7)

The  first  major  result which may be obtained straight-forwardly from the action principle is the law of conservation of energy.  To derive this result we let the virtual displacement $\delta q$ at  each  instant coincide with the actual displacement $\delta q = \dot{q} dt$ which  takes  place during the infinitesimal time interval dt. This  variation alters the coordinates also at the two endpoints $t_1$ , $t_2$ , so (1-6) is no longer valid and must be replaced by

$$\delta I = [ \partial L / \partial q^A . \delta q^A ]_1^2 .$$

If we now adopt the notation

$$p_A \equiv \partial L / \partial \dot{q}^A \qquad\qquad -- (1-8)$$

then we have $\qquad\qquad \delta I = [p_A \delta q^A]_1^2$ .

Assuming that our system is conservative (t does not appear explicitly in the functional form of the Lagrangian), our expression for $\delta I$ becomes

$$dt.[L]_1^2 = dt.[p_A \dot{q}^A]_1^2 ,$$

giving $\qquad p_A \dot{q}^A - L = E = const. \qquad\qquad -- (1-9)$

Now the potential energy of the system is independent of the velocities, so that $p_A = m\dot{q}_A$ and the first term of (1-9) is simply twice the kinetic energy. Hence (1-9) expresses the fact that T+V=E, ie, that the total energy of the system is constant. In the light of this the quantities $p_A$, defined by (1-8), will be called the _generalised_ _momenta_ of the system.

W.R.Hamilton (1817-85) achieved an important modification of analytical mechanics by transforming the Lagrangian problem of N second-order differential equations into an equivalent problem of 2N first-order equations. This he did by replacing the N velocities of the Lagrangian formulation by the N corresponding generalised momenta, according to the following procedure:

i. Introduce the new variables $p_A = \partial L / \partial \dot{q}^A$ .

ii. Introduce the _Hamiltonian_ H which in the case of a conservative system corresponds to the total energy:

$$H \equiv p_A \dot{q}^A - L . \qquad\qquad -- (1-10)$$

iii. Express the new function H in terms of the new variables $p_A$ by solving for the $\dot{q}^A$ as functions of the q's and p's, and substituting into the expression (1-10) for the Hamiltonian.

This transformation procedure, embodied in the equations (1-8) and (1-10), is called a _Legendre_ _transformation_. It has the interesting property that variations of the Hamiltonian are

totally independent of variations of the velocities q, since if we vary the momenta alone then

$$\delta H = \dot{q}^A \delta p_A + p_A \delta \dot{q}^A - \delta \dot{q}^A . \partial L / \partial \dot{q}^A \qquad \text{-- (1-11)}$$
$$= \dot{q}^A \delta p_A$$

from (1-8), where $\delta \dot{q}^A$ is the variation in the velocities brought about by a variation of the momenta. So we see that $\delta H$ depends solely on the variations $\delta p_A$. The Legendre transformation thus forms an injection from the space of coordinates and velocities onto the space of coordinates and momenta, provided the relation (1-8) is nonsingular, ie, iff

$$\left| \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right| \neq 0 \quad . \qquad \text{-- (1-12)}$$

The elegant dual nature of the Legendre transformation is displayed in the following scheme:

| Lagrangian formulation | Hamiltonian formulation |
|---|---|
| $L = L(q^1, \ldots, q^N, \dot{q}^1, \ldots, \dot{q}^N ; t)$ | $H = H(q^1, \ldots, q^N, p_1, \ldots, p_N ; t)$ |
| $p_A = \partial L / \partial \dot{q}^A$ | $\dot{q}^A = \partial H / \partial p_A$ |
| $H = p_A \dot{q}^A - L$ | $L = p_A \dot{q}^A - H$ |

$$\text{-- (1-13)}$$

In addition we have the two relations

$$\partial L / \partial q^A = -\partial H / \partial q^A \quad ; \quad \partial L / \partial t = -\partial H / \partial t \quad . \quad \text{-- (1-14)}$$

Using the momenta we can now rewrite the Euler-Lagrange equations (1-7) in the form

$$\dot{p}_A = \partial L / \partial q^A$$
$$= -\partial H / \partial q^A \quad ,$$

by application of (1-14). Thus we have finally replaced the Lagrangian equations of motion by a new set of differential equations called the _canonical equations_:

$$\dot{q}^A = \partial H / \partial p_A \quad ; \quad \dot{p}_A = -\partial H / \partial q^A \quad . \qquad \text{-- (1-15)}$$

The coordinates and momenta are known collectively as the _canonical variables_ (q,p), the variables q and p being said to be

<u>conjugate</u>. The 2N-dimensional space of all canonical variables is called the <u>phase space</u>, and each point of phase space represents an instantaneous <u>state</u> of the system. The canonical equations are entirely equivalent (assuming (1-12)) to the original Euler-Lagrange equations, being merely a mathematically new form. Yet the new equations are superior in a variety of ways, not least of which is the easy transition to quantum theory made possible by the Hamiltonian formulation. In addition the Hamiltonian equations are of a more usable theoretical form than the Euler-Lagrange equations, since the lack of dependence of the Hamiltonian on the velocities means that all time derivatives appear on the LHS of the equations. A solution (q(t),p(t)) of the canonical equations will be called a <u>trajectory</u>.

If we make the transformation

$$q \to -p \quad ; \quad p \to q \qquad\qquad -- (1-16)$$

then the transformed canonical equations become

$$-\dot{p}_A = \partial H/\partial q^A \quad ; \quad \dot{q}^A = \partial H/\partial p_A \quad ,$$

which are exactly equivalent to the original equations of motion. Because of this it seems that neither position nor momentum may be considered more fundamental than the other, and so we now consider a more general type of transformation than was possible in the Lagrangian case. Consider the following transformation, in which all canonical variables are involved on an equal basis:

$$\left.\begin{array}{c} q \to Q(q,p,t) \quad ; \quad p \to P(q,p,t) \quad ; \\[4pt] H(q,p,t) \to \overline{H}(Q,P,t) \quad . \end{array}\right\} \qquad -- (1-17)$$

Clearly the most significant such transformations will be those which leave the equations of motion invariant - such transformations are called <u>canonical transformations</u> (CTs).

For a CT we have

$$\delta \int \{ p_A \dot{q}^A - H \} dt = \delta \int \{ P_A \dot{Q}^A - \overline{H} \} dt$$

$$= 0 \equiv \delta \int (dF/dt).dt ,$$

for an arbitrary function F. Hence we obtain the following sufficient condition that the transformation (1-17) be canonical:

$$\{ p_A \dot{q}^A - H \} - \{ P_A \dot{Q}^A - \overline{H} \} = dF/dt . \qquad -- (1-18)$$

Here F will in general be a function of the 4N+1 variables $(q^A, p_A, Q^A, P_A, t)$, but because of the relations (1-17) we can reduce this to t and any 2N of the others. Condition (1-18) is also necessary, modulo simple dilations and reflections of phase space (see Sudarshan & Mukunda,1974). We now consider a number of possibilities for the function F.

## Case 1

Consider the case

$$F = F_1 (q,Q,t) ,$$

then

$$dF_1/dt = \dot{q}^A \partial F_1 / \partial q^A + \dot{Q}^A \partial F_1 / \partial Q^A + \partial F_1 / \partial t .$$

If we regard (q,Q,t) as independent variables we find

$$\left. \begin{array}{l} p_A = \dfrac{\partial}{\partial q^A} F_1 (q,Q,t) \quad ; \quad P_A = -\dfrac{\partial}{\partial Q^A} F_1 (q,Q,t) \quad ; \\[2ex] \overline{H} - H = \dfrac{\partial}{\partial t} F_1 (q,Q,t) . \end{array} \right\} \qquad -- (1-19a)$$

This system is (in principle) soluble for (Q,P) as functions of (q,p), so we see that F determines a transformation of the canonical variables. Indeed, F may be shown (see chapter 4) to be essentially uniquely determined by a given transformation, and so is called the generating function of the transformation (1-19a).

## Case 2

We can obtain the case

$$F = F_2 (q,P,t)$$

from case 1 by applying the Legendre transformation

$$F = F_1 (q,Q,t) + Q^A P_A ,$$

which has the effect of replacing the $Q^A$ by the $P_A$. Considering the same transformation as above,

$$\{P_A \dot{q}^A - H\} - \{P_A \dot{Q}^A - \overline{H}\} = dF_1/dt$$
$$= d/dt.\{F_2 - Q^A P_A\} \quad .$$

Continuing as before this leads to the transformation equations

$$P_A = \partial F_2/\partial q^A \quad ; \quad Q^A = \partial F_2/\partial P_A \quad ; \quad \overline{H} - H = \partial F_2/\partial t \qquad -- (1-19b)$$

Since we have considered the same transformation as in case 1, we would expect these equations to be identical to the transformation equations of case 1, and this is indeed so. The first and last of each are identical since

$$\partial F_1/\partial t = \partial F_2/\partial t \quad \text{and} \quad \partial F_1/\partial q^A = \partial F_2/\partial q^A$$

and although the second of each appear different, they are in fact rearrangements of one another. The remaining two cases are essentially repetitions of the above working:

## Case 3

We obtain

$$q^A = -\partial F_3/\partial P_A \quad ; \quad P_A = -\partial F_3/\partial Q^A \quad ; \quad \overline{H} - H = \partial F_3/\partial t \qquad -- (1-19c)$$

from the Legendre transformation

$$F_3(p,Q,t) = F_1(q,Q,t) - P_A q^A \quad .$$

## Case 4

We obtain

$$q^A = -\partial F_4/\partial P_A \quad ; \quad Q^A = \partial F_4/\partial P_A \quad ; \quad \overline{H} - H = \partial F_4/\partial t \qquad -- (1-19d)$$

from

$$F_4(p,P,t) = F_1(q,Q,t) + Q^A P_A - q^A P_A \quad .$$

Historically, the whole point of studying CT's was originally in order to transform the canonical equations into a more easily soluble form. The <u>Hamilton-Jacobi method</u> explicitly determines a generating function from which can be derived a transformation to a new system in which

$$Q^A = \text{const} = \alpha^A \quad ; \quad P_A = \text{const} = \beta_A \quad .$$

The new constant quantities may be the 2N initial values $(q_o, p_o)$ at $t = 0$, in which case the transformation equations are precisely the solution of the dynamical problem:

$$q = q(q_o, p_o, t) \quad ; \quad p = p(q_o, p_o, t) \quad .$$

Since we are at liberty to choose any of the above four cases to represent a given CT we shall suppose that this special generating function, S, is of the type considered in case 2:

$$S = S(q, P, t) = S(q, \beta, t) \quad ,$$

and since the transformation is canonical we have

$$\dot{Q}^A = \partial\bar{H}/\partial P_A = 0 \quad ; \quad \dot{P}_A = -\partial\bar{H}/\partial Q^A = 0 \quad . \qquad -- \; (1-20)$$

If we now also require that the new Hamiltonian contain no explicit time dependence $(\partial\bar{H}/\partial t = 0)$ then $\bar{H}$ is a constant which may be arbitrarily set to zero. In this case the equations (1-19b) become

$$H(q, p, t) + \partial S/\partial t = 0 \quad ; \quad p_A = \partial S/\partial q^A$$

$$\Rightarrow \qquad H(q, \partial S/\partial q, t) + \partial S/\partial t = 0 \quad . \qquad -- \; (1-21)$$

(1-21) is a first-order partial differential equation called the <u>Hamilton-Jacobi</u> (H-J) equation; it may be written down explicitly for any particular problem, since H will be a known function of the $(q^A, p_A, t)$. Since the H-J equation involves the N+1 independent variables $(q^A, t)$, a complete integral (see Pearson & Carrier, 1976) of the equation will contain N+1 arbitrary constants. However, if $S_o$ is a solution then clearly $S = S_o + \text{const}$ is also a solution. Thus we can reduce the number of arbitrary constants to N, since only the derivatives of S appear in the theory. These N constants may be identified with the $\beta_A$:

$$S = S(q, \beta, t)$$

iff this substitution satisfies the relations (1-20). Our derivation has shown this to be so for the first of (1-20), and it is straightforwardly shown to be true for the second (see for

example Goldstein,1981). The values of the constants $\beta_A$ are then found by substituting the initial values of the problem - $q^A(t_o)$, $p_A(t_o)$ - into $p_A = \partial S/\partial q^A$, and the $\alpha^A$ may be calculated from $\alpha^A = \partial S/\partial \beta_A$.

Jacobi saw the above method simply as a means of simplifying practical problems, but in fact the H-J equation may be developed (Rund,1966) as a third type of equation of motion, distinct from the canonical and Euler-Lagrange equations. An indication of the theoretical significance of the Hamilton-Jacobi theory is obtained by applying the condition (1-18) to the generating function S:

$$dS/dt = \{p_A \dot{q}^A - H\} - \{P_A \dot{Q}^A - \bar{H}\}$$

$$= \{p_A \dot{q}^A - H\} = L$$

$\Rightarrow$ $\qquad$ $S = \int L dt + const.$ $\qquad$ -- (1-22)

Thus we see that up to an additive constant, S is simply the action measured along the trajectory of the system. S is called the 2-point characteristic function of Hamilton, of which more will be said in chapter 5.

In the case where the Hamiltonian of a problem does not involve the time explicitly it is possible to perform a separation of variables to obtain a simpler equation. Omitting the explicit time dependence from (1-21) we have

$$\partial S/\partial t + H(q, \partial S/\partial q) = 0 \quad .$$

The first term involves only t-dependence, while the second involves only q-dependence. Therefore we can separate the time variable by assuming a solution for S of the form

$$S(q, \alpha, t) = W(q, \alpha) - \beta_N t \quad .$$

Substituting this trial solution we obtain the time-independent (H-J) equation:

$$H(q, \partial W/\partial q) = \beta_N \quad , \qquad -- (1-23)$$

in which one of the constants of integration ($\beta_N$) is thus equal

to the constant value of H (usually the energy). Here W acts as a generating function of a transformation to a system in which one of the momenta is equal to the Hamiltonian:

$$P_N = H(q,p) = \overline{H}(Q,P) \quad .$$

A particularly simple example of a case 2 canonical transformation is when $F = P_A q^A$. Here we obtain from (1-19b)

$$p = \partial F/\partial q = P \; ; \; Q = \partial F/\partial P = q \; ; \; \overline{H} = H + \partial F/\partial t = H, \quad -- \quad (1-24)$$

from which we see that this choice of F generates the identity transformation. This enables us to generate infinitesimal transformations by means of the small parameter $\varepsilon$ (independent of q and p) and the arbitrary function G:

$$F = q^A P_A + \varepsilon G(q,P) \quad .$$

From this we have

$$Q^A = \partial F/\partial P_A = q^A + \varepsilon \partial G/\partial P_A \; ; \; p_A = \partial F/\partial q^A = P_A + \varepsilon \partial G/\partial q^A$$

$$\Rightarrow \qquad \delta q^A = \varepsilon \partial G/\partial P_A \; ; \; \delta p_A = -\varepsilon \partial G/\partial q^A \quad .$$

Since $\delta p$ is infinitesimal we may replace P by p in G to obtain

$$\delta q^A = \varepsilon \frac{\partial G}{\partial p_A}(q,p) \; ; \; \delta p_A = -\varepsilon \frac{\partial G}{\partial q^A}(q,p) \quad . \qquad -- \quad (1-25)$$

(1-25) is called an _infinitesimal contact transformation_, and it is usual to call the function G the _generator_ of the transformation. As a particular case of the above, consider

$$\varepsilon = dt \; ; \; G = H \quad ,$$

then
$$\delta q^A = dt.\partial H/\partial p_A = dt.\dot{q}^A \; ; \; \Big\rbrace$$
$$\delta p_A = -dt.\partial H/\partial q^A = dt.\dot{p}_A \quad . \; \Big\rbrace \qquad -- \quad (1-26)$$

Here the Hamiltonian is generating the actual changes which occur in the system due to its motion. Thus the evolution of a system may be regarded, in the words of Hamilton, as the "continuous unfolding" of an infinitesimal contact transformation generated by the Hamiltonian of the system.

The final aspect of analytical mechanics which we will look

at in this *resumé* is the idea of the Poisson bracket. Let F(q,p,t) be some dynamical variable of a system, then we can calculate the rate of change of F along a trajectory:

$$\dot{F}(q,p,t) = dF/dt$$

$$= \frac{\partial F}{\partial q^A} \dot{q}^A + \frac{\partial F}{\partial p_A} \dot{p}_A + \frac{\partial F}{\partial t}$$

$$= \left[\frac{\partial F}{\partial q^A}\frac{\partial H}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial H}{\partial q^A}\right] + \frac{\partial F}{\partial t}$$

$$= \{F,H\} + \partial F/\partial t \ , \qquad\qquad \text{-- (1-27)}$$

where 
$$\{X,Y\} \equiv \frac{\partial X}{\partial q^A}\frac{\partial Y}{\partial p_A} - \frac{\partial X}{\partial p_A}\frac{\partial Y}{\partial q^A} \qquad\qquad \text{-- (1-28)}$$

is the <u>Poisson bracket</u> (PB) of the two phase space quantities X and Y. The following identities follow immediately from the definition of the PB:

$$\{X,Y\} = -\{Y,X\} \quad ; \quad \{X,X\} = 0 \quad ;$$

$$\{X,Y+Z\} = \{X,Y\} + \{X,Z\} \quad ; \quad \{X,YZ\} = Y\{X,Z\} + \{X,Y\}Z \ .$$

Also, 
$$\{q^A,q^B\} = \{p_A,p_B\} = 0 \quad ; \quad \{q^A,p_B\} = \delta^A_B \ . \qquad\qquad \text{-- (1-29)}$$

Equations (1-29) are called the <u>fundamental PB relations</u>: from them may be built up all other PB relations between dynamical quantities.

An important property of PB's is that they are invariant under CT's, ie, if X,Y are two scalar quantities defined on phase space then

$$\{X,Y\} = \{X,Y\}' \ , \qquad\qquad \text{-- (1-30)}$$

where the PB is evaluated in the undashed and dashed systems respectively. The quantities X,Y may have the same <u>value</u>, but not necessarily the same <u>form</u> in the two systems, yet their PB is still an invariant. The proof of (1-30) consists in proving the invariance of the fundamental PB's (1-29) under each of the four types of CT in (1-19), and hence (1-30) follows by building up arbitrary PB's from these. In view of this invariance it becomes

unnecessary to distinguish between PB's evaluated in different systems and we can omit the dash in (1-30).

From (1-29) we see that provided F does not contain an explicit time-dependence, it is a constant of the motion iff {F,H}=0 (whether or not H is a constant of the motion). Special cases of (1-29) are

$$\dot{q}^A = \{q^A,H\} \quad ; \quad \dot{p}_A = \{p_A,H\} \quad ; \quad \dot{H} = \partial H/\partial t \quad , \qquad \text{-- (1-31)}$$

which gives us an alternative, more symmetrical, form of the canonical equations. A further property of the PB is the <u>Jacobi identity</u>:

$$\{X,\{Y,Z\}\} + \{Y,\{Z,X\}\} + \{Z,\{X,Y\}\} = 0 \quad , \qquad \text{-- (1-32)}$$

which may be used to construct new constants of motion from old ones. For suppose Z=H, and X,Y are constants of the motion, then

$$\{X,\{Y,H\}\} + \{Y,\{H,X\}\} + \{H,\{X,Y\}\} = 0$$

$$=> \qquad\qquad\qquad\qquad \{H,\{X,Y\}\} = 0$$

and so {X,Y} is also a constant. Note, however, that this method of producing constants is rarely fruitful in practice, since the constants so produced are often simply new combinations of old ones.

The PB also enables us to express the contact transformation equations (1-25) in a more symmetrical form. Let G be the gener-ator of a contact transformation and let X be some dynamical variable, then the change in X is given by (1-25) as

$$\delta X = \delta q^A . \partial X/\partial q^A + \delta p_A . \partial X/\partial p_A$$

$$= \epsilon . \{X,G\} \quad . \qquad\qquad\qquad \text{-- (1-33)}$$

In the specific case where X is one of the coordinates or momen-ta, we recover the contact transformation equations in PB form:

$$\delta q^A = \epsilon . \{q^A,G\} \quad ; \quad p_A = \epsilon . \{p_A,G\} \quad . \qquad \text{-- (1-34)}$$

Also, if X=H in (1-33), then $\delta H=\epsilon\{H,G\}$; hence a constant of the motion (for which {H,G}=0) generates an infinitesimal transform-

ation which leaves H invariant. This is the basis of Noether's theorem, which we shall look at more closely in chapter 5.

## 1.3 The relativistic model

In this section we shall briefly outline the relativistic model to be used in this thesis and the various notations and conventions which we shall adopt. The model of the universe used in general relativity is a 4-dimensional manifold which amalgamates the classical concept of space (coordinates $x^1, x^2, x^3$) with the fourth dimension of time ($x^4 = t$). This amalgamation, first introduced by Minkowski (1908), is in sharp contrast to the classical viewpoint as expressed in Newton's "Principia":

> "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year."

In relativity time becomes for the first time a participant in the transformations induced by the motions of the observer, and is to be treated simply as one more dimension in the manifold. Of course, certain properties do distinguish time from the other coordinates, and this is expressed by the fact that we use a Lorentz metric $g_{ab}$ on the manifold with signature 2. If $T_p(M)$ denotes the space of tangent vectors to the manifold M at the point $p \epsilon M$, then we can use the above metric to divide the elements of $T_p(M)$ into three classes: a nonzero vector $X \epsilon T_p(M)$ is said to be timelike, null or spacelike according to whether $g_{ab} X^a X^b$ is negative, zero or positive respectively (the Einstein summation convention is taken to apply). In relativity timelike vectors represent the instantaneous motion of physical observers, and null vectors the possible paths of light signals. The set of

all null vectors at p is called the <u>null cone</u> at p. A <u>hyper-</u><u>surface</u> is a subspace of the spacetime manifold, which is termed <u>timelike</u> if it contains one timelike direction; <u>null</u> if it contains no timelike directions but one null direction; and <u>spacelike</u> if it contains only spacelike directions.

The second notable departure from classical mechanics concerns the "flatness" of the universe, referred to implicitly in the Principia:

> "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable."

It is assumed without question in Newtonian mechanics that this "absolute space" is Euclidean, but in general relativity the strict flatness of the Euclidean manifold is abandoned in favour of a series of weaker restrictions on the manifold. For the physical reasons behind the properties given in the following definition, see Hawking and Ellis (1973).

<u>Definition (1.1)</u>: A <u>spacetime</u> (M,g) is a connected, 4-dimensional, oriented, time-oriented, paracompact, Hausdorff manifold M together with a global Lorentz metric g and the associated Levi-Civita connection.

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Hawking and Ellis also make the added assumption that the space-time is <u>inextendible</u>, ie, it cannot be embedded isometrically into a larger spacetime. Hence we now make the following definition:

<u>Definition (1.2)</u>: The <u>general theory of relativity</u> consists of:

(i) An inextendible spacetime (M,g) whose metric is at least $c^2$. The points of the spacetime are called <u>events</u>.

(ii) Let $M_o$ be a convex normal neighbourhood in M, containing the points $p,q \in M$. Then a signal can be sent in $M_o$

between p and q iff p and q can be joined by a $C^1$ curve lying entirely within $M_0$, whose tangent vector is everywhere nonspacelike. This is the postulate of <u>local causality</u>.

(iii) There exists a symmetric (0,2) tensor $T_{ab}$ (the <u>energy-momentum</u> tensor) which depends only on the matter fields, their covariant derivatives and the metric such that:

(a) $T_{ab}=0$ on an open subset U of M iff all matter fields vanish on U.

(b) If $u^a$ are the components of a timelike vector at p$\varepsilon$M then

$$T_{ab}u^a u^b \leqslant 0 \qquad\qquad \text{-- (1-35)}$$

and $T_{ab}u^b$ are the components of a non-spacelike vector at p.

This is the <u>dominant energy condition</u>.

(iv) The metric coefficients at a point satisfy the following set of second-order partial differential equations:

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = -\kappa T_{ab}, \qquad \text{-- (1-36)}$$

where $G_{ab}$ is the <u>Einstein tensor</u>, $R_{ab} = R^d{}_{abd}$ is the <u>Ricci tensor</u>, $R=R^a{}_a$ is the <u>Ricci scalar</u> and $\kappa$ is the <u>gravitational constant</u>. The equations (1-36) are the <u>Einstein field equations</u>, and in the case where $T_{ab}=0$ are called the <u>vacuum equations</u>.

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A few remarks should be added to the above definition. First of all we note that from the definition in (1-36), $G_{ab}$ has zero divergence, so from the field equations we must have

$$-\kappa T_a{}^b{}_{;b} = G_a{}^b{}_{;b} = 0. \qquad \text{--(1-37)}$$

Hence $T_{ab}$ also has zero divergence. We will see in the next

section that $T_{ab}$ in fact represents the negative of the energy-momentum density. This minor inconvenience comes about as a result of our other conventions. The second point to note is that Hawking and Ellis also consider a further condition on spacetime called the <u>stable causality condition</u>. This requires first of all that the spacetime does not admit any closed timelike curves, and secondly that this is also the case for every "sufficiently small" variation of the metric. The idea of this condition is to ensure that quantum fluctuations of the metric do not violate global causality. Thirdly the paths, or <u>worldlines</u>, of neutral, nonrotating physical bodies are the timelike geodesics of M.

Finally we note the following notation conventions used in this thesis. Round and square brackets denote respectively the usual symmetrisation and antisymmetrisation of a tensor over the enclosed indices. Ordinary partial differentiation with respect to the coordinates $x^a$ will be denoted by a comma preceding the derivative indices and covariant differentiation by a semi-colon. All Latin indices $(a,b,c,...)$ will run from 1 to 4 and block indices $(A,B,C,...)$ will run from 1 to N (see chapter 2). Greek indices $(\alpha,\beta,\gamma,...)$ will run from 1 to 3 if Latin indices are currently being used, and from 1 to N-1 if block indices are currently being used - which of these cases applies will be made clear from the context. We will also use the bare tensor symbol, stripped of its indices, to denote the tensor in abstraction from any coordinate system, and the underlining of a tensor will always mean the spatial component of the tensor, ie, the first three components. For example,

$$r^a = (r^1, r^2, r^3, r^4)$$
$$= (r^\alpha, r^4)$$

$$= (\underline{r}, r^4).$$

The word "classical" will at all times be reserved to refer to the mechanical structure underlying the theories of Newtonian, Lagrange and Hamilton. That is, it will mean "discrete" and "non-relativistic", rather than the usual meaning of "non-quantum". Thus the phrase "classical quantum theory" will become meaningful in later chapters.

When there exists a coordinate frame in which the metric coefficients are constant, there is no distinction between partial and covariant derivatives for a certain class of frames. This is the case in <u>Minkowski spacetime</u>, where the metric is given by

$$(g_{ab}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \qquad \text{-- (1-38)}$$

Here the special class of frames is the set of frames in which (1-38) is true. The transformations between these frames are the <u>Lorentz transformations</u>. All of the work in the remainder of this chapter will be confined to Minkowski spacetime.

## 1.4 Electromagnetism in relativity

Maxwell's equations (1-1) can be expressed in covariant form in Minkowski spacetime if we regard $\underline{j}$ and $\varrho$ as the components of a 4-vector $j^a$ and $\underline{E}$ and $\underline{B}$ as the components of an antisymmetric tensor $F_{ab}$, such that

$$(j^1, j^2, j^3) = \underline{j} \; ; \quad j^4 = \varrho, \qquad \text{-- (1-39)}$$

$$(F_{ab}) = \begin{bmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix} . \qquad \text{-- (1-40)}$$

Throughout this thesis we shall use a rationalised system of units in which $\mu_0 = 4\pi$ and $c = 1$. In this case $\epsilon_0 = 1/4\pi$ and the equations (1-1) become

$$\operatorname{div} \underline{E} = 4\pi\varrho \ , \quad \operatorname{curl} \underline{E} = -\partial \underline{B}/\partial t,$$

$$\operatorname{div} \underline{B} = 0 \quad , \quad \operatorname{curl} \underline{B} = 4\pi \underline{j} + \partial \underline{E}/\partial t.$$

We can now express the first and last of these as

$$F^{ab}{}_{,b} = 4\pi j^a \qquad\qquad -- (1\text{-}41a)$$

and the other two as

$$F_{[ab,c]} = 0, \qquad\qquad -- (1\text{-}41b)$$

or alternatively, $F_{ab,c} + F_{bc,a} + F_{ca,b} = 0.$

From here onwards we shall refer to the system (1-41) as Maxwell's equations - (1-41b) is the homogeneous Maxwell equation and (1-41a) is the inhomogeneous Maxwell equation. It is also possible to write the Lorentz force law (1-2) in the covariant form

$$K_b = F_{ab} j^a , \qquad\qquad -- (1\text{-}42)$$

where the space components of $K_b$ correspond to the density of Newtonian force exerted by the electromagnetic field on the charge distribution $j^a$. In order to interpret the time component of $K_b$ we note that according to (1-42), (1-39) and (1-40)

$$K_4 = -\underline{E} \cdot \underline{j},$$

which is the classical expression for the rate of decrease of field energy density due to work done by the field on the charge distribution. For a charged particle e with 4-velocity $u^a$, (1-42) takes the form

$$K_b = eF_{ab} u^a , \qquad\qquad -- (1\text{-}43)$$

where K is now the 4-force on the particle, rather than the 4-force density.

Using the law (1-42) together with the Maxwell equations (1-41) we obtain

$$K_b = F_{ab} j^a$$

$$= 1/4\pi . F_{ab} F^{as}{}_{,s}$$

$$= 1/4\pi . \{(F_{ab} F^{as})_{,s} - F_{ab,s} F^{as}\}.$$

Now
$$F^{as} F_{ab,s} = 1/2 . F^{as}(F_{ab,s} + F_{bs,a})$$

$$= -1/2 . F^{as} F_{sa,b}$$

$$= 1/4 . (F^{as} F_{as})_{,b}$$

and so
$$K_b = 1/4\pi . \{F_{sb} F^{sa} - 1/4 . \delta^a_b F_{rs} F^{rs}\}_{,a} .$$

We define
$$E_b{}^a \equiv 1/4\pi . \{F_{bs} F^{sa} + 1/4 . \delta^a_b F_{rs} F^{rs}\} ,$$

so that
$$K_b = -E_b{}^a{}_{,a} .$$


We call $E_b{}^a$ the _electromagnetic energy-momentum tensor_. In the form required for the Einstein field equations (1-36) it is given by the clearly symmetric expression

$$E_{ab} = 1/4\pi . \{F_{as} F^s{}_b + 1/4 . g_{ab} F^{rs} F_{rs}\}.$$

In general $E_{ab}$ cannot express the entire energy-momentum content of spacetime, since in the Einstein field equations $G_{ab}$ has zero divergence, while we have just shown that $E_{ab}$ does not. However, in the absence of charges and all non-electromagnetic matter fields we have $j^a = 0$ and may represent the entire energy-momentum field by

$$T_{ab} = 1/4\pi . \{F_{as} F^s{}_b + 1/4 . g_{ab} F^{rs} F_{rs}\}, \qquad -- \quad (1-44)$$

for which $T^{ab}{}_{,b} = 0$.

The first point to note about (1-44) is that it is manifestly traceless ($T^a{}_a = 0$). Indeed some treatments use the zero trace of the electromagnetic energy tensor as one of a series of defining characteristics, since it is related to the zero rest-mass of the photon. In order to interpret the various components of $T_{ab}$ we now write them out explicitly by substituting (1-40)

into (1-44):

$$T_{11} = 1/8\pi . [ (E_1)^2 - (E_2)^2 - (E_3)^2 + (B_1)^2 - (B_2)^2 - (B_3)^2 ],$$

$$T_{22} = 1/8\pi . [ -(E_1)^2 + (E_2)^2 - (E_3)^2 - (B_1)^2 + (B_2)^2 - (B_3)^2 ],$$

$$T_{33} = 1/8\pi . [ -(E_1)^2 - (E_2)^2 + (E_3)^2 - (B_1)^2 - (B_2)^2 + (B_3)^2 ],$$

$$T_{44} = -1/8\pi . [ B^2 + E^2 ],$$

$$T_{23} = 1/4\pi . [ E_2 E_3 + B_2 B_3 ], \quad T_{31} = 1/4\pi . [ E_3 E_1 + B_3 B_1 ],$$

$$T_{12} = 1/4\pi . [ E_1 E_2 + B_1 B_2 ],$$

$$T_{14} = 1/4\pi . [ E_2 B_3 - B_2 E_3 ], \quad T_{24} = 1/4\pi . [ E_3 B_1 - B_3 E_1 ],$$

$$T_{34} = 1/4\pi . [ E_1 B_2 - B_1 E_2 ].$$

We can interpret these terms in the following way:

$$\underline{S} \equiv 1/4\pi . (\underline{B} \times \underline{E}) = (-T_{14}, -T_{24}, -T_{34})$$

is *Poynting's* *vector* of classical electromagnetic theory. It represents the flux of radiant energy through a surface element $\underline{ds}$ per unit time: $\underline{S}.\underline{ds}$ .

$$-T_{44} = 1/2 . 1/4\pi . (E^2 + B^2)$$

is the local energy density of the field and the pure space components $T_{\alpha\beta}$ represent the rate of flux of momentum through a surface element $\underline{ds}$:

$$T^{\alpha\beta} (\underline{ds})_\beta.$$

The $T_{\alpha\beta}$ form the components of the *Maxwell* *stress* *tensor*, introduced in Maxwell's original theory. It is only a tensor with respect to spatial rotations in the restframe of the observer.

If we contract $-T_{ab}$ with a velocity vector $u^a = dz^a/ds$, where ds is the element of spacetime interval $ds = \sqrt{(dx^a dx_a)}$, then we obtain a vector $P_b$. Choosing an observer for whom $u^\alpha = 0$ we have

$$P_a \equiv -T_{ab} u^b \qquad\qquad -- (1-45)$$

$$= (\underline{S}, -T_{44}),$$

which combines the above results into one tensor equation. Because of this we assert in general that P represents the net

flow of energy according to an observer with 4-velocity u in Minkowski spacetime.

In order to express Maxwell's equations (1-41) in an alternative form we now put

$$F_{ab} = A_{a,b} - A_{b,a} \; , \qquad\qquad \text{-- } (1\text{-}46)$$

where the 4-vector A is called the _electromagnetic potential_ for the field. Such a potential always exists as a consequence of the homogeneous equation (1-41b), and conversely any tensor given by (1-46) will automatically satisfy (1-41b). Substitution of (1-46) into (1-41a) gives the field equation for the electromagnetic potential

$$\nabla^2 A_a - A^b{}_{,ab} = 4\pi j_a \; , \qquad\qquad \text{-- } (1\text{-}47)$$

where $\qquad\qquad \nabla^2 A_a \equiv A_{a,bc} \, g^{bc} \; .$

According to (1-46) A is not completely determined by the electromagnetic field tensor F. Indeed F remains unchanged under the transformation

$$A_a \rightarrow \bar{A}_a = A_a + B_{,a}$$

where B is an arbitrary scalar function. The above transformation of the electromagnetic potential is called a _gauge transformation_, and may be used to simplify the field equation (1-47). To do this we simply choose the function B such that

$$\nabla^2 B = -A^a{}_{,a} \; ,$$

in which case we have

$$\bar{A}_{a,a} = 0 \; . \qquad\qquad \text{-- } (1\text{-}48)$$

Thus the electromagnetic potential for a given field can always be chosen in such a way that the _Lorentz gauge condition_ (1-48) is satisfied. With the use of this condition (1-47) reduces to the inhomogeneous wave equation

$$\nabla^2 A_a = 4\pi j_a \; . \qquad\qquad \text{-- } (1\text{-}49)$$

According to the Maxwell equations the electromagnetic field

depends on the presence of charges in the field, but even in the absence of charges the field does not in general vanish, because then (1-49) merely reduces to the homogeneous wave equation

$$\nabla^2 A_a = 0 \ . \qquad \text{-- (1-50)}$$

A solution of (1-50) can be obtained by a superposition of plane waves of the form

$$A_a = a_a \cos(1^b x_b + \delta) \ , \qquad \text{-- (1-51)}$$

where $1^b$ is a null vector. Since (1-51) must satisfy the Lorentz condition (1-48) it follows that

$$1^b a_b = 0 \ . \qquad \text{-- (1-52)}$$

In order to examine the properties of these various components we choose the space axes so that the null vector 1 is given by

$$1^b = c(1,0,0,1) \ ,$$

where c is a constant. In this case we have

$$A_b = a_b \cos\{c(x^1-t) + \delta\} \qquad \text{-- (1-53)}$$

and the Lorentz condition (1-52) takes the form

$$a_1 + a_4 = 0 \ . \qquad \text{-- (1-54)}$$

Using (1-53) and (1-54) we then arrive at the following expression for the Maxwell tensor:

$$(F_{ab}) = \begin{bmatrix} 0 & a_2 & a_3 & 0 \\ -a_2 & 0 & 0 & a_2 \\ -a_3 & 0 & 0 & a_3 \\ 0 & -a_2 & -a_3 & 0 \end{bmatrix} c \, \sin\{c(x^1-t) + \delta\} \ . \qquad \text{-- (1-55)}$$

Thus even in the absence of charges the field can contain (amongst other forms) plane electromagnetic waves whose energy is given by substituting (1-55) into (1-44). One immediate point to note concerning the expression (1-55) is that it admits the null eigenvector 1 with zero eigenvalue. For this and other reasons such a field is called a <u>null field</u>. Using the form (1-40) we

find from the above that

$$E_1 B_1 + E_2 B_2 + E_3 B_3 = 0 \; ; \quad E_1 = B_1 = 0 \; ,$$

which shows that the electric and magnetic fields due to a plane electromagnetic wave are perpendicular to each other as well as to the spatial part of l. The solution (1-55) represents a plane wave moving along the positive $x^1$-direction with the speed of light. The direction l is thus called the direction of propagation, and the 2-plane containing $\underline{B}$ and the spatial part of l is called the plane of polarisation of the wave.


## 1.5 The functional calculus

To end this introductory chapter we now give a short summary of functional calculus. Classical Hamiltonian mechanics deals with systems of an arbitrary but finite number of degrees of freedom. For a system with a countable infinity of dynamical variables $q_j$, $p_j$ (j=1,2,...) the state of the system at a given instant of time consists in a specification of all the countable infinity of variables at that instant. However, this is often not the most convenient way of characterising such systems, as, for example, in the case of fields defined on 3-dimensional Euclidean space. Here the state is specified by giving the values of a set of functions at each point of space at a given instant. The general time-dependent variables $q_j(t)$, $p_j(t)$ are replaced by the functions $\psi_A(\underline{x},t)$, $\pi_A(\underline{x},t)$, where instead of the discrete index j we now use the Cartesian position vector $\underline{x}$ to enumerate the infinite degrees of freedom, and the block index A numbers the field components (A=1,2,...,N).

These two ways of counting the degrees of freedom can be put in correspondence, as we shall now see. Given the countable set $q_j$, $p_j$ $(1 \leqslant j < \infty)$ we define $\psi(\underline{x})$, $\pi(\underline{x})$ as the real linear combina-

tions

$$\psi(\underline{x}) = \Sigma_j u_j(\underline{x})q_j \quad ; \quad \pi(\underline{x}) = \Sigma_j v_j(\underline{x})p_j \quad , \quad -- \quad (1-56)$$

in terms of some set of functions $u_j, v_j$ yet to be determined.

Assuming these functions form a canonical set of variables we can

legitimately calculate the fundamental PB conditions (1-29) bet-

ween them, which by substitution from (1-56) yields the result

$$\{\psi(\underline{x}), \psi(\underline{y})\} = \{\pi(\underline{x}), \pi(\underline{y})\} = 0 \quad ;$$

$$\{\psi(\underline{x}), \pi(\underline{y})\} = \Sigma_j u_j(\underline{x})v_j(\underline{y}) \quad .$$

We now demand that the linear combinations (1-56) be nonsingular

in the sense that we can solve them for $q_j$, $p_j$ in the form

$$q_j = \int_S U_j(\underline{x})\psi(\underline{x})d^3x \quad ; \quad p_j = \int_S V_j(\underline{x})\pi(\underline{x})d^3x \quad , \quad -- \quad (1-57)$$

where $U_j, V_j$ is some appropriate collection of functions. Here S

is some 3-dimensional spacelike hypersurface indexed by the

vector $\underline{x}$ relative to some fixed origin and on which $\psi$ and $\pi$ are

defined. Now from (1-56) and (1-57) we must have the following

relations between the $u_j$, $v_j$, $U_j$, $V_j$ :

$$\psi(\underline{x}) = \int_S \Sigma_j u_j(\underline{x})U_j(\underline{y})\psi(\underline{y})d^3y \quad ; \quad \pi(\underline{x}) = \int_S \Sigma_j v_j(\underline{x})V_j(\underline{y})\pi(\underline{y})d^3y \quad ;$$

$$q_j = \Sigma_i \int_S U_j(\underline{x})u_i(\underline{x})q_i d^3x \quad ; \quad p_j = \Sigma_i \int_S V_j(\underline{x})v_i(\underline{x})p_i d^3x \quad .$$

All these relations are satisfied if we choose $u_j$, $v_j$ to be

complete reciprocal bases for functions of $\underline{x}$ in the sense

$$U_j(\underline{x}) = v_j(\underline{x}) \quad ; \quad V_j(\underline{x}) = u_j(\underline{x}) \quad ;$$

$$\Sigma_j u_j(\underline{x})v_j(\underline{y}) = \delta^3(\underline{x}-\underline{y}) \quad ; \quad \int_S u_j(\underline{x})v_i(\underline{x})d^3x = \delta_{ij} \quad .$$

(Here $\delta_{ij}$ is the Kroenecker delta and $\delta^3$ is the 3-dimensional

Dirac function.) This finally results in the fundamental Poisson

bracket relations

$$\{\psi(\underline{x}), \psi(\underline{y})\} = \{\pi(\underline{x}), \pi(\underline{y})\} = 0; \quad \{\psi(\underline{x}), \pi(\underline{y})\} = \delta^3(\underline{x}-\underline{y}). \quad -- \quad (1-58)$$

Thus we see a situation arising in which a system with an

infinite number of degrees of freedom is represented by a

Hamiltonian _field_ system. In the transition from the discrete

system to the continuous we find the discrete index j replaced by

a continuous index $\underline{x}$. The summations become integrals and the Kroenecker delta becomes the Dirac delta-function in the appropriate number of dimensions. This situation brings us to the notion of a _functional_. Much of the initial work on functionals was done by Volterra (1931), and while his work was far more general than ours shall be, the generalisation to his work from ours is straightforward.

Functionals are, by definition, functions of functions. That is, a functional establishes a mapping from the space of functions into some other set - maybe the real numbers or maybe back into the original function space. The following example illustrates this.

Example (1.3) -

i. $F[\psi;x] = \psi(x)$ : In this trivial example F takes the function $\psi$ and produces the same function evaluated at the point x. The notation $F[\psi;x]$ means that F depends on the global shape of the function $\psi$ as well as on the point x.

ii. $F[\psi] = \int_x \int_y \psi(x) f(x,y) \psi(y) . dydx$ : Here F takes the function $\psi$ defined on some domain in $\underline{R}^2$ coordinatised by x and y, and produces a number. The notation $F[\psi]$ means that this number depends only on the overall shape of the function $\psi$, and not on the specific point x. Already in these two examples we see the close correspondence between functional calculus and normal differential calculus. Example (ii) is a quadratic functional of $\psi$, and we see how integration over a repeated continuous index (here x,y) eliminates that index from the value of the functional in exactly the same way as summation eliminates discrete dummy indices in the differential calculus. (ii) may be thought of as providing a generalised norm of the function $\psi$, with $f(x,y)$ forming a metric in function space.

iii. $F[\psi_A(\underline{x},t),\partial\psi_A/\partial t] = \int_S \phi(\psi_A,\psi_{A,b}).d\tau$ : This final example

illustrates the sort of functional we shall mainly be concerned

with. Here $\phi(\psi_A,\psi_{A,b})$ is some scalar defined on Minkowski space

in terms of the variables $\psi_A$, $\psi_{A,b}$ and S is some spacelike

3-dimensional hypersurface with 3-volume element $d\tau$. The

integration over S will eliminate the space coordinates $x^\alpha$ from

the result, so that F will be a functional indexed by t and

dependent upon the variables $\psi_A$ and their time derivatives $\psi_{A,4}$.

Here it must be made clear that although the time derivative of

$\psi_A$ is treated as being independent of $\psi_A$, being a sort of gener-

alised velocity, the spatial derivatives $\psi_{A,\beta}$ are not new indep-

endent quantities but are, in a generalised sense, functions of

the basic variables $\psi_A(\underline{x},t)$. This somewhat arbitrary separation

of the time derivatives from the space derivatives stems from the

use of the hypersurface S to eliminate all local dependence of F

on the spatial variables. Clearly if the numerical values of $\psi_A$

are prescribed for all $\underline{x}$ at a given time t, then we also know the

values of $\psi_{A,\beta}$ at that time. It is important to realise this in

order to calculate legitimately the "partial derivatives" of F

with respect to its arguments.

———————

In keeping with the close analogy between the functional

calculus and the ordinary differential calculus we now seek a

functional equivalent of the partial derivative. Rather than

finding the rate of change of a function with respect to the j-th

coordinate, we now wish to calculate the rate of change of a

functional $F[\psi_A]$ with respect to a small change in the argument

$\psi_A$ at the point $\underline{x}$ only. Accordingly we define the functional

derivative of a functional $F[\psi_A]$ with respect to the variable

$\psi_A(\underline{x})$ by

$$\frac{\delta F[\psi]}{\delta \psi_A(\underline{x})} \equiv \lim_{\delta\omega_A \to 0} \frac{1}{\delta\omega_A} \{F[\ldots,\psi_A + \delta_{\underline{x}}\psi_A,\ldots] - F[\ldots,\psi_A,\ldots]\}. \quad --(1-59)$$

Here $\psi_A(\underline{y})$ is defined all over a spacelike 3-surface S and $\delta_{\underline{x}}\psi_A$ is a variation of the A-th variable alone in the vicinity of the point $\underline{x}\varepsilon S$. It is assumed that this variation is zero outside some $\varepsilon$-neighbourhood of $\underline{x}$. $\delta\omega_A$ is then the 4-dimensional volume element contained between the 3-surfaces $\psi_A(\underline{y})$ and $\psi_A(\underline{y})+\delta_{\underline{x}}\psi_A(\underline{y})$. (Note that these four dimensions are _not_ all spacetime dimensions, but rather three spatial dimensions parametrised by $\underline{y}$ and one function dimension parametrised by $\psi_A$ - see fig. 1.4).

fig. 1.4
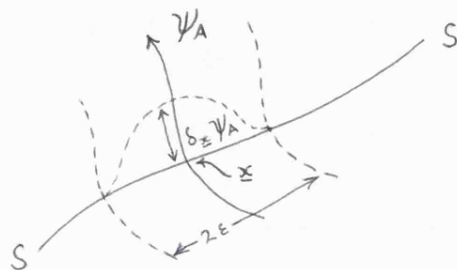


In the limit $\delta\omega_A \to 0$ the surface $\psi_A+\delta_{\underline{x}}\psi_A$ tends towards $\psi_A$ in such a way that $\varepsilon$ tends to zero, $\delta\omega_A$ always containing the point $\underline{x}$. It is convenient to describe this process using the 3-dimensional $\delta$-function:

$$\delta_{\underline{x}}\psi_A(\underline{y}) = \delta\omega_A \delta^3(\underline{y}-\underline{x}), \qquad\qquad -- (1-60)$$

which is consistent with the fact that the integral of $\delta_{\underline{x}}\psi_A(\underline{y})$ over the whole of S is simply $\delta\omega_A$.

Now let us calculate the functional derivative of the variable $\psi_A$ itself. According to (1-59) and (1-60) we have

$$\frac{\delta\psi_A(\underline{y})}{\delta\psi_B(\underline{x})} = \delta^B_A \delta^3(\underline{y}-\underline{x}), \qquad\qquad -- (1-61)$$

which is what we would expect, considering the analogies brought out earlier between discrete and continuous indices. For the purposes of calculating actual expressions for functional derivatives such as that in (1-61) we now look at the integral

$$\frac{\delta F[\psi]}{\delta\psi_A(\underline{y})}\delta_{\underline{x}}\psi_A(\underline{y})d^3y = \lim_{\delta\omega_A\to 0}\int\{F[\ldots,\psi_A+\delta_{\underline{x}}\psi_A,\ldots]-F[\ldots,\psi_A,\ldots]\}\delta^3(\underline{y}-\underline{x})d^3y$$

$$= \lim_{\delta\omega_A \to 0} \{F[\ldots,\psi_A + \delta_{\underline{x}}\psi_A,\ldots] - F[\ldots,\psi_A,\ldots]\}$$

$$= \delta_{\underline{x}} F \ .$$

First suppose that there is no implied summation in this calculation, then $\delta_{\underline{x}}F$ is the change in F due to the variation of the A-th variable alone. But this means that summation over A gives the entire change in F for arbitrary variations of $\psi_A$ <u>at the point</u> $\underline{x}$. Clearly we shall in general be concerned not just with variations at a single point, so to obtain the overall change in F we must integrate over all the point variations:

$$\delta F = \int \delta_{\underline{x}} F d^3 x$$

$$= \int\int \frac{\delta F[\psi]}{\delta\psi_A(\underline{y})} \delta_{\underline{x}}\psi_A(\underline{y}) d^3 y d^3 x$$

$$= \int \frac{\delta F[\psi]}{\delta\psi_A(\underline{y})} \{\int \delta_{\underline{x}}\psi_A(\underline{y}) d^3 x\} d^3 y$$

$$= \int \frac{\delta F[\psi]}{\delta\psi_A(\underline{y})} \delta\psi_A(\underline{y}) d^3 y \ . \qquad\qquad -- \ (1-62)$$

Now let us return to the example (iii) above. We can imagine altering the $\psi_A(\underline{x},t)$ and the "velocities" $\psi_{A,4} = \dot{\psi}_A(\underline{x},t)$ by small amounts $\delta\psi_A$ and $\delta\dot{\psi}_A$ respectively, these increments being independent of one another at any one time, and computing the change in F without altering the functional form of F. This change is given by

$$\delta F = \int\int \left\{\frac{\delta F}{\delta\psi_A(\underline{x},t)} \cdot \delta\psi_A(\underline{x},t) + \frac{\delta F}{\delta\dot{\psi}_A(\underline{x},t)} \cdot \delta\dot{\psi}_A(\underline{x},t)\right\} d\tau \ . \quad -- \ (1-63)$$

Since, by definition, F depends on the spatial derivatives of $\psi$, we will at first encounter in $\delta F$ terms involving the gradient $(\partial/\partial\underline{x})\delta\psi_A$, but by means of integration by parts all such terms can eventually be put into the form (1-63) (see chapter 2). Thus (1-62) gives us a way of calculating the $\delta F/\delta\psi_A$ in terms of partial derivatives, and in fact some authors <u>define</u> the partial functional derivatives of F with respect to $\psi_A$ and $\dot{\psi}_A$ as being

the coefficients of the independent increments $\delta\psi_A$, $\delta\dot{\psi}_A$ in

(1-63). Clearly the exact nature of the change $\delta F$ will depend on

the types of variation $\delta\psi$ occurring in (1-63).

Before proceeding we note a convention which we shall adopt

for ease of notation. Since the spatial coordinates $\underline{x}$ and $\underline{y}$

serve simply as dummy indices, with volume elements $d^3x$, $d^3y$

respectively, we shall omit mention of these variables whenever

the meaning is clear. Instead we adopt the convention that an

undashed quantity is a function of x ($\psi_A = \psi_A(\underline{x})$), while a dashed

quantity is a function of the independent variable x'

($\psi'_A = \psi_A(\underline{x}')$); the volume elements $d\tau$, $d\tau'$ will correspond to the

elements $d^3x$ and $d^3x'$ respectively. Note that, as we saw in the

derivation of (1-62), except in certain pathological cases we can

reverse the order of these integrations - just as we may evaluate

the summation of repeated discrete indices in any order.

Finally in this section we look at the generalisation of the

Legendre transformation to the case of functionals. Let $F[u_A]$ be

a functional of N functions $u_A(\underline{x})$ defined on the indexing

coordinates $x^\alpha$. We introduce a new set of variables $v^A(\underline{x})$ on the

same index space by means of the following transformation:

$$v^A = \delta F/\delta u_A \ . \qquad\qquad \text{-- (1-64)}$$

We assume the "Hessian" $\left|\dfrac{\delta^2 F}{\delta u_B \delta u_A}\right|$ to be nonzero, thus ensuring the

independence of the N variables $v^A$. In this case the equations

(1-64) are soluble for the $u_A$ as functions of the $v^A$ (Volterra,

1931). A new functional G is now defined by

$$G = \int u_A v^A d\tau - F \ . \qquad\qquad \text{-- (1-65)}$$

We can express the $u_A$ in terms of the $v^A$ and substitute into

(1-65), and the functional G can then be expressed in terms of

the new variables $v^A$ alone:

$$G = G(v^A) \ . \qquad\qquad -- \ (1-66)$$

We now consider the infinitesimal variation of G brought about by arbitrary variations of the $v^A$. The combination of (1-65) and (1-64) gives

$$\delta G = \int \frac{\delta G}{\delta v^{\cdot B}} \ \delta v^{\cdot B} \ d\tau'$$

$$= \int \left\{ \int \left[ \int u_A \frac{\delta v^A}{\delta v^{\cdot B}} \ d\tau + \int v^A \frac{\delta u_A}{\delta v^{\cdot B}} \ d\tau \right] \delta v^{\cdot B} d\tau' - \delta F \right.$$

$$= \int \left\{ u_A \delta v^A + \delta u_A \left[ v^A - \frac{\delta F}{\delta u_A} \right] \right\} d\tau \ . \qquad -- \ (1-67)$$

Since G is a functional of the $v^A$ alone, we should express the $u_A$ as functions of the $v^A$ in (1-67). However, examination of (1-67) shows that this elimination is rendered unnecessary by the fact that the coefficient of $\delta u_A$ is automatically zero from the definition (1-64) of $v^A$. From (1-67) we then see immediately that

$$u_A = \delta G / \delta v^A \ . \qquad\qquad -- \ (1-68)$$

One further generalisation of the above occurs when F is a functional not only of the $u_A$, but also of a set of N variables $w_K$ : $F = F[w_1, \ldots, w_M; u_1, \ldots, u_N]$. The $w_K$ (K=1,..,M) are independent of the $u_A$. They occur in F as parameters but do not participate in the transformation. We call $u_A$ the <u>active</u>, and $w_K$ the <u>passive</u>, variables of the transformation and find

$$\delta F / \delta w_K = -\delta G / \delta w_K \ . \qquad\qquad -- \ (1-69)$$

# CHAPTER 2

## FIELD THEORY

In this chapter we look at the description of fields as a variational system. The methods of analytical mechanics presented in section 1.2 are easily generalised to a form which is capable of describing systems of fields, although we shall see that the step from Lagrangian to Hamiltonian field theory presents certain problems. As a specific example of the methods of field theory we shall make particular reference to the canonical formulation of electromagnetic theory against the background of general relativity. This approach is particularly convenient for the study of symmetries and conservation laws.

## 2.1 Lagrangian field theory

Field theory is based upon a number of field variables $\psi_A(x)$ $(A=1,2,..,N)$ defined over spacetime. Here the block indices $A, B, \ldots$ are very general, encompassing one or more small Latin indices, as, for example, when the field variables are the metric coefficients $g_{ab}$. Other possible field variables are the N generalised coordinates of Lagrangian mechanics or the components of the potential field $A_a$. We will at all times assume that the $\psi_A(x)$ are differentiable and that the Lagrangian is a differentiable function L of the $\psi_A$ and their derivatives $\psi_{A,a}$ (we use bold lettering for the Lagrangian for reasons which will become clear later). L is a scalar quantity and is both local (depending only on the $\psi_A$ and their first derivatives) and conservative (possessing no explicit dependence on the $x^a$). The field equations are assumed to be derivable from a variational principle

$$\delta I = 0 \quad ; \quad I = \int_V L(\psi_A, \psi_{A,a}) d^4x \ , \qquad \text{-- (2-1)}$$

where V is an arbitrary fixed 4-dimensional region of spacetime. The variation $\delta I$ in the action integral is caused by a completely arbitrary variation of the $\psi_A$ - a variation which is assumed to vanish on the boundary of V. We then find

$$\delta I = \int_V \delta L d^4x$$

$$= \int_V \{L[\psi_A + \delta\psi_A, (\psi_A + \delta\psi_A)_{,a}] - L(\psi_A, \psi_{A,a})\} d^4x$$

$$= \int_V \left\{ \frac{\partial L}{\partial \psi_A} \delta\psi_A + \frac{\partial L}{\partial \psi_{A,a}} \delta(\psi_{A,a}) \right\} d^4x$$

$$= \int_V \left\{ \frac{\partial L}{\partial \psi_A} \delta\psi_A + \left[ \frac{\partial L}{\partial \psi_{A,a}} \delta\psi_A \right]_{,a} - \left[ \frac{\partial L}{\partial \psi_{A,a}} \right]_{,a} \delta\psi_A \right\} d^4x$$

$$= \int_V \left\{ \frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,a}} \right]_{,a} \right\} \delta\psi_A d^4x + \int_V \left\{ \frac{\partial L}{\partial \psi_{A,a}} \delta\psi_A \right\}_{,a} d^4x$$

$$= 0 \ ,$$

where we have used the fact that the variation $\delta$ commutes with integration and partial differentiation with respect to the coordinates $x^a$. The right-hand term in the above vanishes by an application of Gauss' theorem and by remembering that all variations vanish on the boundary of V. Hence we arrive at the following Euler-Lagrange field equations:

$$L^A \equiv \frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,a}} \right]_{,a} = 0 \ . \qquad \text{-- (2-2)}$$

If field sources are present then the field equations are modified to

$$L^A = P^A \ , \qquad \text{-- (2-3)}$$

where the $P^A$ are again assumed differentiable and represent the field sources. It is not assumed that L is an invariant density, but it is defined invariantly in each coordinate system.

In an important paper Bergmann (1949) considered a very general field theory derived from a Lagrangian containing only the $\psi_A$ and their first derivatives, although the theory can easily be extended to include second or higher derivatives. He

showed how the assumption that L is an invariant density leads to a set of identities (the generalised Bianchi identities) which in turn yield differential conservation laws for the system. It would not be appropriate to go deeply into Bergmann's work here, since we are concerned rather with Hamiltonian theory, but it is worth noting that he turned Lagrangian field theory into a very general and powerful tool. In his paper Bergmann made use of the method of infinitesimal coordinate transformations of the form

$$x^a \rightarrow \bar{x}^a = x^a + \xi^a(x) ,$$

or to first order,

$$x^a = \bar{x}^a - \xi^a(\bar{x}) .$$

-- (2-4)

Now when this transformation has been carried out it defines a new function $\bar{\psi}$ according to the relation

$$\bar{\psi}(\bar{x}) \equiv \psi(x) .$$

This function can then be applied to any point, including that whose _new_ coordinates are $x^a$. This produces the result $\bar{\psi}(x)$ and enables us to introduce the _substantial_ variation $\delta$, defined by

$$\delta\psi_A \equiv \bar{\psi}_A(x) - \psi_A(x) .$$

$\delta$ measures the total change in functional form of $\psi_A(x)$; it is far more restricted than the variation used to derive (2-2), possessing only the four degrees of freedom represented by the $\xi^a$. We can also express the substantial variation in the form

$$\delta\psi_A \equiv [\bar{\psi}_A(\bar{x}) - \psi_A(x)] - [\bar{\psi}_A(\bar{x}) - \bar{\psi}_A(x)] .$$

The first expression in brackets is the _local_ variation $\bar{\delta}\psi_A$, so called because it refers to the total change in $\psi_A$ at the single geometrical point x under the coordinate transformation (2-4), and can be evaluated by the known transformation law for the $\psi_A$. In all practical cases the local variation may be expressed to first order as

$$\bar{\delta}\psi_A = F_{A\,r}{}^{B\,s} \xi^r{}_{,s} \psi_B ,$$

where the $F_{A\,r}{}^{B\,s}$ are constants. This form is not a necessary

consequence of the above, and so is not the most general law
which may be encountered in geometrical objects; it does, how-
ever, describe all types of tensors and tensor densities and also
spinors.   On the other hand the above expression reflects only
the changes in $\psi_A$ brought about by the transformation (2-4).   If
in addition we vary the $\psi_A$ independently of (2-4) then we must
include an extra term:

$$\delta\psi_A = F_{Ar}{}^{Bs}\xi^r{}_{,s}\psi_B + \delta_o\psi_A .$$

The term $\delta_o\psi_A$ does not arise from (2-4), but is the <u>independent</u>
variation, describing only changes in the functional form of the
$\psi_A$ which are independent of the underlying index space - usually
a linear mixing of the $\psi_A$ among themselves.   We shall give an
example of this type of variation directly.   Collecting together
these results we have the following first-order expressions for
the variations:

$$
\left.
\begin{aligned}
\delta\psi_A &= \bar{\psi}_A(x) - \psi_A(x) \\
&= \delta\psi_A - \psi_{A,r}\xi^r , \\
\text{where} \quad \bar{\delta}\psi_A &= \bar{\psi}_A(\bar{x}) - \psi_A(x) \\
&= F_{Ar}{}^{Bs}\xi^r{}_{,s}\psi_B + \delta_o\psi_A .
\end{aligned}
\right\} \quad\text{-- (2-5)}
$$

A simple example of a pure independent variation of the
field variables is the <u>gauge transformation of the first kind</u>.
These transformations form a group, and are defined by

$$\delta\psi_A = -ie\varepsilon\eta^{(A)}\psi_A \quad ; \quad \delta x^a = 0 , \qquad\text{-- (2-6)}$$

where e is a constant which we shall later identify with charge,
and $\varepsilon$ is an infinitesimal constant characterising the transform-
ation.   $\eta^{(A)}$ is a sign function which takes the value 0 when $\psi_A$
is real; if $\psi_A$ is one of a complex conjugate pair then $\eta^{(A)}$ takes
the value +1 or -1, depending on which of the pair is chosen as
the standard (see example 2.3).   The brackets on the index of
$\eta^{(A)}$ indicate that no summation is performed over the repeated

index A: $\eta^{(A)}$ is simply a constant factor determining the sign of complex fields and eliminating real fields. For a given value of $\varepsilon$ we see that

$$\delta\psi_A = \delta\psi_A = \delta_0\psi_A$$
$$= -ie\varepsilon\eta^{(A)}\psi_A ,$$

expressing the fact that (2-6) is a transformation which merely affects the functional form of $\psi$. On account of the infinitesimal nature of this transformation we may write it in the alternative form

$$\bar{\Psi}_A = \psi_A + \delta\psi_A = [1 - ie\varepsilon\eta^{(A)}]\psi_A$$
$$= \exp\{-ie\varepsilon\eta^{(A)}\}y_A . \qquad\qquad -- (2-7)$$

This form reveals that the gauge transformation of the first kind is simply a rotation in the complex $\psi_A$-plane through an angle $-e\varepsilon\eta^{(A)}$. We shall see later that these transformation are closely linked with charge conservation in the variational formalism.

As an application of Lagrangian field theory we now look at how the free-space electromagnetic field can be described within the Lagrangian framework. We have already seen the basic equations for the electromagnetic field in Minkowski space, and we now seek a Lagrangian from which we can derive these equations via a variational principle in the more general spacetime of general relativity. Maxwell's equations (1-41) may easily be expressed (see Adler, Bazin & Schiffer, 1975) in general relativistic form:

$$F^{ab}{}_{;b} = 4\pi j^a ;$$
$$F_{[ab,c]} = 0 , \qquad\qquad -- (2-8)$$

and we shall see that these are derivable from the Lagrangian

$$L \equiv -\kappa\sqrt{(-g)} \{1/8\pi.F_{ab}F^{ab} + 2A_a j^a\} ;$$
$$F_{ab} \equiv A_{a;b} - A_{b;a} = A_{a,b} - A_{b,a} . \qquad\qquad -- (2-9)$$

A full description of the dynamics of a charge distribution would involve inclusion of an additional Lagrangian describing the gravitational field, but since this would take us too far from our work we shall merely investigate (2-9). The homogeneous equation in (2-8) is immediately valid as a result of the definition of $F_{ab}$, so it remains only to investigate the effect of varying the four field variables $A_a$ and their derivatives $A_{a,b}$:

$$\frac{1}{\kappa\sqrt{(-g)}}\cdot\frac{\partial L}{\partial A_{a,b}} = \frac{-2}{8\pi}\left[\frac{\partial}{\partial A_{a,b}}\{A_{c,d}A^{c,d} - A_{c,d}A^{d,c}\}\right]$$

$$= -1/2\pi.(A^{a,b} - A^{b,a}) = -1/2\pi.F^{ab} \quad ,$$

so $\qquad L^A = L^a = \dfrac{\partial L}{\partial A_a} - \left[\dfrac{\partial L}{\partial A_{a,b}}\right]_{,b} = 0$

becomes $\quad \kappa\ \{-2j^a\sqrt{(-g)} + 1/2\pi.(\sqrt{(-g)}F^{ab})_{,b}\} = 0$

$\Rightarrow \qquad (\sqrt{(-g)}F^{ab})_{,b} = 4\pi\sqrt{(-g)}j^a \quad , \qquad\qquad$ -- (2-10)

which is precisely equivalent (see Adler,Bazin & Schiffer,1975) to the required equation (2-8). If we confine our attention to the source-free field then we are dealing with the new Lagrangian

$$L = -\kappa/8\pi.\sqrt{(-g)}F_{ab}F^{ab} \quad , \qquad\qquad \text{-- (2-11)}$$

which clearly yields the source-free form of (2-10). Note that (2-11) also involves a dependence on the determinant g, so that in a complete description including the gravitational interaction we would be forced to take account of the $g_{ab}$ as additional field variables. In this context it is worth calculating the effect on the action integral of varying the $g_{ab}$, using the standard result for the variation of the factor $\sqrt{(-g)}$ (see Adler,Bazin & Schiffer,1975):

$$\delta\sqrt{(-g)} = -1/2.\sqrt{(-g)}.g_{rs}.\delta g^{rs}$$

$\Rightarrow \qquad \delta\int_V L d^4x = \kappa/4\pi.\int_V \{F_r{}^a F_{as} + 1/4.g_{rs}F^{ab}F_{ab}\}\sqrt{(-g)}\delta g^{rs}d^4x$

$$= \kappa.\int_V\sqrt{(-g)}T_{rs}\delta g^{rs}d^4x \quad . \qquad\qquad \text{-- (2-12)}$$

We see that $T_{rs}$ is simply the energy-momentum tensor (1-44) evaluated for a source-free electromagnetic field. Because of this the Bianchi identities derived in Bergmann's work lead to the well-known conservation law

$$T_a{}^b{}_{;b} = 0 \quad .$$

## 2.2 Hamiltonian field theory

We have now seen that Lagrangian field theory represents a very neat and powerful generalisation of the ideas of analytical mechanics to continuous systems which has the advantage as far as relativity is concerned of treating time on an equal footing with the three space coordinates. When we come to the transition to a Hamiltonian theory, however, we find that the theory becomes far less aesthetically pleasing. This was, indeed, the original motivation for this thesis, since while many textbooks proudly display their respective treatments of Lagrangian field theory, most fall strangely silent on the subject of Hamiltonians. This silence has a disturbing effect on the student - he comes to feel that Hamiltonian field theory must be an incredibly complex theory to whose level he can never hope to aspire. In fact the reason is simple: there exists, as yet, no convincing Hamiltonian theory which wholeheartedly accepts the basic precepts of relativity as set out in chapter 1. In this section we shall present the version most frequently used. It originates in the work of Heisenberg and Pauli (1929;30), who were intent upon producing relativistic commutation relations for the purpose of field quantisation, following the work of Dirac (1927). Their work was then developed by Weiss (1936;38a;38b), Fuchs (1939) and Chang (1945;46) into a form which still constitutes a central theme of quantum field theory today.

Our exposition of the theory is based largely on the work of Weiss (1936;38a), and considers the case of N field variables $\psi_A$ dependent upon the four indexing variables $x^a$, which we shall assume form the coordinates for a <u>Minkowski</u> manifold. The theory presented here is more restricted than that of Weiss in that he considers a more general index space comprising $v$ variables. He does, however, require that space to form a flat Lorentz manifold, so the generalisation to Weiss' theory is straightforward from that presented here. We start from the action integral

$$I = \int_V L(\psi_A, \psi_{A,b}) d^4 x \quad , \qquad \qquad -- \ (2\text{-}13)$$

where $V$ is now the more specific 4-dimensional region contained between two spacelike hyperplanes $S_1$, $S_2$ and a cylindrical timelike surface intersecting both $S_1$ and $S_2$. We assume $S_2$ to lie at some finite time <u>later</u> than $S_1$. The Lagrangian L is now required to be a scalar density, and is again both local and conservative. We now cause a substantial variation of the action integral (2-13) by first performing an infinitesimal transformation of the coordinates:

$$x^a \rightarrow \overline{x}^a = x^a + \xi^a(x) \qquad \qquad -- \ (2\text{-}14a)$$

and simultaneously performing an independent variation $\delta_0 \psi_A$ of the field variables (see (2-5)) to obtain

$$\psi(x) \rightarrow \overline{\psi}(x) = \psi(x) + \delta\psi(x) \ . \qquad \qquad -- \ (2\text{-}14b)$$

Note that we are no longer requiring these variations to vanish on the boundary of V. This is a point gone into in some detail by Weiss. We include the timelike portion of the boundary V because we wish to look at the time-evolution of the field quantities $\psi_A$. However, we wish to be able to ignore terms on the timelike surface and reduce all boundary terms to dependence on $S_1$ and $S_2$. Now study of the Cauchy problem in relativity shows (see Pearson & Carrier, 1976) that correctly set data for

our dynamical problem has the following form:

  i. Initial data, $\psi_A$ <u>and</u> $\psi_{A,4}$, on the initial spacelike
    hypersurface $S_1$.

  ii. Boundary data, $\psi_A$ <u>or</u> $\psi_{A,4}$, on the timelike region.

  iii. No data on the remaining spacelike region $S_2$.

Such data are called "mixed data". In order to set our data
correctly (and to enable a later smooth transition to the
Hamiltonian formalism) we therefore now choose a special coord-
inate system for the index space such that the $x^4$ coordinate runs
along the timelike tube and the $x^\alpha$ coordinatise each of the
surfaces $S_1$, $S_2$. We arrange this system such that the cross-
section of the tube on both $S_1$ and $S_2$ has the same $x^\alpha$ coordin-
ates, the two cross-sections differing only in the coordinate $x^4$
(see fig.(2.1)). Thus we choose a slicing of spacetime such that
t=const. surfaces are Cauchy surfaces, giving a 3+1 decomposition
of the field.
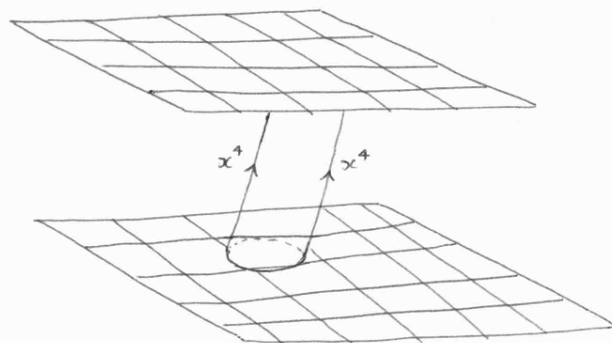


fig.(2.1)

In this coordinate system we are free to prescribe all the var-
iations (2-14) arbitrarily on $S_1$, whilst setting them to zero on
the timelike region. It then follows that the time derivatives
of the variations are also zero on the timelike region and we can
therefore consider all boundary variations as arising solely from
variations on $S_1$ and $S_2$.

  Let us now calculate the specific form of the variation $\delta I$

brought about in the action integral (2-13) as a result of the coordinate variations (2-14). Denoting varied quantities with a bar, the general form of the variation is

$$\delta I = \int_{\bar{V}} L[\bar{\psi}(\bar{x})]d^4\bar{x} - \int_{V} L[\psi(x)]d^4x$$

$$= \int_{\bar{V}} \{L[\bar{\psi}(\bar{x})] - L[\psi(\bar{x})]\}d^4\bar{x}$$

$$+ \int_{\bar{V}} L[\psi(\bar{x})]d^4\bar{x} - \int_{V} L[\psi(x)]d^4x , \quad -- \quad (2-15)$$

remembering that the coordinates of the boundary of V have changed as a result of (2-14a). Noting that the Jacobian $|\partial\bar{x}/\partial x|$ is given by $1+\xi^a_{,a}$ we have

$$\int_{\bar{V}} L[\psi(\bar{x})]d^4\bar{x} = \int_{V} L[\psi(x+\xi)](1+\xi^a_{,a})d^4x$$

$$= \int_{V} \{L[\psi(x)] + (L\xi^a)_{,a}\}d^4x \quad -- \quad (2-16)$$

with neglect of higher order terms. In addition to first order we have

$$\int_{\bar{V}} \{L[\bar{\psi}(\bar{x})]-L[\psi(\bar{x})]\}d^4\bar{x} = \int_{V} \{L[\bar{\psi}(x)]-L[\psi(x)]\}d^4x$$

$$= \int_{V} \left[\frac{\partial L}{\partial\psi_A}\delta\psi_A + \frac{\partial L}{\partial\psi_{A,b}}(\delta\psi_A)_{,b}\right]d^4x \quad -- \quad (2-17)$$

where we have again neglected second order terms and above, and remembering that, by definition, $\delta\psi_{A,b} = (\delta\psi_A)_{,b}$. Hence, collecting together (2-15), (2-16) and (2-17), and performing an integration by parts, we obtain

$$\delta I = \int_{V} \left\{ \left[ \left[\frac{\partial L}{\partial\psi_A}\right] - \left[\frac{\partial L}{\partial\psi_{A,b}}\right]_{,b} \right]\delta\psi_A \right.$$

$$\left. + \left[\frac{\partial L}{\partial\psi_{A,b}}\delta\psi_A + L\xi^b\right]_{,b} \right\} d^4x . \quad -- \quad (2-18)$$

The second term of the integrand in (2-18) is the one we eliminated in the Lagrangian approach of the previous section by transforming it to a surface integral. Repeating this procedure now we find the following form for the second term:

$$\int_{V} \left\{\frac{\partial L}{\partial\psi_{A,b}}\delta\psi_A + L\xi^b\right\}_{,b} d^4x = F(S_2) - F(S_1) ,$$

where

$$F(S) \equiv \int_{S} (\pi^{Ab}\delta\psi_A + L\xi^b) d^3S_b \quad -- \quad (2-19)$$

and

$$\pi^{Ab} \equiv \partial L/\partial\psi_{A,b} . \quad -- \quad (2-20)$$

Hence by taking account of (2-19) and (2-5) we may bring (2-18) into the final form

$$\delta I = \int_V \left\{ \frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,b}} \right]_{,b} \right\} \delta \psi_A \; d^4 x + F(S_2) - F(S_1)$$

$$\text{where } F(S) = \int_S \{ \pi^{Ab} \delta \psi_A - (\pi^{Ab} \psi_{A,c} - \delta_c^b L) \xi^c \} d^3 S_b . \qquad \text{-- (2-21)}$$

Having obtained this form for $\delta I$ we are now in a position to apply the Principle of Stationary Action, which states, exactly as before, that the only fields occurring in nature are those for which the variation $\delta I$ is independent of variations $\delta \psi$ of the field _within_ V. Now from (2-21) we see that $\delta I$ arises from two distinct sources: (a) from variations at the boundaries $S_1$, $S_2$, and (b) from variations within the interior of V. Thus the principle of stationary action requires that the volume integral in (2-21) vanish:

$$\delta I = \int_V \left\{ \frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,b}} \right]_{,b} \right\} \delta \psi_A \; d^4 x = 0 \quad .$$

The variations $\delta \psi_A$ are arbitrary functions independent of each other, so we are left with exactly the same Euler-Lagrange equations as before

$$\frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,b}} \right]_{,b} = 0 \qquad\qquad \text{-- (2-22)}$$

and the following expression for $\delta I$:

$$\delta I = F(S_2) - F(S_1) . \qquad\qquad \text{-- (2-23)}$$

This form of the action is particularly useful for the study of integral conservation laws, although we shall not be concerned with these here. These laws are not new physical laws, but follow as a consequence either of the field equations (2-22) or directly from the action functional (2-13). To see this we look at the expression (2-23) for $\delta I$, which holds when $\psi$ is a solution of (2-22). (2-23) represents the change in the action functional as a result of the variations $\xi^a$ and $\delta_o \psi$ of the index variables

and the fields. It may happen that I is invariant with respect to a certain group of infinitesimal transformations of the $x^a$ or $\psi_A$. In this case $\delta I = 0$ and

$$F(S_1) = F(S_2) \qquad \text{-- (2-24)}$$

for this particular type of transformation. Take, for example, the gauge transformations of the first kind (2-6). In this case the functional F(S) becomes

$$F(S) = -ie\epsilon\eta^{(A)}\int_S \frac{\partial L}{\partial \psi_{A,b}} \psi_A \, dS_b \quad .$$

Introducing the current-density vector

$$j^b = -ie\eta^{(A)}\frac{\partial L}{\partial \psi_{A,b}}\psi_A \qquad \text{-- (2-25)}$$

and the scalar $\qquad Q(S) = \int_S j^b dS_b \qquad$ -- (2-26)

we can express F in the form

$$F(S) = \epsilon Q(S) \quad .$$

Thus if the action is invariant with respect to the gauge transformations then we have the conservation law

$$Q(S_1) = Q(S_2) \quad , \qquad \text{-- (2-27)}$$

which we shall later see is related to charge conservation.

We now return to the Lagrangian field equations (2-22) to reformulate them in a way which is akin to the classical Hamiltonian formulation of particle mechanics. Making use of the definition (2-20) the field equations become

$$\partial L/\partial\psi_A - (\pi^{Ab})_{,b} = 0 \quad . \qquad \text{-- (2-28)}$$

Continuing to use the convenient coordinate system described above, we now define the momentum $\pi^A$ conjugate to $\psi_A$

$$\pi^A \equiv \pi^{A4} = \partial L/\partial\psi_{A,4} \qquad \text{-- (2-29)}$$

and separate the field equations (2-28) into their "spatial" and "temporal" parts:

$$\pi^A_{,4} = \frac{\partial L}{\partial\psi_A} - \left[\frac{\partial L}{\partial\psi_{A,\beta}}\right]_{,\beta} \quad . \qquad \text{-- (2-30)}$$

The Hamiltonian density is now defined as

$$H(\psi_A, \psi_{A,\beta}, \pi^A) \equiv \pi^A \psi_{A,4} - L \quad , \qquad \text{-- (2-31)}$$

for which we have

$$\delta H = \frac{\partial H}{\partial \pi^A} \delta \pi^A + \frac{\partial H}{\partial \psi_A} \delta \psi_A + \frac{\partial H}{\partial \psi_{A,\beta}} \delta \psi_{A,\beta} \qquad \text{-- (2-32)}$$

$$= \psi_{A,4} \delta \pi^A - \left[ \frac{\partial L}{\partial \psi_A} \delta \psi_A + \frac{\partial L}{\partial \psi_{A,\beta}} \delta \psi_{A,\beta} \right] \quad .$$

Thus $\quad \dfrac{\partial H}{\partial \pi^A} = \psi_{A,4} \quad ; \quad -\dfrac{\partial H}{\partial \psi_A} = \dfrac{\partial L}{\partial \psi_A} \quad ; \quad -\dfrac{\partial H}{\partial \psi_{A,\beta}} = \dfrac{\partial L}{\partial \psi_{A,\beta}} \qquad \text{-- (2-33)}$

and so, combining these with (2-30), we arrive at

$$\frac{\partial H}{\partial \psi_A} - \left[ \frac{\partial H}{\partial \psi_{A,\beta}} \right]_{,\beta} = -\pi^A_{,4}$$

and

$$\partial H / \partial \pi^A = \psi_{A,4} \quad . \qquad\qquad\qquad \text{-- (2-34)}$$

This is our first form of the canonical field equations.

The equations (2-34) do not at first glance seem to resemble the canonical equations of classical point mechanics. However, their canonical nature may be brought out by the introduction of two new notations into the formalism. First of all we introduce the dot notation for coordinate time derivatives: $\dot{F}_A = F_{A,4}$ for any field quantity $F_A$. This corresponds to the dot notation used for total differentiation with respect to universal Newtonian time in Hamiltonian point mechanics. The second notation we introduce is that of the functional calculus, introduced in section 1.5. From the work of that section we see that the RHS of (2-30) is simply the functional derivative of the total (integrated) Lagrangian:

$$L = \int_S L d\tau \quad ;$$

$$\frac{\delta L}{\delta \psi_A} = \frac{\partial L}{\partial \psi_A} - \left[ \frac{\partial L}{\partial \psi_{A,\beta}} \right]_{,\beta} \quad ; \quad \frac{\delta L}{\delta \dot{\psi}_A} = \frac{\partial L}{\partial \dot{\psi}_A} \quad . \qquad \text{-- (2-35)}$$

Here we have adopted the convention which will hold throughout this chapter, that a **bold** letter will denote a density, while a quantity written in normal script will be a tensor. Throughout

the remainder of this chapter $\tau$ will stand for the 3-volume

element of the hypersurfaces S. Thus in this notation (2-30)

becomes

$$\pi^A{}_{,4} = \delta L/\delta\psi_A \quad ; \quad \pi^A = \delta L/\delta\dot{\psi}_A \quad .$$

The transition to the total Hamiltonian

$$H = \int_S H d\tau = \int \pi^A \psi_{A,4} d\tau - L$$

corresponding to (2-31) is simply a special case of the Legendre

transformations considered in section 1.5, with $\psi_A$ the passive

variables of the transformation and the $\pi^A$ replacing the $\dot{\psi}_A$.

Thus from (2-33) and (2-35) we have

$$\frac{\delta H}{\delta\psi_A} = \frac{\partial H}{\partial\psi_A} - \left[\frac{\partial H}{\partial\psi_{A,\beta}}\right]_{,\beta} \quad ; \quad \frac{\delta H}{\delta\pi^A} = \dot{\psi}_A \quad ,$$

and this gives us our final form of the canonical field equations

(2-34):

$$\delta H/\delta\psi_A = -\dot{\pi}^A \quad ; \quad \delta H/\delta\pi^A = \dot{\psi}_A \quad . \qquad\qquad \text{-- (2-36)}$$

Note that in keeping with its rôle as the total energy of the

system, we find from (2-36) and the definition (1-59) of the

functional derivative that

$$\frac{dH}{dt} = \int \left[\frac{\delta H}{\delta\psi_A}\dot{\psi}_A + \frac{\delta H}{\delta\pi^A}\dot{\pi}^A\right] d\tau$$

$$= 0 \quad .$$

## 2.3 Bracket expressions and canonical transformations

Now let $F=\int F d\tau$ be an arbitrary functional of the field var-

iables $\psi_A,\psi_{A,\beta},\pi^A,\pi^A{}_{,\beta}$, then we find in the general case

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \int \left[\frac{\delta F}{\delta\psi_A}\dot{\psi}_A + \frac{\delta F}{\delta\pi^A}\dot{\pi}^A\right] d\tau$$

or, using the canonical field equations (2-36),

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \int \left[\frac{\delta F}{\delta\psi_A}\frac{\delta H}{\delta\pi^A} - \frac{\delta F}{\delta\pi^A}\frac{\delta H}{\delta\psi_A}\right] d\tau$$

for physical fields. This algebraic expression closely resembles

(1-27), and for that reason we now define the Poisson bracket of

two functionals as

$$\{F,G\} \equiv \int \left[ \frac{\delta F}{\delta \psi_A} \frac{\delta G}{\delta \pi^A} - \frac{\delta F}{\delta \pi^A} \frac{\delta G}{\delta \psi_A} \right] d\tau \quad , \qquad \qquad -- (2\text{-}37)$$

which yields the expression

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\} \quad . \qquad \qquad -- (2\text{-}38)$$

If the functional F has no explicit time dependence then the term $\partial F/\partial t$ drops out of this expression, and one immediate consequence of (2-38) is that any total Hamiltonian H possessing no explicit time dependence will necessarily be constant in time.

In order to calculate PB's between the field variables $\psi$ and $\pi$ we express the variables in functional form in the following way:

$$\psi_A (\underline{x},t) = \int \psi_A (\underline{x}',t) \delta^3 (x-x') d\tau' \quad ;$$

$$\pi^A (\underline{x},t) = \int \pi^A (\underline{x}',t) \delta^3 (x-x') d\tau' \quad , \qquad \qquad -- (2\text{-}39)$$

so that $\quad \dfrac{\delta \psi_A (x,t)}{\delta \psi^A_B (x',t)} = \dfrac{\delta \pi^B (x,t)}{\delta \pi^A (x',t)} = \delta^B_A \delta^3 (x-x') \quad .$

For convenience we shall adopt the convention in what follows that a dash against a quantity signifies its dependence upon the 3-vector $\underline{x}'$, so that $F=F(\underline{x},t)$, $F'=F(\underline{x}',t)$, and so on. The time coordinate will at all times remain unaltered. Using (2-39) we can now obtain the fundamental PB's:

$$\{\psi_A , \psi'_B\} = \{\pi^A , \pi'^B\} = 0 \qquad \qquad -- (2\text{-}40a)$$

$$\{\psi_A , \pi'^B\} = \int \frac{\delta \psi_A}{\delta \psi''_C} \frac{\delta \pi'^B}{\delta \pi''^C} d\tau'' = \int \delta^C_A \delta^3 (x-x'') \delta^B_C \delta^3 (x'-x'') d\tau''$$

$$= \delta^B_A \delta^3 (x-x') \qquad \qquad -- (2\text{-}40b)$$

$$\{\psi'_A , \pi^B_{,\beta}\} = \delta(\pi^B_{,\beta})/\delta \pi'^A = \delta^B_A \delta^3 (x-x')_{,b} \qquad \qquad -- (2\text{-}40c)$$

and all further PB's are essentially obtainable from these by algebraic processes. In particular it is easily seen that

$$\{\psi_A ,F\} = \delta F/\delta \pi^A \quad ; \quad \{\pi^A ,F\} = -\delta F/\delta \psi_A \quad , \qquad \qquad -- (2\text{-}41)$$

which gives the concise form of the field equations:

$$\{\psi_A, H\} = \dot\psi_A \quad ; \quad \{\pi^A, H\} = \dot\pi^A \quad . \qquad \text{-- (2-42)}$$

We now turn our attention to transformations of the field variables $\psi$ and $\pi$. As in section 1.2, if we make the substitution

$$\psi_A \to -\pi^A \quad ; \quad \pi^A \to \psi_A$$

then the canonical field equations remain unaltered, which suggests that we consider general transformations of the form

$$\psi_A \to \Psi_A(\psi, \pi, t) \quad ; \quad \pi^A \to \Pi^A(\psi, \pi, t) \quad .$$

Note that we allow for a possible time dependence in this transformation, even though t itself must remain unaffected. The most significant such transformations will be those which leave the field equations form-invariant, ie:

$$\dot\Pi^A = -\delta\overline{H}/\delta\Psi_A \quad ; \quad \dot\Psi_A = \delta\overline{H}/\delta\Pi^A \quad ,$$

where
$$\overline{H} = \int\{\Pi^B\dot\Psi_B - \overline{L}\}\, d\tau$$

and $\overline{L}$ is a Lagrangian density which gives the correct field equations with the new field variables $\Psi, \Pi$. Such transformations will again be called canonical transformations (CT's), and amount to a (possibly time dependent) relabelling of the 2N-dimensional phase space spanned by the $\psi_A, \pi^A$.

Now for form-invariance of the field equations we will require the relation between the action integral before and after the transformation to be such that if we have an arbitrary set of solutions $(\psi, \pi)$ within a volume V of the form shown in fig.(2.1), then the change in the action depends only on the boundary of V. Due to the considerations at the beginning of this chapter the only such contributions come from the spacelike surfaces $S_1, S_2$:

$$\overline{I}(V) - I(V) = F(t_1) - F(t_2).$$

If this equation is to hold for all V then we can identify integrands to obtain the following sufficient condition for form-invariance of the canonical field equations:

$$\int \pi^{\cdot A} \dot{\psi}^{\cdot}_A d\tau' - \overline{H} = \int \pi^A \dot{\psi}_A d\tau - H + dF/dt$$

$$= \int \pi_A \int \left[ \frac{\delta \psi_A}{\delta \Psi^{\cdot}_B} \dot{\psi}^{\cdot}_B + \frac{\delta \psi_A}{\delta \pi^{\cdot B}} \dot{\pi}^{\cdot B} \right] d\tau' d\tau + \int \pi^A \partial \psi_A / \partial t . d\tau$$

$$- H + \left[ \frac{\delta F}{\delta \Psi^{\cdot}_A} \dot{\psi}^{\cdot}_A + \frac{\delta F}{\delta \pi^{\cdot A}} \dot{\pi}^{\cdot A} \right] d\tau' + \partial F/\partial t .$$

By remembering that $\pi^A$ is dependent upon x, and not x', we can take the $\pi^A$ in the first term on the RHS under the integral over $\tau'$ and reverse the order of integration. By equating terms in $\dot{\psi}^{\cdot}$ and $\dot{\pi}^{\cdot}$ we then obtain

$$\int \left[ \pi^{\cdot A} \dot{\psi}^{\cdot}_A - \frac{\delta F}{\delta \Psi^{\cdot}_A} \dot{\psi}^{\cdot}_A - \int \pi^B \frac{\delta \psi_B}{\delta \Psi^{\cdot}_A} \dot{\psi}^{\cdot}_A d\tau \right] d\tau' = 0 \quad ;$$

$$\left[ \frac{\delta F}{\delta \pi^{\cdot A}} \dot{\pi}^{\cdot A} + \int \pi^B \frac{\delta \psi_B}{\delta \pi^{\cdot A}} \dot{\pi}^{\cdot A} d\tau \right] d\tau' = 0 \quad ;$$

$$\overline{H} = H - \int \pi^A \partial \psi_A / \partial t . d\tau - \partial F/\partial t . \qquad \text{-- (2-43)}$$

The first two of these equations may be simplified by seeing that the integrand of each must be zero and removing $\Psi^{\cdot}$ and $\pi^{\cdot}$ from under the integral over $\tau$:

$$\delta F/\delta \Psi^{\cdot}_A = \pi^{\cdot A} - \int \pi^B \delta \psi_A / \delta \Psi^{\cdot}_A . d\tau \quad ; \quad \Big\}$$
$$\delta F/\delta \pi^{\cdot A} = -\int \pi^B \delta \psi_B / \delta \pi^{\cdot A} . d\tau \quad . \qquad \Big\} \qquad \text{-- (2-44)}$$

If the function F can be found, satisfying the equations (2-44), then we can use it in (2-43) to obtain $\overline{H}$. The necessary and sufficient conditions for the existence of F is a set of three integrability conditions:

$$\frac{\delta}{\delta \Psi^{\cdot \cdot}_C} \left[ \pi^{\cdot A} - \int \pi^B \frac{\delta \psi_B}{\delta \Psi^{\cdot}_A} d\tau \right] = \frac{\delta}{\delta \Psi^{\cdot}_A} \left[ \pi^{\cdot \cdot C} - \int \pi^B \frac{\delta \psi_A}{\delta \Psi^{\cdot \cdot}_C} d\tau \right] \qquad \text{-- (2-45a)}$$

$$\frac{\delta}{\delta \pi^{\cdot \cdot C}} \left[ \int \pi^B \frac{\delta \psi_B}{\delta \pi^{\cdot A}} d\tau \right] = \frac{\delta}{\delta \pi^{\cdot A}} \left[ \int \pi^B \frac{\delta \psi_B}{\delta \pi^{\cdot \cdot C}} d\tau \right] \qquad \text{-- (2-45b)}$$

$$\frac{\delta}{\delta \pi^{\cdot \cdot C}} \left[ \pi^{\cdot A} - \int \pi^B \frac{\delta \psi_B}{\delta \Psi^{\cdot}_A} d\tau \right] = \frac{\delta}{\delta \Psi^{\cdot}_A} \left[ \int \pi^B \frac{\delta \psi_B}{\delta \pi^{\cdot \cdot C}} d\tau \right] \qquad \text{-- (2-45c)}$$

These immediately simplify to the following form:

$$\int \left[ \frac{\delta\psi_B}{\delta\Psi'_A} \frac{\delta\pi^B}{\delta\Psi''_C} - \frac{\delta\psi_B}{\delta\Psi''_C} \frac{\delta\pi^B}{\delta\Psi'_A} \right] d\tau = 0 \qquad\qquad \text{-- } (2\text{-}46a)$$

$$\int \left[ \frac{\delta\psi_B}{\delta\pi^{\cdot A}} \frac{\delta\pi^B}{\delta\pi''^C} - \frac{\delta\psi_B}{\delta\pi''^C} \frac{\delta\pi^B}{\delta\pi^{\cdot A}} \right] d\tau = 0 \qquad\qquad \text{-- } (2\text{-}46b)$$

$$\int \left[ \frac{\delta\psi_B}{\delta\Psi'_A} \frac{\delta\pi^B}{\delta\pi''^C} - \frac{\delta\psi_B}{\delta\pi''^C} \frac{\delta\pi^B}{\delta\Psi'_A} \right] d\tau = \delta^A_C \delta^3(x'-x'') \qquad \text{-- } (2\text{-}46c)$$

The characteristic structure of these expressions leads us to define a new bracket called the _Lagrange bracket_ (LB). If $\omega_1(\psi,\pi),\ldots,\omega_{2N}(\psi,\pi)$ is a set of 2N functions on phase space such that the $\psi_A,\pi^A$ can all be written as functions of the $\omega$'s, then the LB of two $\omega$'s computed with respect to $(\psi,\pi)$ is defined by

$$[\omega'_K,\omega''_L] \equiv \int \left[ \frac{\delta\psi_B}{\delta\omega'_K} \frac{\delta\pi^B}{\delta\omega''_L} - \frac{\delta\psi_B}{\delta\omega''_L} \frac{\delta\pi^B}{\delta\omega'_K} \right] d\tau , \qquad \text{-- } (2\text{-}47)$$

where we reserve the indices K,L,M provisionally to run from 1 to 2N. The LB's and PB's of a set of 2N independent functions $\omega_K$ form matrices which are essentially inverses of one another:

$$\Sigma_K \int [\omega_K,\omega'_L]\{\omega_K,\omega''_M\} d\tau = \delta^L_M \delta^3(x'-x'') . \qquad \text{-- } (2\text{-}48)$$

We shall prove this below as a theorem, but in the meantime (2-48) enables us to write (2-46) either in LB form:

$$[\Psi'_A,\Psi''_C] = [\Pi^{\cdot A},\Pi''^C] = 0 \; ; \; [\Psi'_A,\Pi''^C] = \delta^A_C \delta^3(x'-x'') \quad \text{-- } (2\text{-}49)$$

or in the equivalent PB form:

$$\{\Psi'_A,\Psi''_C\} = \{\Pi^{\cdot A},\Pi''^C\} = 0 \; ; \; \{\Psi'_A,\Pi''^C\} = \delta^C_A \delta^3(x'-x''). \; \text{-- } (2\text{-}50)$$

In both cases the relevant brackets are computed with respect to the original field variables $\psi_A,\pi^A$.

Thus CT's satisfy the relations (2-49) or equivalently (2-50). Only such transformations will leave the Hamiltonian scheme invariant (modulo trivial dilations and reflections). If we consider the identity transformation $\Psi_A=\psi_A$, $\Pi^A=\pi^A$, equations (2-49) and (2-50) are trivially satisfied; we therefore characterise a CT by saying that it preserves the values of the

fundamental PB's and LB's.

THEOREM (2.2) - Given the 2N quantities $\omega_K$ we may calculate the LB and PB of any two. These brackets then obey the relation (2-48):

$$\Sigma_K \int [\omega_K, \omega'_L] \{\omega_K, \omega''_M\} \, d\tau = \delta^L_M \delta^3(x'-x'') \quad ,$$

where K,L,M = 1,2,...,2N. Note that from the definition (2-47), the LB $[\omega_K, \omega'_L]$ is contravariant in K and L. Since this fact is not expressed in our notation, the $\Sigma$ symbol has been inserted for summation over K.

PROOF:　　　Using the definitions of the PB and LB, the LHS of (2-48) becomes

$$\Sigma_K \int \left\{ \int \left[ \frac{\delta\psi_C(y)}{\delta\omega_K} \frac{\delta\pi^C(y)}{\delta\omega'_L} - \frac{\delta\psi_C(y)}{\delta\omega'_L} \frac{\delta\pi^C(y)}{\delta\omega_K} \right] d^3y \right\}$$

$$\times \left\{ \int \left[ \frac{\delta\omega_K}{\delta\psi_A(y')} \frac{\delta\omega''_M}{\delta\pi^A(y')} - \frac{\delta\omega_K}{\delta\pi^A(y')} \frac{\delta\omega''_M}{\delta\psi_A(y')} \right] d^3y' \right\} d\tau \quad .$$

Here we have used y,y' for convenience as additional dummy parameters. Since y,y' are independent variables we may multiply the two integrands together and integrate the product as a double integral over $d^3y \, d^3y'$. We also make use of the immediate relations

$$\Sigma_K \int \frac{\delta\psi_C(y)}{\delta\omega_K} \frac{\delta\omega_K}{\delta\pi^A(y')} \, d\tau = 0 \quad ; \quad \Sigma_K \int \frac{\delta\pi^C(y)}{\delta\omega_K} \frac{\delta\omega_K}{\delta\psi_A(y')} \, d\tau = 0 \quad ;$$

$$\Sigma_K \int \frac{\delta\psi_C(y)}{\delta\omega_K} \frac{\delta\omega_K}{\delta\psi_A(y')} \, d\tau = \delta^A_C \delta^3(y-y') \quad ;$$

$$\Sigma_K \int \frac{\delta\pi^C(y)}{\delta\omega_K} \frac{\delta\omega_K}{\delta\pi^A(y')} \, d\tau = \delta^C_A \delta^3(y-y') \quad ,$$

to arrive at the following form:

$$\int\int \left[ \frac{\delta\pi^C(y)}{\delta\omega'_L} \frac{\delta\omega''_M}{\delta\pi^A(y')} \delta^A_C \delta^3(y-y') + \frac{\delta\psi_C(y)}{\delta\omega'_L} \frac{\delta\omega''_M}{\delta\psi_A(y')} \delta^C_A \delta^3(y-y') \right] d^3y \, d^3y'$$

$$= \int \left[ \frac{\delta \pi^C(y)}{\delta \omega^{\cdot}_L} \frac{\delta \omega^{\cdot \cdot}_M}{\delta \pi^C(y)} + \frac{\delta \psi_C(y)}{\delta \omega^{\cdot}_L} \frac{\delta \omega^{\cdot \cdot}_M}{\delta \psi_C(y)} \right] d^3 y$$

$$= \delta^L_M \delta^3 (x^{\cdot} - x^{\cdot \cdot}) \quad .$$

QED

Finally we now look back at the PB conditions (2-50) for a CT:

$$\{\pi^A, \pi^{\cdot B}\} = \{\psi_A, \psi^{\cdot}_B\} = 0 \quad ; \quad \{\psi_A, \pi^{\cdot B}\} = \delta^B_A \delta^3 (x-x^{\cdot}) \quad \text{-- (2-51)}$$

are the fundamental PB relations and, as seen earlier, are preserved under CT's. Now by differentiating the second and third of these relations with respect to x we obtain

$$\{\psi_{A,\beta}, \psi^{\cdot}_B\} = 0 \quad ; \quad \{\psi_{A,\beta}, \pi^{\cdot B}\} = \delta^B_A \delta^3 (x-x^{\cdot})_{,\beta} \quad . \quad \text{-- (2-52)}$$

Now let $F(\pi^A, \psi_A, \psi_{A,\beta})$ be an arbitrary density and set

$$F = \int F d\tau \quad .$$

Using (2-51) and (2-52) we can evaluate the PB's of $\pi^A$ and $\psi_A$ with F to obtain the relations (2-41) quoted earlier:

$$\{\psi_A, F\} = \partial F / \partial \pi^A \quad ; \quad \{\pi^A, F\} = (\partial F / \partial \psi_{A,\beta})_{,\beta} - \partial F / \partial \psi_A \quad .$$

In this manner we can evaluate any PB's using solely the relations (2-51), the relations (2-52) being a consequence of (2-51). It follows, therefore, that all PB's between arbitrary quantities are preserved under CT's.


## 2.4 Examples

The canonical theory of fields may be developed further along the lines of the previous section in a way which is exactly analogous to the development of particle mechanics in section 1.2. Although it would be inappropriate to go into it in great detail here, we have separately developed a formalism involving generating functions and generators which is in every way analogous to that of section 1.2. We now give a number of examples

which demonstrate applications of the above theory and end up with a discussion of the merits of the theory.

**Example (2.3)** - As our first example we consider the simplest case of a non-interacting field, that is, a single real scalar field $\varphi(x)$ described by the action functional

$$I = \int L d^4 x \quad ; \quad L = -1/2.(\varphi_{,a}\varphi'^{a} + m^2\varphi^2) \quad , \qquad \text{-- (2-53)}$$

where m is a real constant. From this we derive

$$\pi = \partial L/\partial\varphi_{,4} = -\varphi'^4 = \dot{\varphi} \quad ;$$

$$H = \dot{\varphi}^2 + 1/2.(\varphi_{,a}\varphi'^{a} + m^2\varphi^2)$$

$$= 1/2.(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2) \quad ,$$

in which case the Hamiltonian equations of motion are

$$\dot{\varphi} = \delta H/\delta\pi = \partial H/\partial\pi = \pi$$

$$-\dot{\pi} = \delta H/\delta\varphi = \partial H/\partial\varphi - (\partial H/\partial\varphi_{,b})_{,b}$$

$$= m^2\varphi - \nabla^2\varphi$$

$$\Rightarrow \qquad\qquad (\square - m^2)\varphi = 0 \quad , \qquad\qquad \text{-- (2-54)}$$

where $\square$ is the d'Alembert operator, defined by

$$\square\varphi \equiv g^{ab}\varphi_{,ab} \quad .$$

Equation (2-54) is called the <u>Klein-Gordon</u> (K-G) equation, and the operator $\square - m^2$ is the Klein-Gordon operator. The K-G equation was first studied as a possible generalisation of the Schroedinger equation for classical particles (see, for example, Feshbach & Villars,1958); we shall come across the K-G equation again in this context in chapter 6.

A simple generalisation of the K-G field is the case of the complex scalar field described by the Lagrangian

$$L = -(\varphi^*_{,a}\varphi'^{a} + m^2\varphi^*\varphi) \quad . \qquad\qquad \text{-- (2-55)}$$

This again yields the K-G equation for each of the fields:

$$(\square - m^2)\varphi = 0 \quad ; \quad (\square - m^2)\varphi^* = 0 \quad ,$$

with

$$\pi = \dot{\varphi}^* \quad ; \quad \pi^* = \dot{\varphi} \quad ;$$

$$H = \dot{\varphi}^*\dot{\varphi} + m^2\varphi^*\varphi + \nabla\varphi^*\nabla\varphi \quad .$$

In this complex case we can define the phase transformations

$$\varphi \rightarrow \exp(-ie\varepsilon)\varphi \quad ; \quad \left.\begin{array}{l} \\ \varphi^* \rightarrow \exp(+ie\varepsilon)\varphi^* \quad . \end{array}\right\} \qquad \text{-- (2-56)}$$

This is a gauge transformation of the first kind and results in the infinitesimal transformations

$$\delta\varphi = -ie\varepsilon\varphi \quad ; \quad \delta\varphi^* = +ie\varepsilon\varphi^* \quad .$$

According to the definitions (2-25) and (2-26) we have

$$j_b = -ie[\varphi\varphi^*_{,b} - \varphi^*\varphi_{,b}] \quad ; \quad Q = \int ie(\varphi\pi - \varphi^*\pi^*)d\tau$$

as the expressions for the current density and total charge of the field. Since the Lagrangian (2-55) is bilinear in $\varphi$ and $\varphi^*$, L is clearly invariant under the transformation (2-56), so that charge is conserved:

$$dQ/dt = 0 \quad ,$$

with the differential form $j^b_{,b} = 0$. In view of this conservation law we see that charge seems to be associated with complex fields. The complex field $\varphi$ is said to carry a charge e, while its conjugate $\varphi^*$ carries a charge -e.

The Lagrangian (2-55) in fact describes what is called the charged scalar meson field, while the Lagrangian (2-53) describes the neutral scalar meson field. Since experience tells us that electric charge is conserved, we deliberately construct a Lagrangian which is invariant under gauge transformations of the first kind and are led to charge conservation: Q means total electric charge of the system. On the other hand we might consider transformations such that protons and neutrons have $\eta^{(A)}=+1$, all pions have $\eta^{(A)}=0$ and antiprotons and antineutrons have $\eta^{(A)}=-1$. The corresponding 'charge' in this case is called the baryon number; experience tells us that this also is conserved, so in physical applications the additional constraint is placed on the Lagrangian that it be invariant also under this

second, independent phase transformation.

**Example (2.4)** - We now look at the case of a 4-component field $A_a$, where $A_a$ is a real 4-vector. We take as our Lagrangian

$$L = -1/2 . (A^{a,b} A_{a,b} + m^2 A^a A_a) \quad ,$$

where again m is a real constant. From this we obtain

$$\pi^{ab} = \partial L / \partial A_{a,b} = -A^{a,b}$$

$$\Rightarrow \qquad \pi^a = A^a_{,4} \quad ;$$

$$H = A^a_{,4} A_{a,4} + 1/2 (A^{c,d} A_{c,d} + m^2 A^c A_c) \quad ,$$

and the field equations are

$$\dot{A}_a = \partial H / \partial \pi^a = 2A_{a,4} + A_a{}^{,4}$$

$$= A_{a,4} \quad ;$$

$$\dot{\pi}^a = \partial H / \partial A_a - (\partial H / \partial A_{a,\beta})_{,\beta}$$

$$= m^2 A^a - (A^{a,\beta})_{,\beta} \quad .$$

Remembering that $\pi^a = (A^a_{,4})_{,4}$ we can rewrite this second field equation as

$$(\Box - m^2) A_a = 0 \quad . \qquad\qquad -- (2-57)$$

This is the _vectorial_ K-G equation, and describes the _vector meson_ field. When m=0 it is simply the homogeneous wave equation, which is also satisfied by the electromagnetic potentials $A_a$ in free space. Thus the electromagnetic field in free space is termed a massless vector meson field.

---

In this chapter we have been concerned solely with the description of fields. We mentioned at the beginning of section 2.2 that historically the motivation for finding a Hamiltonian formulation of field theory has always been to develop a quantum theory of fields, with the result that with the exception of Weiss (1936) no author has taken the trouble to present a systematic exposition of canonical field theory as attempted in this

chapter. In most cases, as soon as anything like a Poisson bracket appears in the theory the author promptly uses it to quantise everything, and the non-quantum theory is left in an incomplete state of limbo. In our work we have seen that a complete development of the theory is in principle possible, but that is not to say that it is particularly plausible. Consider the canonical field equations (2-36). At first sight they seem reasonable enough, but consider that their original form (2-34) was decidely less convincing. The only reason that (2-36) is more appealing is because the theory has been crammed into the canonical mould by means of the introduction of the dot notation and the functional calculus. Added to this basically aesthetic objection is the fact that the splitting of the derivatives implied in (2-34) (a) singles out time as a preferred coordinate and (b) is totally unfeasible in general relativity. It must be admitted that this theory has performed sterling service in many areas of physics, but in the transition to the canonical form-alism the elegant theory of section 2.1 has grown so clumsy that we cannot accept it as anything like a reasonable description of nature.

The question now arises: which way are we to turn next? In answer we appeal to the original motivation for this work, which was the problem of quantisation. The only reason for attempting to quantise fields is because of the wave/particle duality which arises in quantum theory. Yet historically this duality did not stem from the study of fields, but rather from the study of the electron. In moving from Heisenberg's quantum theory to quantum field theory two changes of attitude are made: a change in the object of study (from particles to fields) and a change in the laws which are to govern its behaviour (from classical to relat-

ivistic mechanics). Maybe the more pedestrian approach will be more fruitful: (a) first attempt a relativistic Hamiltonian theory of particles, and (b) only then investigate the possible extension of the theory to the description of fields. (a) will occupy our attention throughout the remainder of this thesis.

## CHAPTER 3

## THE HOMOGENEOUS CANONICAL FORMALISM

### 3.1 Time as an additional coordinate

One of the most far-reaching innovations involved in the transition from Newtonian physics to relativity theory is the inclusion of time as an additional coordinate in the 4-dimensional spacetime structure. Now the canonical formalism offers two possibilities for doing this. In chapter 2 we saw how the structure of the Euler-Lagrange equations in field theory suggests the idea of reducing the three space coordinates from the status of field variables to join the time coordinate and form a sort of "4-dimensional time parameter". Thus in field theory all four spacetime coordinates $x^a$ are relegated to the background, or parameter, space, which led to the somewhat unsatisfactory results of canonical field theory. In this chapter we shall investigate the alternative possibility, where time is elevated to the status of field variable and some other parameter is adopted. We ought perhaps at this point to draw attention to the fact that we shall from here onwards use the terms "coordinate" and "field variable" almost interchangeably. Each is indexed by the block indices A,B,C,... introduced in chapter 2 and each obeys some kind of Euler-Lagrange equation. The parameter space, on the other hand, is the background space of arguments of the field variables, and in this chapter will have dimension 1.

Classical Hamiltonian mechanics is not parameter-invariant in the sense that the action integral and the resulting Euler-Lagrange equations are not independent of the choice of t (in keeping with Newtonian "absolute, true and mathematical time").

The coordinates, on the other hand, are completely arbitrary, even though position and time are treated as geometrical entities of the same kind in configuration space. We therefore look now at the modification of the classical theory obtained by <u>con-sistently</u> regarding time as an additional coordinate. We consider initially the case of a Lagrangian $L^*$ dependent on the $N-1$ independent coordinates $q^\alpha$ ($\alpha=1,2,\ldots,N-1$), the (classical) parameter $t$ and the derivatives $q'^\alpha \equiv dq^\alpha/dt$ (we use dashes here to denote differentiation with respect to $t$ for later convenience). Let $C:q^\alpha=q^\alpha(t)$ be any curve in the space $\underline{R}^N$ of the variables $(q^\alpha,t)$ joining two points $P_1$, $P_2$ with parameter values $t_1$, $t_2$ respectively. The corresponding action integral

$$I = \int_C L^* dt \qquad\qquad -- (3-1)$$

can now be written in a different form involving the new parameter $\tau(t)$, with the restriction that $\tau(t)$ be of class $C^1$ with $d\tau/dt>0$ at all points of $C$. We shall use a dot to denote derivatives with respect to $\tau$, rather than the usual meaning of derivatives with respect to time, in which case we have $\dot{q}^\alpha=q'^\alpha(d\tau/dt)^{-1}$. Now we can rewrite (3-1) as

$$I = \int_C L^*(t,q^\alpha,\dot{q}^\alpha.d\tau/dt).dt/d\tau.d\tau \ .$$

This suggests that we write $t=q^N$, so that the coordinates of $\underline{R}^N$ can be denoted simply by $q^A=(q^\alpha,t)$ ($A=1,2,\ldots,N$). We then have by supposition $dt/d\tau=\dot{q}^N\neq0$, so that (3-1) becomes

$$I = \int_C L(q^A,\dot{q}^A)d\tau \qquad\qquad\left.\begin{array}{c}\\[1em]\end{array}\right\}$$
$$\text{with} \qquad L(q^A,\dot{q}^A) = L^*(q^\alpha,t,\dot{q}^\alpha/\dot{q}^N).\dot{q}^N \ . \qquad -- (3-2)$$

The integral (3-2) is identical with (3-1), but possesses the two additional properties that it is (i) parameter invariant and (ii) positively homogeneous of the first degree in the $\dot{q}^A$, irrespective of the form of the given Lagrangian $L^*$. We shall see in the following that parameter-invariant integrals are characterised by

integrands of this type.

Having seen how we can change to a scheme in which time forms an additional coordinate we now shift our attention entirely to the N-dimensional manifold M of the $q^A$. In this manifold a set of equations of the type $q^A = q^A(\tau)$, where $\tau$ is an arbitrary parameter, represents a curve C in M. If these functions are of class $C^1$ then we can form the derivatives $\dot{q}^A = dq^A/d\tau$ forming the components of a tangent vector to C. The Lagrangian is supposed to be an arbitrary ($C^3$) function of the coordinates and velocities: $L = L(q^A, \dot{q}^A)$. Given any curve C in M we can form the action integral

$$I = \int_C L(q^A, \dot{q}^A) d\tau , \qquad\qquad \text{-- (3-3)}$$

about which we now make the following central assumption of this chapter:

ASSUMPTION (3.1) - We assume that the value of the integral (3-3)
    is invariant under arbitrary parameter transformations
    of the form $s = s(\tau)$, where the function s is $C^1$ and such
    that

$$\dot{s} = ds/d\tau > 0 . \qquad\qquad \text{-- (3-4)}$$

This assumption means that the resulting theory will be invariant not only under transformations of the coordinates but also under transformations (3-4) of the parameter.

Performing the transformation (3-4), assumption (3.1) becomes

$$I = \int_C L(q^A, \dot{q}^A) d\tau$$

$$= \int_C L(q^A, dq^A/ds . \dot{s}) ds/\dot{s}$$

$$\equiv \int_C L(q^A, dq^A/ds) ds$$

for any curve C. But this is only possible if

$$L(q, \lambda\dot{q}) = \lambda L(q, \dot{q}), \text{ for all } \lambda > 0. \qquad\qquad \text{-- (3-5)}$$

That is, our central assumption implies the condition (3-5), which says that L must be positively homogeneous of the first degree in q. Conversely, assumption (3.1) is fulfilled if (3-5) holds for all $\lambda > 0$. Thus we shall assume throughout this and the following section that $L(q,\dot{q})$ is positively homogeneous of the first degree in $\dot{q}$ and does not depend explicitly on the parameter $\tau$. We have already seen above that this restriction is not as severe as one might at first think, since any variational problem can be reduced to one involving the homogeneous Lagrangian (3-2) without altering the value of the action integral.

We shall now see that there are a number of fundamental difficulties involved in moving to this homogeneous setup from the classical theory of chapter 1, which will cause us to look at an alternative picture due to Rund in the following section. Firstly, by Euler's theorem on homogeneous functions we have

$$\frac{\partial L(q,\dot{q})}{\partial \dot{q}^B} \cdot \dot{q}^B = L(q,\dot{q}) \ , \qquad \text{-- (3-6)}$$

where the derivatives $\partial L/\partial \dot{q}^B$ are now positively homogeneous of degree zero in $\dot{q}$. Hence by applying Euler's theorem again we obtain

$$\frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^A \partial \dot{q}^B} \cdot \dot{q}^B \equiv 0 \qquad \text{-- (3-7)}$$

$$\Rightarrow \qquad \det \left| \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right| = 0 \ . \qquad \text{-- (3-8)}$$

As we can see from chapter 1, this contravenes the fundamental assumption (1-12) of the classical, non-homogeneous theory. Furthermore, if we write $p_A = \partial L/\partial \dot{q}^A$ as the momentum and substitute into expression (1-10) for the Hamiltonian, then we obtain

$$H = p_A \dot{q}^A - L(q,\dot{q}) = 0 \ , \qquad \text{-- (3-9)}$$

from (3-6). Thus we see that an alternative formalism is requir-ed. The one we shall study in the following section forms the

basis for all attempts at a Hamiltonian formulation of relativity which we shall study in this thesis, and seems to originate in the work of Carathéodory (1935). However, his work has been developed more recently by both Rund (1959; 1966) and Synge (1960), and we shall confine our attention to these more recent developments. All these approaches are heavily dependent for their theoretical basis upon the concept of a _Finsler space_.

## 3.2 Finsler geometry and the theory of Rund

**Definition (3.2):** Let M be a real N-dimensional manifold with tangent space $T_q(M)$ at the point $q \epsilon M$. Let $q^A$ (A=1,2,...,N) be local coordinates of q and $u^A$, $v^A$ be the corresponding coordinates of $u, v \epsilon T_q(M)$ in the natural basis for $T_q(M)$. M is then called a _Finsler space_ if there is given a global scalar function F(q,u) on the tangent bundle over M such that the following conditions hold:

i. The function F(q,u) is positively homogeneous of degree 1 in u:

$$F(q, \lambda u) = \lambda F(q,u), \text{ for all } \lambda > 0 .$$

ii. F(q,u) is positive for all nonzero $u \epsilon T_q(M)$.

iii. The quadratic form

$$\frac{\partial^2 F^2(q,u)}{\partial u^A \partial u^B} v^A v^B$$

is positive definite (ie, positive for all nonzero v). Note the appearance of $F^2$ in this definition, rather than the simple occurrence of F.

---

**Example (3.3)** - A particular case of a Finsler space is where the function F is of the form

$$F(q,u) = [g_{AB}(q)u^A u^B]^{1/2} ,$$

where the $g_{AB}$ are the (symmetric) components of the metric tensor, which is independent of the $u^A$. M is in this case called a Riemannian space. Forming the quadratic form of condition (iii) above we find

$$\frac{\partial^2 F^2(q,u)}{\partial u^A \partial u^B} v^A v^B = 2g_{AB}(q)v^A v^B ,$$

which is twice the square of the usual Riemannian norm.

---

The idea of a Finsler space will form the basis of the homogeneous canonical formalism. We shall need to be careful with the definition as it stands, however, since we wish to express the equations of mechanics in terms of a set of coordinates on a Lorentz manifold, and the metric of such a manifold is _not_ positive definite, as required in definition (3.2). Hence our model may need to relax the condition (iii) above, and consequently also condition (ii). Rund's solution to the problem is to confine his attention to regions of M in which the metric is sign definite; this is the same as looking only at either completely timelike or completely spacelike trajectories. It is clear that the Lagrangian L satisfying (3-5) is a suitable contender for the rôle of the function F. Thus we arrive finally at a model of Lagrangian mechanics which consists of the space M together with the function L(q,q̇), satisfying the homogeneity condition (3-5). This model is valid since we can always regard an arbitrary element of $T_q$ as the tangent vector to some curve through q (see for instance Hawking & Ellis,1973).

By analogy with example (3.3) we now define a set of quantities $g_{AB}$ by the equations

$$g_{AB}(q,q̇) \equiv \frac{1}{2} \frac{\partial^2 L^2(q,q̇)}{\partial q^A \partial q̇^B} . \qquad\qquad -- (3-10)$$

It follows directly from the definitions that the $g_{AB}$ form the components of a rank 2 symmetric covariant tensor over M. Carrying out the differentiation in (3-10) and applying (3-6) and (3-7) we obtain

$$g_{AB}(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}^A}\frac{\partial L}{\partial \dot{q}^B} + L\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}$$

$\Rightarrow \qquad g_{AB}\dot{q}^B = L\,\partial L/\partial \dot{q}^A \qquad\qquad$ -- (3-11a)

and $\qquad g_{AB}\dot{q}^A\dot{q}^B = L^2$ . $\qquad\qquad$ -- (3-11b)

Now (3-11b) is distinctly reminiscent of the case in Riemannian geometry:

$$L^2(q,\dot{q}) = g_{AB}(q,\dot{q})\dot{q}^A\dot{q}^B \,, \qquad\qquad \text{-- (3-12)}$$

except that now the $g_{AB}$ possess an additional dependence on the velocities, rather than simply on the coordinates. Thus if we define the __magnitude__ of an arbitrary $u \varepsilon T_q$ as

$$|u| = L(q,u)$$

then it follows from (3-12) that all magnitudes in $T_q$ may be expressed in terms of the $g_{AB}$, which we shall regard as the __metric__ __tensor__ on $T_q$. This interpretation throws a whole new light on the significance of the action integral (3-3). Since the length of a small displacement dq at a point $q \varepsilon M$ can be defined as

$$dw = |dq| = L(q,dq) = [g_{AB}(q,dq)dq^A dq^B]^{1/2}$$

in terms of the metric on $T_q$, it is now natural to impose a metric on M by defining the __length__ of a curve $C:q=q(\tau)$ connecting the two points $q(\tau_1),q(\tau_2)\varepsilon M$ as the integral

$$w \equiv \int_{\tau_1}^{\tau_2} L(q,dq) = \int_{\tau_1}^{\tau_2} L(q,\dot{q})d\tau \qquad\qquad \text{-- (3-13)}$$

$$= I \,.$$

We thus arrive at an interpretation in which the action I is simply the length of the particular curve C chosen to connect

$q(\tau_1)$ and $q(\tau_2)$, and our variational problem becomes identical with the standard geodesic problem of differential geometry, with the difference that the Finsler metric $g_{AB}(q,dq)$ depends not only on the position q, but also on the displacement dq. A Riemannian space with positive definite metric is therefore a particular case of a Finsler space whose metric tensor $g_{AB}$ is independent of direction.

The metric is said to be <u>non-degenerate</u> at q if there is no nonzero vector $u \varepsilon T_q$ such that $g_{AB} u^A v^B = 0$ for all vectors $v \varepsilon T_q$. In terms of the metric components alone, the metric is nondegenerate iff the matrix $(g_{AB})$ of components of g is nonsingular, ie, iff

$$\det (g_{AB}) = \frac{1}{2} \left| \frac{\partial^2 L^2 (q,\dot{q})}{\partial \dot{q}^A \partial \dot{q}^B} \right| \neq 0 \quad . \qquad \text{-- (3-14)}$$

From now on we shall <u>always</u> assume the metric tensor is non-degenerate; this assumption (3-14) is fundamental to Rund's theory in the same way that assumption (1-12) was fundamental to the classical theory. Note that in the strict Finsler space definition (3.2) the possibility of a singular metric is prohibited by condition (iii). However, Rund states the assumption separately and we must in any case always be aware of the fact that we may want to relax this condition to enable the description of relativistic mechanics. By virtue of (3-14) we may associate with each arbitrary contravariant vector $\dot{q} \varepsilon T_q$ a <u>unique</u> covariant vector p with components

$$p_A = g_{AB} (q,\dot{q})\dot{q}^B , \qquad \text{-- (3-15)}$$

where we note that the directional argument of the $g_{AB}$ must coincide with the vector $\dot{q}^B$ under consideration. It follows from (3-15) that the $p_A$ are positively homogeneous of the first degree in $\dot{q}$. The momentum vector p at the point q is a covariant vector lying in the cotangent space $T_q^*(M)$, or simply $T_q^*$.

Before continuing further we note from the definition (3-10) that the $g_{AB}$ are positively homogeneous of degree zero in $\dot{q}$, so that we have

$$\frac{\partial g_{AB}(q,\dot{q})}{\partial \dot{q}^C} \dot{q}^C = \frac{\partial g_{AC}(q,\dot{q})}{\partial \dot{q}^B} \dot{q}^C = 0 , \qquad \text{-- (3-16)}$$

where we have used the fact that the derivatives $\partial g_{AB}/\partial \dot{q}^C$ are symmetric in the three indices A,B,C. Now combining this result with the transformation (3-15) we find

$$\frac{\partial p_A}{\partial \dot{q}^B} = \frac{\partial g_{AC}}{\partial \dot{q}^B} \dot{q}^C + g_{AC} \delta_B^C = g_{AB} ,$$

so that the Jacobian of (3-15) is just the determinant of the matrix $(g_{AB})$. In this case the transformation can be inverted to give the $\dot{q}^A$ as functions of the $p_A$. Furthermore, according to (3-14) the matrix $(g_{AB})$ possesses an inverse which by substitution for $\dot{q}$ can be written $(g^{AB}(q,p))$ such that

$$g^{AB}(q,p)g_{BC}(q,\dot{q}) = \delta_C^A , \qquad \text{-- (3-17)}$$

whenever $\dot{q}$ and $p$ are related by (3-15). From (3-17) we can see that the $g^{AB}$ must be positively homogeneous of degree zero in $\dot{q}$, and by combining this with the equations (3-15) we can solve (3-15) explicitly for the $\dot{q}^A$ in the form

$$\dot{q}^A = g^{AB}(q,p)p_B . \qquad \text{-- (3-18)}$$

Again it is easily seen that the $g^{AB}$ form the components of a symmetric second rank contravariant tensor.

The obvious symmetry between covariant and contravariant tensor relations now prompts us to define the <u>Hamiltonian function</u>

$$H^2(q,p) \equiv g^{AB}(q,p)p_A p_B \qquad \text{-- (3-19)}$$

by analogy with (3-12). Substituting into this from (3-15) we deduce from (3-17) that

$$H^2(q,p) = g_{AB}(q,\dot{q})\dot{q}^A \dot{q}^B = L^2(q,\dot{q}) .$$

Since the definition (3-19) is ambiguous with regard to the sign

of H we now require that the signs of L and H coincide, so that

$$H(q,p) = L(q,\dot{q}) \qquad\qquad -- (3-20)$$

whenever p and $\dot{q}$ are related by (3-15). This is Rund's sub-

stitute for the Legendre transformation.

Now differentiating (3-17) with respect to $\dot{q}^D$ we obtain

$$\frac{\partial g^{AB}}{\partial \dot{q}^D} g_{BC} + g^{AB} \frac{\partial g_{BC}}{\partial \dot{q}^D} = 0$$

$$=> \qquad \frac{\partial g^{AB}}{\partial p_E} g_{ED} g_{BC} + \frac{\partial g_{BC}}{\partial \dot{q}^D} g^{AB} = 0 .$$

Multiplying by $\dot{q}^C$ and taking into account (3-15) and (3-16) this

leads to

$$\frac{\partial g^{AB}}{\partial p_E} p_B g_{ED} = 0 ,$$

which in view of (3-14) gives us

$$\frac{\partial g^{AB}}{\partial p_C} p_B = 0 . \qquad\qquad -- (3-21)$$

Thus if (3-19) is differentiated successively with respect to $p_A$

and $p_B$ we find in complete analogy to (3-10) that

$$g^{AB}(q,p) = \frac{1}{2} \frac{\partial^2 H^2(q,p)}{\partial p_A \partial p_B} . \qquad\qquad -- (3-22)$$

It follows that H(q,p) is positively homogeneous of the first

degree in p, and we have from (3-18) and (3-22)

$$\dot{q} = H\partial H/\partial p = 1/2.\partial H^2/\partial p \qquad\qquad -- (3-23a)$$

as the counterpart to (3-11a) in the form

$$p = L\partial L/\partial \dot{q} = 1/2.\partial L^2/\partial \dot{q} . \qquad\qquad -- (3-23b)$$

Finally, if we differentiate (3-20) with respect to $q^A$

keeping the $p_A$ constant, we find from (3-15) and (3-23) that

$$\frac{\partial H}{\partial q^A} = \frac{\partial L}{\partial q^A} + \frac{\partial L}{\partial \dot{q}^B}\frac{\partial \dot{q}^B}{\partial q^A}(q,p) = \frac{\partial L}{\partial q^A} + \frac{p_B}{L}\frac{\partial}{\partial q^A}\left[H\frac{\partial H}{\partial p_B}\right] ,$$

But H is positively homogeneous of degree one in p and so sat-

isfies the relations

$$P_B \frac{\partial H}{\partial P_B} = H \quad ; \quad P_B \frac{\partial^2 H}{\partial q^A \partial P_B} = \frac{\partial H}{\partial q^A} \quad ,$$

which then gives us

$$\frac{\partial H}{\partial q^A} = \frac{\partial L}{\partial q^A} + \frac{2}{L}\left[\frac{\partial H}{\partial q^A} H\right]$$

$$\Rightarrow \quad \frac{\partial H(q,p)}{\partial q^A} = - \frac{\partial L(q,\dot{q})}{\partial q^A} \quad , \qquad\qquad -- \quad (3-24)$$

where we have used (3-20), and (in analogy with the classical canonical equations) we regard the $(q,\dot{q})$ and the $(q,p)$ as the independent variables of the right- and left-hand sides respectively. Furthermore, if p is a unit vector, ie, if

$$H(q,p) = 1 \quad ,$$

then equations (3-23) become

$$p_A = \partial L/\partial \dot{q}^A \quad ; \quad \dot{q}^A = \partial H/\partial p_A \quad . \qquad -- \quad (3-25)$$

The close similarity between the canonical relations (1-15) of classical mechanics and equations (3-25) is immediately obvious, _provided_ that the function H is interpreted as a Hamiltonian. Clearly the normalisation condition on p can always be enforced by the choice of parameter $\tau=w$ defined in (3-13), so we have here a theory which provides a beautifully straightforward geometrical interpretation of the relations (3-25). These relations provide us with a correspondence between the tangent space $T_q$ and its dual $T_q^*$, while from the point of view of mechanics they represent the correspondence between the velocity and momentum variables of the dynamical system.

Having established the above analogy between dynamics and Finsler geometry Rund's next step is to seek the geodesic curves of the Finsler space. These are defined by the requirement that the length of the curve $C:q=q(\tau)$ be stationary under arbitrary variations $\delta q^A$ of the points of C which vanish at the endpoints:

$$\delta I = \delta \int_C dw$$

$$= \delta \int_C L(q, dq)$$

$$= \delta \int_C L(q, \dot{q}) d\tau = 0 \quad . \qquad \qquad -- \text{(3-26)}$$

But (3-26) is precisely the variational problem (1-6) of mechanics, and by exactly the same argument as in chapter 1 we arrive at the Euler-Lagrange equations, which are now seen to characterise the geodesics of the system:

$$\frac{\partial L}{\partial q^A} - \frac{d}{\partial \tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = 0 \quad , \qquad \qquad -- \text{(3-27)}$$

for an arbitrary parameter $\tau$ along the curve. The curves C which satisfy (3-27) are called <u>extremals</u>, and will hopefully correspond to the physical evolution of systems occurring in nature.

We can express the equations (3-27) in a form which displays more clearly the geometrical aspect of the situation if we differentiate (3-12) with respect to $q^A$, $\dot{q}^A$, taking account of (3-16), in which case we obtain

$$\frac{\partial L}{\partial q^A} = \frac{1}{2L}\frac{\partial g_{BC}}{\partial q^A}\dot{q}^B\dot{q}^C \quad ; \quad \frac{\partial L}{\partial \dot{q}^A} = \frac{1}{L}g_{AB}\dot{q}^B$$

$$\Rightarrow \frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = \frac{1}{L}\left\{\left[\frac{1}{2}\frac{\partial g_{BC}}{\partial q^A} - \frac{\partial g_{AC}}{\partial q^B}\right]\dot{q}^B\dot{q}^C - g_{AB}\ddot{q}^B\right\}$$

$$+ \frac{1}{L^2}\frac{dL}{d\tau}g_{AB}\dot{q}^B \qquad \qquad -- \text{(3-28)}$$

In analogy with the Christoffel symbols of Riemannian geometry we now define the following 3-index symbols for the generalised Finsler space:

$$\gamma_{ABC} \equiv \frac{1}{2}\left[\frac{\partial g_{AB}}{\partial q^C} + \frac{\partial g_{AC}}{\partial q^B} - \frac{\partial g_{BC}}{\partial q^A}\right] \quad ,$$

$$\gamma^A_{BC} = g^{AD}(q,p)\gamma_{DBC}(q,\dot{q}) \quad , \qquad \qquad -- \text{(3-29)}$$

where, as always, we assume that q and p are related by (3-15). Using (3-29) we can now express (3-28) as

$$\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = \frac{-1}{L}g_{AD}\left[\ddot{q}^D + \gamma^D_{BC}\dot{q}^B\dot{q}^C - \frac{1}{L}\frac{dL}{d\tau}\dot{q}^D\right]$$

$$= 0 \quad . \qquad \qquad -- \text{(3-30)}$$

Now these equations are not in general soluble for q (Rund,1966), but if we make the __specific__ choice of parameter τ=w, defined by (3-13), then we have L(q,q̇)=1. Consequently dL/dτ=0 and (3-30) reduces to

$$\frac{\delta \dot{q}^A}{\delta w} \equiv \frac{d^2 q^A}{dw^2} + \gamma^A_{BC} \frac{dq^B}{dw}\frac{dq^C}{dw} = 0 \quad , \qquad \text{-- (3-31)}$$

where $\delta \dot{q}^A/\delta w$ must be a contravariant vector, since the LHS of (3-30) is such. The accelerations $\ddot{q}^A$ appear explicitly in (3-31), but this is true __only__ for the special choice of parameter τ=w.

The vector $\delta \dot{q}^A/\delta w$ is the generalisation to Finsler space of the absolute derivative of $\dot{q}^A$. However, it must be borne in mind that the γ-symbols defined in (3-29) are __not__ identical with the Christoffel symbols of Riemannian geometry, and so cannot in general be used to define the covariant derivative of an arbitrary vector field. The general expression for the covariant derivative of a vector field $X^A$ is given by Rund (1959) as

$$X^A_{;B}(q,\dot{q}) = \frac{\partial X^A}{\partial q^B} + \left\{ {A \atop B\ C} \right\} X^C \quad , \qquad \text{-- (3-32)}$$

where $\left\{ {A \atop B\ C} \right\}(q,\dot{q}) = g^{AD}(q,p)[D,BC](q,\dot{q}) \;;$

$[A,BC] = \gamma_{ABC} - \dot{q}^D\{C_{EAB}P^E_{DC} + C_{EAC}P^E_{DB} - C_{EBC}P^E_{DA}\};$

$P^E_{DA} = \gamma^E_{DA} - C^E_{DH}\gamma^H_{GA}\dot{q}^G \;;$

$C^E_{DH} = g^{EG}C_{GDH} \;;$

$C_{GDH} = \frac{1}{2}\frac{\partial g_{DH}}{\partial \dot{q}^G} = \frac{1}{4}\frac{\partial^3 L^2}{\partial \dot{q}^G \partial \dot{q}^D \partial \dot{q}^H} \quad .$

$$\left. \right\} \qquad \text{-- (3-33)}$$

It is easily seen that the formula (3-32) reduces to its Riemannian counterpart under two important circumstances. Firstly, if the metric is Riemannian, ie, if the $g_{AB}$ are independent of $\dot{q}$, then from the definitions (3-33) we see that

$$\left\{ {A \atop B\ C} \right\}(q) = \gamma^A_{BC}(q) \quad . \qquad \text{-- (3-34)}$$

Secondly, since according to (3-33) the $C_{ABC}$ are homogeneous of degree -1 in $\dot{q}$, we have

$$\dot{q}^C \left\{ \begin{matrix} A \\ B \ C \end{matrix} \right\} (q,\dot{q}) = \dot{q}^C P^A_{BC}(q,\dot{q}) = \dot{q}^C \gamma^A_{BC}(q,\dot{q}) \quad , \qquad \text{-- (3-35)}$$

so that (3-31) does indeed represent the absolute derivative of $\dot{q}$ along the curve in Finsler space to which it is a tangent vector. It also follows from (3-35) that the geodesics (3-31) are obtained by parallel displacement of the tangent vector $\dot{q}$, exactly as in the Riemannian case.

Returning now to the Euler-Lagrange equations (3-27) for the homogeneous problem, we can rewrite them in terms of the $p_A$ using the expression (3-23b):

$$L \left\{ \frac{\partial L}{\partial q^A} - \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}^A} \right] \right\} = 0$$

$$\Rightarrow \quad L \frac{\partial L}{\partial q^A} - \frac{d}{d\tau} \left[ L \frac{\partial L}{\partial \dot{q}^A} \right] + \frac{dL}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}^A} \right] = 0$$

$$\Rightarrow \quad \dot{p}_A \equiv \frac{dp_A}{d\tau} = L \frac{\partial L}{\partial q^A} + \frac{dL}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}^A} \right] \quad . \qquad \text{-- (3-36)}$$

Now, using (3-24) and making again the special choice of parameter $\tau=w$, we have $H \equiv L \equiv 1$, so that we obtain

$$\dot{p}_A = -\partial H / \partial q^A \quad .$$

This combines with the relations (3-25) to give the homogeneous counterpart of the canonical equations _for the specific choice_ $\tau=w$:

$$\dot{q}^A = \partial H / \partial p_A \quad ; \quad \dot{p}_A = -\partial H / \partial q^A \quad . \qquad \text{-- (3-37)}$$

We should perhaps draw attention here to the fact that Rund (1966) gives an alternative pair of equations to (3-37), including a sign ambiguity in $\dot{q}$ and $\dot{p}$ according as H is positive or negative (the possibility L=0 is excluded). This arises from the fact that Rund permits negative values of L, so the normalisation condition $H(q,p)=1$ must be replaced by the condition $|H(q,p)|=1$.

However, the restriction to positive Lagrangians (and Hamiltonians) will suffice for our purposes in this chapter, and has the advantage of eliminating this sign ambiguity.

Using the canonical equations for a general parameter $\tau$ we can now find the homogeneous counterpart of the Poisson bracket (1-28). Let $F(q,p)$ be some $C^1$ function of the canonical variables $(q,p)$ and consider its derivative with respect to $\tau$ along some extremal (geodesic). We have, using (3-23a) and (3-36),

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial q^A} \dot{q}^A + \frac{\partial F}{\partial p_A} \dot{p}_A$$

$$= \frac{\partial F}{\partial q^A} \left[ H\frac{\partial H}{\partial p_A} \right] + \frac{\partial F}{\partial p_A} \left[ L\frac{\partial L}{\partial q^A} + \frac{dL}{d\tau}\frac{\partial L}{\partial \dot{q}^A} \right]$$

$$= H\left[ \frac{\partial F}{\partial q^A}\frac{\partial H}{\partial p_A} - \frac{\partial H}{\partial q^A}\frac{\partial F}{\partial p_A} \right] + \frac{p_A}{H} \cdot \frac{dH}{d\tau}\frac{\partial F}{\partial p_A}$$

along an extremal. Using the definition

$$\{F,G\} \equiv \frac{\partial F}{\partial q^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial q^A} \qquad \text{-- (3-38)}$$

of the Poisson bracket (PB) of two functions F and G, the expression for $dF/d\tau$ now becomes

$$\frac{dF}{d\tau} = H\{F,H\} + p_A \frac{\partial F}{\partial p_A} \cdot H^{-1}\frac{dH}{d\tau} \quad . \qquad \text{-- (3-39)}$$

Applying the usual normalisation condition $H(q,p)=1$ this becomes finally

$$dF/dw = \{F,H\} \quad , \qquad \text{-- (3-40)}$$

illustrating once again that the choice of parameter $\tau=w$ ensures that the homogeneous formalism coincides with that of the classical (nonhomogeneous) canonical formalism of chapter 1.


## 3.3 A modification of Rund's theory

We now explore a possible simplification of the theory presented above and developed by Rund (1959;66); the resulting modified theory is a formalised version of a relativistic

Hamiltonian formalism proposed in a number of textbooks, for instance in Goldstein (1981). This simplification arises mainly from an attempt to develop the theory of canonical transformations within Rund's formalism, where it turns out to be more convenient to study the invariance of the integral of $L^2$, rather than the action integral itself. However $L^2$ is no longer homogeneous of degree 1 in $\dot{q}$, but of degree 2. Because of this we now look back more closely at the assumptions of parameter independence leading to the homogeneity condition (3-5).

In assumption (3.1) Rund requires the action integral (3-3) to be invariant under arbitrary parameter transformations, but this assumption may be questioned on two counts. Firstly we are not in fact concerned with the actual <u>value</u> of the action integral, but simply its first variation, or equivalently the corresponding Euler-Lagrange equations. Secondly the general type of parameter transformation considered by Rund is not altogether appropriate to the relativistic situation. General relativity is based on an affine geometry of geodesics on which the parameter is defined up to linear transformations. Consequently we shall try restricting our attention to affine parameters $\sigma, \tau$ on the possible trajectories, where $\sigma$ and $\tau$ are related by transformations of the form

$$\sigma = a\tau + b \qquad \qquad \text{-- (3-41)}$$

and a,b are constants. These transformations are characterised by the fact that $d\sigma/d\tau$ is a constant - a fact which will now be seen to be of some importance.

The above two considerations suggest that we consider a far less stringent situation than that of Rund's theory, embodied in the following.

<u>ASSUMPTION (3.4)</u> - We assume that the action principle:

$$\delta \int_C L(q,\dot{q}) d\tau = 0$$

for extremals $C:q=q(\tau)$ is invariant under all (non-singular) affine transformations of the parameter $\tau$.

Performing the affine parameter transformation, assumption (3.4) becomes

$$\delta \int_C L(q,\dot{q}) d\tau = 0 \iff \delta \int_C L(q,q') d\sigma = 0 \quad ,$$

where a dash signifies differentiation with respect to $\sigma$ and a dot differentiation with respect to $\tau$. This is satisfied if L is positively homogeneous of any degree n in $\dot{q}$, where n is a positive integer, since then

$$\delta \int L(q,\dot{q}) d\tau = \delta \int L(q,q'\dot{\sigma}) d\sigma / \dot{\sigma}$$

$$= \delta \int L(q,q') \dot{\sigma}^{n-1} d\sigma$$

$$= \dot{\sigma}^{n-1} \delta \int L(q,q') d\sigma \quad ,$$

$\sigma$ being a constant for the affine parameter transformations under consideration. Thus the requirement that the action <u>principle</u> (as opposed to the action integral) be <u>affine</u> parameter independent can be satisfied by the condition that L be positively homogeneous of <u>any</u> degree in $\dot{q}$.

Since we are now free to choose the degree of homogeneity of L, we make the specific choice $n=2$ and define the quantities

$$g_{AB}(q,\dot{q}) = \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \quad , \qquad \qquad -- (3-42)$$

which form the components of a rank 2 symmetric covariant tensor. From Euler's theorem on homogeneous functions we have

$$\dot{q}^A \partial L / \partial \dot{q}^A = 2L \quad , \qquad \qquad -- (3-43)$$

and since $\partial L / \partial \dot{q}^A$ must be homogeneous of degree 1 in $\dot{q}$ we must also have the following identity:

$$\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \dot{q}^B = \frac{\partial L}{\partial \dot{q}^A} \qquad \qquad -- (3-44)$$

$$= g_{AB} \dot{q}^B \quad .$$

Finally, by substituting (3-44) into (3-43), we find

$$g_{AB} \dot{q}^A \dot{q}^B = 2L \quad .$$

We see that the situation developing here is very close to the theory of Rund, with the difference that L no longer represents the metric function on M, but rather half its square:

$$L(q,\dot{q}) = 1/2 \cdot g_{AB}(q,\dot{q}) \dot{q}^A \dot{q}^B \quad . \qquad \text{-- (3-45)}$$

Again we can define the length of a curve $C:q=q(\tau)$ in M as the integral

$$w \equiv \int_C \sqrt{|2L(q,dq)|} = \int_C \sqrt{|2L(q,\dot{q})|} d\tau \quad . \qquad \text{-- (3-46)}$$

If we now specify that w is to be one possible parametrisation of C, this will ensure that all parameters obtained from w by the transformation (3-41) will in fact be affine in the usual sense.

Again we shall assume that the metric $g_{AB}$ is nondegenerate, and we note that this key condition now corresponds exactly to the assumption (1-12) of the classical theory. By virtue of this condition we can associate with each contravariant vector $\dot{q} \varepsilon T_q$ a unique covariant vector $p \varepsilon T_q^*$ with components

$$p_A = g_{AB}(q,\dot{q}) \dot{q}^B \quad , \qquad \text{-- (3-47)}$$

where again the directional argument of the $g_{AB}$ must coincide with the vector $\dot{q}$ under consideration. By an exact repeat of the arguments of section 3.2 we find that we can invert (3-47) explicitly to obtain the $\dot{q}^A$ :

$$\left. \begin{array}{l} \dot{q}^A = g^{AB}(q,p) p_B \quad ; \\ g^{AB}(q,p) g_{BC}(q,\dot{q}) = \delta^A_C \quad , \end{array} \right\} \qquad \text{-- (3-48)}$$

whenever $\dot{q}$ and p are related by (3-47). We define the Hamiltonian by analogy with the relation (3-45):

$$H(q,p) = 1/2 \cdot g^{AB}(q,p) p_A p_B \quad , \qquad \text{-- (3-49)}$$

and by substituting into this from (3-47) we arrive at the identity

$$H(q,p) \equiv L(q,\dot{q}) \quad . \qquad \qquad \text{-- } (3-50)$$

Because of this identity we can write (3-45) in the form

$$H(q,p) = \dot{q}^A p_A - L(q,\dot{q}) \quad , \qquad \qquad \text{-- } (3-51)$$

which is simply the Legendre transformation corresponding to the change of variable implied by (3-47):

$$p_A = \partial L / \partial \dot{q}^A \quad . \qquad \qquad \text{-- } (3-52a)$$

The inverse equation $\quad \dot{q}^A = \partial H / \partial p_A \qquad \qquad$ -- (3-52b)

follows in precisely the same way as (3-23a) of Rund's theory, and as before we obtain the relations

$$g^{AB}(q,p) = \frac{\partial^2 H}{\partial \dot{q}^A \partial \dot{q}^B} \quad ; \qquad \qquad \text{-- } (3-53)$$

$$\partial H / \partial q^A = -\partial L / \partial q^A \quad . \qquad \qquad \text{-- } (3-54)$$

Note that the key equations (3-52) are already in the classical canonical form for the general affine parameter $\tau$: the specific choice of parameter $\tau = w$ is no longer required to produce the canonical form. Indeed our original interest in the affine parameter approach arose from a desire to eradicate the factor L from the expression (3-23b) to bring it into the more canonical form (3-52a). We still have the desirable correspondence between the geometrical and dynamical viewpoints, characterised by (3-47) and (3-52) respectively, but now the dynamical equations (3-52) and (3-54) give a canonical interpretation for any affine parameter.

The problem of finding the extremals with the second degree Lagrangian is essentially the same as that of section 3.2, but with the difference that we no longer have the straightforward geometrical interpretation of the integral which was possible in Rund's theory. Since any affine parameter $\tau$ may be obtained from the arclength parameter w by a linear transformation (3-41), we see that

$$\delta \int L(q,\dot{q})d\tau = \delta \int 1/2 \cdot g_{AB} \frac{dq^A}{dw} \frac{dq^B}{dw} \dot{w}^2 \frac{dw}{w}$$

$$= 1/2 \cdot \dot{w} \, \delta \int dw \quad,$$

so that the variational condition

$$\delta I \equiv \delta \int L(q,\dot{q})d\tau = 0 \qquad \qquad -- (3-55)$$

is still equivalent to the geodesic condition

$$\delta \int dw = 0$$

despite the fact that the action can no longer be identified with the arclength w. The extremals are still the (non-null) geodesics of the space.

By the usual argument we obtain from (3-55) the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = 0 \qquad \qquad -- (3-56)$$

for an arbitrary affine parameter $\tau$ on the extremal. All the definitions (3-29), (3-32) and (3-33) carry over into the modified theory provided every occurrence of $1/2 \cdot L^2$ in Rund's theory is replaced by L in the new equations. Hence by the same argument as before we can obtain the geodesic equations

$$\frac{\delta \dot{q}^A}{\delta \tau} \equiv \frac{d^2 q^A}{d\tau^2} + \gamma^A_{BC} \frac{dq^B}{d\tau} \frac{dq^C}{d\tau} = 0 \quad, \qquad -- (3-57)$$

which now hold for the general affine parameter $\tau$. In particular we once again find that

$$\left\{\begin{matrix} A \\ B \ C \end{matrix}\right\}(q) = \gamma^A_{BC}(q)$$

when the metric is Riemannian, and (3-57) again represents the absolute derivative of $\dot{q}$ along the extremal. The passage from the Euler-Lagrange equations to Hamilton's equations is now immediate as in the classical theory. Collecting together (3-52), (3-54) and (3-56) we arrive at the canonical equations of motion:

$$\dot{q}^A = \partial H/\partial p_A \quad; \quad \dot{p}_A = -\partial H/\partial q^A \quad. \qquad -- (3-58)$$

We have now developed a formalism based on a Lagrangian

homogeneous of degree two in the velocities to the same level as our exposition of Rund's theory in the previous section. The slightly more general nature of the parameter involved in the canonical equations (3-58) seems to recommend the second degree theory, but the final test of any theory must be its practical application to physical problems. For this reason we now compare these two theories in the case of a relativistic charged particle in an electromagnetic field.


## 3.4 Relativistic particle mechanics

In this section we restrict the general considerations above to the specific case of relativistic particle mechanics in which a single particle is describe by the four coordinates $z^a(s)$, where s is the proper time defined by $-ds^2 = g_{ab} dz^a dz^b$. The components of the 4-velocity of the particle are $u^a = dz^a/ds$, amongst which holds the following relation:

$$u^a u_a = -1 \quad . \qquad\qquad -- \ (3\text{-}59)$$

But this immediately poses a problem for the formulation of a relativistic Hamiltonian theory. For in the calculus of variations it is assumed that the coordinates and velocities appearing in L are independent, which in (3-59) is certainly not the case. One way of dealing with this problem is via the theory of constraints, but this suffers from a similar disadvantage to that of the field theory of chapter 2, since it involves picking out a preferred coordinate to be dependent on the others and so is suited only to special relativity. An alternative approach is to demand that the Lagrangian be at least affine parameter invariant. For an arbitrary affine parameter $\tau$ the quantities $\dot{z}^a = dz^a/d\tau$ are not related by (3-59), but rather by

$$\dot{z}^a \dot{z}_a = -(ds/d\tau)^2 = -k^2 = \text{const.} \qquad -- \ (3\text{-}60)$$

The constant k is dependent upon which affine parameter we choose, but until we have specified our choice of parameter it remains undetermined. Thus the demand of _affine_ parameter invariance gives us precisely sufficient freedom for the $\dot{z}^a$ to be independent, but leaves no leeway for variation of the parameter after the four velocity components have initially been given. This is a very satisfactory state of affairs. Note that in Rund's theory this problem does not arise, since he does not demand that $\tau$ be affine. Since Rund's parameter is completely arbitrary the velocities obey no relation of the form (3-60), ensuring that a properly set variational problem ensues.

However, there is still a second obstruction to the development of a variational formalism. The requirement of affine parameter invariance specifies only that the Lagrangian be homogeneous of degree n in $\dot{z}$. Thus from (3-60) we may multiply L by the factor $(-1/k^2 . \dot{z}^a \dot{z}_a )^m$ without altering the value of L or the action integral. On the other hand such a multiplication would certainly affect the Euler-Lagrange equations of motion. The choice of any specific degree of homogeneity for the Lagrangian eliminates this problem, since any homogeneity properties would be destroyed by multiplication with powers of $(-k^{-2} . \dot{z}^a \dot{z}^a )$. This means that both Rund's first degree theory and the modified second degree theory ensure uniqueness of the Lagrangian. Having settled these difficulties we now look at the problem of the relativistic particle and its treatment according to the two theories presented above. Since we have already seen that the Euler-Lagrange equations are in both cases capable of defining geodesics in general, curved, Finsler space, we will now content ourselves with the simpler case of Minkowski space, where the $g_{ab}$ are constants.

i. The second degree Lagrangian

In the second degree theory the free particle is described by the following Lagrangian:

$$L = m/2 . \dot{z}^a \dot{z}_a \quad . \qquad \qquad -- (3-61)$$

This leads to the momenta

$$p_a = \partial L / \partial \dot{z}^a = m \dot{z}_a$$

and the first canonical equation

$$\dot{p}_a = -\partial H / \partial z^a = 0 \quad ,$$

as expected for the free particle. The second canonical equation then yields the trivial equation

$$\dot{z}^a = \partial H / \partial p_a = 1/2m . \partial (g^{bc} p_b p_c ) / \partial p_a$$
$$= 1/m . g^{ab} p_b \quad .$$

These results for the free particle are satisfactory, but this was to be expected since the second degree Lagrangian (3-61) is of precisely the same form as the pure kinetic energy Lagrangian $1/2 . mv^2$ of classical mechanics. Indeed when we study the second degree theory in the practical context of relativistic mechanics we find that it is simply a naive wholesale transfer of classical expressions into a covariant phrasing (see for example Goldstein, 1981). But the crucial failing in this theory lies in the very triviality of the above example. Having decided upon a Lagrangian which is homogeneous of degree two in the velocities we cannot now add interaction terms of any degree other than two. Hence the simple case of a charged particle in an electromagnetic field proves the theory to be unusable, since there does not seem to exist a satisfactory interaction term based on the electromagnetic potentials $A_a$ which is homogeneous of degree two. It seems that despite certain desirable features this theory must be abandoned. Yet it must be remembered that the choice of a second degree Lagrangian was prompted by the wish to remove the awkward

occurrence of the factor L in Rund's expression (3-23b). If we are going to abandon the affine parameter theory in favour of Rund's then we must accept the frequent L's scattered throughout our equations.

## ii. The first degree Lagrangian

In Rund's theory the general problem of the relativistic particle is described by the Lagrangian

$$L = L_0 + L_I \quad , \qquad \qquad -- \ (3-62)$$

where $L_0$ is the free particle Lagrangian

$$L_0(z,dz) = -m(-g_{ab}dz^a dz^b)^{1/2} \qquad \qquad -- \ (3-63)$$

and $L_I$ is the interaction Lagrangian. Rund (1966) has shown that the variational problem defined by (3-63) in fact yields maximum values of the action integral for extremals $z(\tau)$, due to the hyperbolic nature of the Lorentz manifold. The interaction Lagrangian depends on the type of external field considered, for example:

a. The scalar field depending on position only is characterised by the Lagrangian

$$L_I = k(-g_{ab}\dot{z}^a \dot{z}^b)^{1/2} \varphi(z) \quad ,$$

where k is some coupling constant.

b. A vector field such as the electromagnetic field may be derived from the Lagrangian

$$L_I = \varphi_a(z)\dot{z}^a \quad .$$

c. A tensor field described by a symmetric tensor $\varphi_{ab}(x)$ again needs the factor $(-g_{ab}\dot{z}^a \dot{z}^b)^{1/2}$ inserting to restore the correct homogeneity:

$$L_I = k(-g_{ab}\dot{z}^a \dot{z}^b)^{-1/2} \varphi_{cd}(z)\dot{z}^c \dot{z}^d \quad .$$

As a concrete example of the above we look now at the extremals of a charged particle of mass m and charge e moving in Minkowski space in the presence of an electromagnetic potential

$A_a(z)$. The Lagrangian we shall use is of the type b:

$$L = L_0 + L_I$$

$$= -m(-g_{ab}\dot{z}^a\dot{z}^b)^{1/2} + eA_a\dot{z}^a \quad , \quad \text{-- (3-64)}$$

which yields the canonical momenta

$$p_a = L\partial L/\partial\dot{z}^a$$

$$= L[(-g_{ab}\dot{z}^a\dot{z}^b)^{-1/2}mg_{ab}\dot{z}^b + eA_a]. \quad \text{-- (3-65)}$$

Multiplying (3-65) by the contravariant tensor $g^{ac}$ we may rearrange it to the form

$$\dot{z}^a = \frac{(-g_{cd}\dot{z}^c\dot{z}^d)^{1/2}}{mH} g^{ab}(p_b + eHA_b) \quad , \quad \text{-- (3-66)}$$

which gives $\dot{z}^a$ in terms of $p_a$. Note that $g_{ab}$ is therefore definitely __not__ the Finsler space metric, since the part played by $g_{ab}$ in (3-65) and (3-66) is very different from that played by the metric in (3-15) and (3-18). The metric in Finsler space may be obtained directly from (3-64):

$$h_{ab} \equiv \frac{1}{2}\frac{\partial^2 L^2}{\partial\dot{z}^a \partial\dot{z}^b} = -m^2 g_{ab} + e^2 A_a A_b + meA_c\dot{z}^c\dot{z}_a\dot{z}_b(-\dot{z}^d\dot{z}_d)^{-3/2}$$

$$+ (-\dot{z}^c\dot{z}_c)^{-1/2}me(A_a\dot{z}_b + \dot{z}_a A_b)$$

$$+ meA_c A^c g_{ab} \quad , \quad \text{-- (3-67)}$$

a complicated expression depending on both z and ż. Life can be made simpler by taking the __Riemannian__ modulus of (3-66):

$$g_{ab}\dot{z}^a\dot{z}^b = \frac{(-g_{cd}\dot{z}^c\dot{z}^d)}{m^2 H^2}g_{ab}g^{ae}g^{bf}(p_e + eHA_e)(p_f + eHA_f)$$

$$\Rightarrow \quad H^2 = -1/m^2 . g^{ab}(p_a + eHA_a)(p_b + eHA_b) \quad . \quad \text{-- (3-68)}$$

Here we see a pattern arising in which the 'free' momenta $p_a$ are replaced by the 'interacting' momenta $p_a + eHA_a$, yet the appearance of the Hamiltonian H on the RHS of (3-68) marks the failure of Rund's theory. The objection to the occurrence of the factor H in (3-23) was purely aesthetic, but its appearance in (3-68) causes a more serious problem. Equation (3-68) does not give an expression for $H^2$, but rather represents a quadratic which must

be solved for H. However, we can see that Rund's theory gives the 'right' answers by studying the Euler-Lagrange equations for the Lagrangian (3-64):

$$\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = 0$$

$$= eA_{a,b}\dot{z}^a - eA_{b,a}\dot{z}^a - \frac{d}{d\tau}\{(-g_{cd}\dot{z}^c\dot{z}^d)^{-1/2}mg_{ab}\dot{z}^a\}.$$

Choosing $\tau=s$, so that $g_{cd}\dot{z}^c\dot{z}^d=-1$ we obtain

$$\ddot{z}_b = eF_{ab}\dot{z}^a \ . \qquad\qquad\qquad -- \ (3-69)$$

This is a vindication of the basis of Rund's theory in first degree homogeneity, but a satisfactory and soluble Hamiltonian formalism is only obtained in the case H=L=1. Thus Rund's theory also suffers from distinct disadvantages: why bother with a parameter-invariant theory when any practical situation demands a highly specific choice of parameter? The theory presented in the next section offers a possible answer to this question.


## 3.5 The theory of Synge

There exists an alternative to Rund's theory which has been developed by Synge (1954;1960). This theory is again based on the idea of a homogeneous Lagrangian function defined on some manifold M, but it avoids the uncomfortable duality of Rund's theory, where the equations of motion are derived from L, whilst defining the momenta in terms of $L^2$ (see (3-23)). Synge keeps in mind at all times the decomposition into the classical Lagrangian, which serves as a guiding influence throughout the theory - at times to the detriment of the relativistic formalism.

We start with the parameter-invariant Lagrangian positively homogeneous of degree 1:

$$L(q,\alpha\dot{q}) = \alpha L(q,\dot{q}) \ ; \quad (\alpha > 0) \ , \qquad\qquad -- \ (3-70)$$

from which are derived the parameter-invariant Euler-Lagrange

equations:

$$\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = 0 \quad . \qquad \text{-- (3-71)}$$

These equations are not all independent, since using the homogeneity of L we find

$$\dot{q}^A \left\{\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right]\right\} = \dot{q}^A \frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\dot{q}^A \frac{\partial L}{\partial \dot{q}^A}\right] + \ddot{q}^A \frac{\partial L}{\partial \dot{q}^A}$$

$$= dL/d\tau - dL/d\tau$$

$$= 0 \quad . \qquad \text{-- (3-72)}$$

Thus we must recognise that the homogeneous formalism adds no new information to the classical formalism, since the number of independent equations of motion is reduced to N-1: precisely the number of classical equations. However, this "slack" is taken up by only demanding that we can determine the ratios of the velocities $\dot{q}^1 : \dot{q}^2 : \ldots : \dot{q}^N$. This is reminiscent of the state of affairs in relativistic mechanics, where $u^a u_a = -1$, and only the ratios $u^1 : u^2 : u^3 : u^4$ need be solved for.

Now since we may use any parameter for the description of mechanics, one possible choice is $\tau = q^N$. Denoting the action by

$$I = \int L d\tau \quad , \qquad \text{-- (3-73)}$$

we then have
$$dI = L d\tau$$

$$= L(q^A, q^{\cdot 1}, \ldots, q^{\cdot N-1}, 1) dq^N \quad ,$$

where a dash represents differentiation with respect to $q^N$:

$$q^{\cdot A} = \dot{q}^A / \dot{q}^N \quad .$$

Defining the function $L^*$ by

$$\left.\begin{array}{l} L^*(q^1, \ldots, q^{N-1}, t, q^{\cdot 1}, \ldots, q^{\cdot N-1}) \equiv L(q^A, q^{\cdot 1}, \ldots, q^{\cdot N-1}, 1); \\ \qquad\qquad t \equiv q^N \end{array}\right\} \quad \text{-- (3-74)}$$

we obtain the <u>classical decomposition</u> of (3-73), which now becomes

$$I = \int L^* dt \quad . \qquad \text{-- (3-75)}$$

We shall call $L^*$ the <u>classical Lagrangian</u>. The functions L, $L^*$

determine one another, since by equating the differentials of (3-73) and (3-75) we have

$$L(q,\dot{q}) = L^*(q^A, q^{\cdot 1}, \ldots, q^{\cdot N-1})\dot{t}$$
$$= L^*(q^A, \dot{q}^1/\dot{q}^N, \ldots, \dot{q}^{N-1}/\dot{q}^N)\dot{q}^N , \qquad -- (3-76)$$

which is of homogeneity 1, as required. To find the correspondence between the partial derivatives of L and $L^*$ we vary $q,\dot{q}$ in (3-76):

$$\frac{\partial L}{\partial q^A}\delta q^A + \frac{\partial L}{\partial \dot{q}^A}\delta \dot{q}^A = \dot{t}\left[\frac{\partial L^*}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L^*}{\partial t}\delta t + \frac{\partial L^*}{\partial q^{\cdot\alpha}}\delta q^{\cdot\alpha}\right] + L^*\delta\dot{t} ,$$

where we have adopted the convention that Greek indices $\alpha, \beta, \ldots$ run from 1 to N-1. But

$$q^{\cdot\alpha} = \dot{q}^\alpha/\dot{t} \quad ; \quad \delta q^{\cdot\alpha} = \delta\dot{q}^a/\dot{t} - \dot{q}^\alpha\delta\dot{t}/\dot{t}^2 ,$$

so by equating coefficients we find

$$\left.\begin{array}{ll} \dfrac{\partial L}{\partial q^\alpha} = \dot{t}\dfrac{\partial L^*}{\partial q^\alpha} & ; \quad \dfrac{\partial L}{\partial q^N} = \dot{t}\dfrac{\partial L^*}{\partial t} \quad ; \\[2ex] \dfrac{\partial L}{\partial \dot{q}^\alpha} = \dfrac{\partial L^*}{\partial q^{\cdot\alpha}} & ; \quad \dfrac{\partial L}{\partial \dot{q}^N} = L^* - q^{\cdot\alpha}\dfrac{\partial L^*}{\partial q^{\cdot\alpha}} \quad . \end{array}\right\} \qquad -- (3-77)$$

Substituting these into the equations of motion (3-71) yields the usual classical equations of motion for the first N-1 components:

$$\frac{\partial L^*}{\partial q^\alpha} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial q^{\cdot\alpha}}\right] = 0 ,$$

while the N-th equation gives the corresponding relation to (3-72):

$$q^{\cdot\alpha}\left\{\frac{\partial L^*}{\partial q^\alpha} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial q^{\cdot\alpha}}\right]\right\} = 0 .$$

We have constructed above a system which admits at least two equivalent descriptions: the classical description in terms of $L^*$ and the homogeneous description in terms of L. At this stage Synge is able to transfer to a Hamiltonian description by defining the momenta in the usual way:

$$p_A = \partial L/\partial \dot{q}^A . \qquad -- (3-78)$$

Synge's momenta are more natural than Rund's definition (3-23b), but as a result the momenta (3-78) are now homogeneous of degree

zero in the $\dot{q}^A$ and so we cannot invert the transformation (3-78) uniquely, because of the resulting equation (3-8). Given the momenta, all we can ascertain are the N-1 ratios $\dot{q}^1 : \dot{q}^2 : \ldots : \dot{q}^N$. Elimination of these ratios from the equations (3-78) yields an equation which we write

$$\Omega(q^A, p_A) = 0 \; . \qquad\qquad -- \; (3-79)$$

Equation (3-79) is a constraint equation, confining the possible values of p at a given point q of M to an N-1 dimensional hypersurface in the cotangent space $T_q^*$: Synge calls it the _energy equation_, and the function $\Omega$ is the _energy function_. The 2N-1 dimensional hypersurface defined in the cotangent bundle by (3-79) is called the _energy surface_.

Along any curve with parameter $\tau$ and tangent $\dot{q}=dq/d\tau$, the element of action is

$$dI = L(q,\dot{q})d\tau = \dot{q}^A . \partial L/\partial \dot{q}^A . d\tau$$
$$= p_A dq^A \; ,$$

where we have used the homogeneity of L. This expresses the element of action entirely in terms of the canonical variables (q,p), and therefore provides a form of the action principle suitable for the canonical formalism - this is known as the _Pfaffian_ form of the element of action. We require that the action integral be stationary with respect to all variations of the path of integration for which the endpoints are kept fixed:

$$\delta I = \delta \int p_A dq^A$$
$$= \int \{\delta p_A dq^A + p_A \delta dq^A\}$$
$$= \int \{\delta p_A dq^A - \delta q^A dp_A\} = 0 \; . \qquad -- \; (3-80)$$

Here we have performed the usual trick of integrating by parts and then eliminating the contribution from the endpoints. In addition to the physical constraint of the action principle (3-80) we suppose that physical paths are also constrained by the

energy equation (3-79). By making use of the method of Lagrange

multipliers (see Goldstein,1981) we therefore have

$$\delta \int (L - \lambda\Omega)d\tau = 0 \quad ,$$

where $\lambda$ is a Lagrange multiplier. From this we obtain

$$\delta p_A dq^A - \delta q^A dp_A - \lambda d\tau \left[ \frac{\partial\Omega}{\partial q^A} \delta q^A + \frac{\partial\Omega}{\partial p_A} \delta p_A \right] = 0$$

$$\Rightarrow \qquad \left. \begin{array}{l} \dot{p}_A = -\lambda\partial\Omega/\partial q^A \quad ; \\[2mm] \dot{q}^A = \lambda\partial\Omega/\partial p_A \quad . \end{array} \right\} \qquad -- (3-81)$$

These form Synge's version of the canonical equations of motion.

It must be remembered that they do not hold throughout the cotan-

gent bundle, but only on the energy surface, and so must _always_

be supplemented the energy equation $\Omega(q,p)=0$. By transferring to

the special choice of parameter $dw=\lambda d\tau$ these are reduced to the

conventional form

$$\left. \begin{array}{l} dp_A/dw = -\partial\Omega/\partial q^A \quad ; \\[2mm] dq^A/dw = \partial\Omega/\partial p_A \quad . \end{array} \right\} \qquad -- (3-82)$$

A curve with a vector p attached at each point which satisfies

(3-81) or (3-82) is called a _trajectory_.

The special parameter w cannot be altered once the energy

function $\Omega(q,p)$ has been given, for dw is determined by the

element of action:

$$dI = p_A dq^A = p_A dw.dq^A/dw = p_A dw.\partial\Omega/\partial p_A \quad , \qquad -- (3-83)$$

but the element dw clearly has different values for different

assignments of the energy function. If we have two different

energy equations $\Omega=0$, $\bar{\Omega}=0$, both expressing the same relationship,

then the corresponding parameters satisfy

$$d\bar{w}/dw = d\Omega/d\bar{\Omega} \quad .$$

If we choose $\Omega$ in such a way that $(\Omega+1)$ is homogeneous of degree

1 in the $p_A$, then

$$p_A \partial\Omega/\partial p_A = p_A \partial(\Omega+1)/\partial p_A = (\Omega+1) = 1 \quad ,$$

so that from (3-83) $w$ is simply the action $I$ of the trajectory under consideration. This indicates a relationship with Rund's more geometrical approach.

In the case of Synge's Lagrangian dynamics we were able to give a straightforward decomposition of the homogeneous theory involving $L$ into the classical theory involving $L^*$. The same is possible for the Hamiltonian formalism developed here. Making use of the decomposition equations (3-77) we see that

$$\left.\begin{array}{l} p_\alpha = \partial L/\partial \dot{q}^\alpha = \partial L^*/\partial \dot{q}^\alpha = p^*_\alpha \ ; \\[2mm] p_N = \partial L/\partial \dot{q}^N = L^* - \dot{q}^\alpha \partial L^*/\partial \dot{q}^\alpha = -H^* \ , \end{array}\right\} \qquad \text{-- (3-84)}$$

where $p^*_\alpha, H^*$ are the classical momenta and Hamiltonian respectively. In addition we can (in principle) solve the energy equation (3-79) for one of the momenta, say $p_N$, to give

$$p_N + \omega(q^A, p_\alpha) = 0 \ ,$$

or $\qquad H^* = \omega(q^\alpha, t, p_\alpha),$ $\qquad\qquad$ -- (3-85)

thus expressing the classical Hamiltonian in terms of $(q^\alpha, t, p_\alpha)$, as is the case in the classical theory. The action principle (3-80) now takes the classical form

$$\delta I = \int\{\delta p_A dq^A - \delta q^A dp_A\}$$

$$= \int\{\delta p_\alpha dq^\alpha - \delta q^\alpha dp_\alpha - \delta H^* dt + \delta t dH^*\}$$

$$= 0 \ .$$

But from (3-85),

$$\delta H^* = \frac{\partial H^*}{\partial q^\alpha}\delta q^\alpha + \frac{\partial H^*}{\partial t}\delta t + \frac{\partial H^*}{\partial p_\alpha}\delta p_\alpha \ ,$$

so $\qquad \delta I = \int\left\{\delta p_\alpha\left[dq^\alpha - \frac{\partial H^*}{\partial p_\alpha}dt\right] - \delta q^\alpha\left[dp_\alpha + \frac{\partial H^*}{\partial q^\alpha}dt\right]\right.$

$$\left. + \delta t\left[dH^* - \frac{\partial H^*}{\partial t}dt\right]\right\} \ .$$

The variations $\delta q^\alpha, \delta t, \delta p_\alpha$ being arbitrary, the action principle

therefore yields Hamilton's equations for the trajectories:

$$q'^{\alpha} = \partial H^*/\partial p_{\alpha} \quad ; \quad p'_{\alpha} = \partial H^*/\partial q^{\alpha} \qquad -- (3-86)$$

and in addition we obtain the equation

$$dH^*/dt = \partial H^*/\partial t \quad ,$$

which tells us that the time-dependence of the classical Hamiltonian is simply its _explicit_ time-dependence. We have seen in (1-31) that this fact is a consequence of the classical equations (3-86), so that these equations present themselves clearly as a system of 2N-2 first-order equations. However, the homogeneous system (3-82) apparently consists of 2N equations. The reason for this is that we have first divided the equations through by $dq^N/dw$ to make t the independent parameter, and then applied the energy equation to obtain the constraint (3-85). Each of these steps reduces the number of independent equations by one, to yield the classical (2N-2) order system given in (3-86).

In classical dynamics the Lagrangian and Hamiltonian formalisms represent two equivalent descriptions of the same state of affairs, and we wish now to see whether the same holds for the homogeneous formalisms. We have already shown how to move from the homogeneous Lagrangian system to the Hamiltonian setup described above, so it now remains to be proven that, given some energy equation

$$\Omega(q,p) = 0 \quad , \qquad\qquad -- (3-87)$$

we can transform to a homogeneous Lagrangian which describes the same physical situation. On an arbitrary curve $q^A = q^A(\tau)$ the velocities $\dot{q}^A$ are automatically determined by the equations of the curve. The momenta $p_A$ may be assigned arbitrarily except that they must obey the energy equation, but Synge restricts them somewhat further to what he calls the _natural_ momenta by imposing the equations

$$\dot{q}^A = \lambda \partial \Omega / \partial p_A \quad , \qquad \qquad -- \quad (3-88)$$

where $\lambda$ is some undetermined multiplier. This restriction is simply the requirement that the momenta satisfy the second of the canonical equations (3-81). We can now solve (3-87) and (3-88) for $p_A$ and $\lambda$ as functions of the $q^A, \dot{q}^A$ and define a Lagrangian

$$L(q, \dot{q}) = p_A \dot{q}^A \quad . \qquad \qquad -- \quad (3-89)$$

The element of action may then be written

$$dI = p_A dq^A = p_A \dot{q}^A d\tau = L d\tau$$

to give the Lagrangian form (3-73) of the action principle. Finally, it is clear that if the equations (3-87), (3-88) and (3-89) are satisfied by a certain set of values $(L, q^A, \lambda)$ then they are also satisfied by the set $(kL, k\dot{q}^A, k\lambda)$ for any number k, so this ensures that the Lagrangian (3-89) is homogeneous of degree unity in the $\dot{q}^A$. Thus we have arrived, as required, at the Lagrangian formalism we started with at the beginning of this section.

In accordance with the Hamilton-Jacobi theory of classical mechanics we now define the following function on the manifold M which is important in the dynamical development of a system in Synge's theory.

**Definition (3.5):** Let $\Gamma$ be a trajectory connecting the point $Q \varepsilon M$ to the point q. Then the _2-point character-istic function_ S(Q,q) is defined as the action measured along $\Gamma$ from Q to q, ie:

$$S(Q,q) \equiv \int_\Gamma L d\tau = \int_\Gamma p_A dq^A \quad .$$

Since the 2-point function is a global function it may well be that the trajectory $\Gamma$ crosses from one coordinate patch of M into another. In this case we need to distinguish between functional symbols at Q and q, which we do by use of a bar:

$$L(q,\dot{q}) \quad ; \quad \overline{L}(Q,\dot{Q}) \quad ; \quad \Omega(q,p) = 0 \quad ; \quad \overline{\Omega}(Q,P) = 0 \quad .$$

Note that the function S is not necessarily single-valued, since there may be more than one trajectory connecting Q to q. However, in practice we are interested in a given arc of a known trajectory. In the cases where S is potentially multiple-valued we shall always assume that some prescription has been given to specify the particular trajectory along which S is to be measured. Thus $S(Q,q)$ will <u>always</u> be assumed single-valued.

We can obtain an expression for the variation of $S(Q,q)$ by repeating the derivation of (3-80) above, remembering that we are now also varying the endpoints of $\Gamma$:

$$\delta S = \int_\Gamma \{\delta p_A dq^A + p_A \delta dq^A\}$$

$$= [p_A \delta q^A]_1^2 + \int_\Gamma \{\delta p_A dq^A - \delta q^A dp_A\} \quad ,$$

where the indices 1 and 2 label the endpoints of $\Gamma$ at $\tau_1, \tau_2$. Since $\Gamma$ is defined to be a trajectory the integral expression here vanishes, so that we are left with

$$\delta S = [p_A \delta q^A]_1^2$$

$$= p_A \delta q^A - P_A \delta Q^A \quad . \qquad\qquad \text{-- (3-90)}$$

If the variations $\delta q^A, \delta Q^A$ are independent this gives

$$\partial S/\partial q^A = p_A \quad ; \quad \partial S/\partial Q^A = -P_A \quad , \qquad \text{-- (3-91)}$$

which on account of the energy equation (3-87) leads to Synge's form of the H-J equation for the case when the initial point is held fixed:

$$\Omega(q, \partial S/\partial q) = 0 \quad . \qquad\qquad \text{-- (3-92)}$$

These equations may also be expressed in classical form, remembering that

$$q^N = t \quad ; \quad p_N = -H^* \quad .$$

By (3-90) we have

$$\delta S = p_\alpha \delta q^\alpha - H^* \delta t - P_\alpha \delta Q^\alpha + \overline{H}^* \delta T$$

$$\Rightarrow \quad \partial S/\partial q^{\alpha} = p_{\alpha} \quad ; \quad \partial S/\partial t = -H^*(q^{\alpha}, p_{\alpha}, t) \quad ; \left.\begin{array}{c} \\ \\ \end{array}\right\} \quad -- \quad (3\text{-}93)$$
$$\partial S/\partial Q^{\alpha} = -P_{\alpha} \quad ; \quad \partial S/\partial T = \overline{H}^*(Q^{\alpha}, P_{\alpha}, T) \quad .$$

Finally, (3-93) leads to the usual classical form (1-21) of the H-J equation (3-92):

$$\partial S/\partial t + H^*(q^{\alpha}, \partial S/\partial q^{\alpha}, t) = 0 \quad . \quad\quad -- \quad (3\text{-}94)$$

Having seen how Synge's theory is developed, we now look at its application to the variational description of relativistic dynamics. First we must note a few facts about the special choice of parameter s, the proper time, which arises naturally in relativistic dynamics. If $\tau = s$ then

$$L(z,\dot{z})d\tau = L(z,dz) = L(z,u)ds \quad ,$$

where $u^a = dz^a/ds$ is the 4-velocity of a particle with coordinates $z^a$. The action is then

$$I = \int L(z,u)ds$$

and Lagrange's equations read

$$\frac{\partial L}{\partial z^a} - \frac{d}{ds}\left[\frac{\partial L}{\partial u^a}\right] = 0 \quad\quad\quad -- \quad (3\text{-}95a)$$

with the special relation

$$u^a u_a = -1 \quad . \quad\quad\quad -- \quad (3\text{-}95b)$$

In relativistic dynamics we can use the 4-velocity form (3-95a) of the Euler-Lagrange equations, but we must be careful not to apply the constraint (3-95b) until after all partial derivatives in (3-95a) have been calculated.

For a single particle of mass m and charge e in an electromagnetic field we assume, as in Rund's theory, that

$$L(z,u) = -m(-g_{ab}u^a u^b)^{1/2} + eA_a u^a \quad , \quad\quad -- \quad (3\text{-}96)$$

where for simplicity we suppose the $g_{ab}$ to be constants. Then

$$p_c = \partial L/\partial u^c = m(-g_{ab}u^a u^b)^{-1/2}g_{cd}u^d + eA_c$$
$$= mu_c + eA_c \quad\quad\quad\quad -- \quad (3\text{-}97)$$

$\Rightarrow$ $\qquad$ $\dot{p}_b = m\dot{u}_b + eA_{b,a}u^a$ .

Also $\qquad$ $\partial L/\partial z^b = eA_{a,b}u^a$ ,

so Lagrange's equations become

$$m\dot{u}^b = eF_{ab}u^a \quad , \qquad\qquad -- (3-98)$$

precisely the required equation of motion. To obtain the energy

equation $\Omega(z,u)=0$ we combine the expression (3-97) for the

momentum with the special condition (3-95b):

$$2\Omega \equiv g^{ab}(p_a - eA_a)(p_b - eA_b) + m^2 = 0 \quad . \quad -- (3-99)$$

Remember that in Synge's theory the energy function is in no way

unique, so that there is an infinite number of choices for $\Omega$;

(3-99) simply happens to be Synge's choice. The canonical

equations are now

$$dz^a/dw = \partial\Omega/\partial p_a = p^a - eA^a \quad ; \qquad\qquad -- (3-100a)$$

$$dp_a/dw = -\partial\Omega/\partial z^a = e(p_b - eA_b)A^b{}_{,a} \quad , \qquad -- (3-100b)$$

of which (3-100a) is the canonical version of (3-97), while

(3-100b) corresponds to (3-98). It seems that Synge's choice of

$\Omega$ is fortunate, since w=s now gives the correct relativistic

equation of motion, but it must be born in mind that this choice

is still essentially arbitrary. Finally, the energy equation

(3-99) together with (3-92) gives us the H-J equation for

relativistic dynamics:

$$g^{ab}\left[\frac{\partial S}{\partial z^a} - eA_a\right]\left[\frac{\partial S}{\partial z^b} - eA_b\right] + m^2 = 0 \quad . \quad -- (3-101)$$

Synge's theory suffers from a number of disadvantages,

notably the ambiguity in the energy function and the need to res-

trict ourselves to the special parameter w. However, we see here

that its answers to the problem of a charged particle in an

electromagnetic field are far more satisfactory than those of the

two theories considered earlier in this chapter. We therefore

feel that it constitutes the most satisfactory existing canonical

formalism to date.

## CHAPTER 4

## HOMOGENEOUS MECHANICS

### 4.1 A synthesis of existing theories

The major feature lacking in Synge's theory, developed in section 3.5, was the geometric viewpoint so clearly brought out in Rund's theory. We saw at the end of chapter 3 that the equations of Synge's theory are more elegant and lead to more convincing results, but the geometric simplicity of Rund's theory cannot be denied. Consequently we shall endeavour in this chapter to approach Synge's theory in a different manner in an attempt to bring out the geometric aspect more explicitly.

Since Synge's Lagrangian is positively homogeneous of degree one in the velocities q̇ we can again construct a Finsler geometry using L(q,dq) as a metric function. In this case all the work on Rund's theory up to and including equation (3-14) can be carried over into the present work, and in particular we can define a metric function

$$g_{AB} = \frac{1}{2} \frac{\partial^2 L^2(q,\dot{q})}{\partial \dot{q}^A \partial \dot{q}^B} \qquad\qquad -- (4-1)$$

satisfying

$$g_{AB}\dot{q}^B = L \, \partial L/\partial \dot{q}^A \quad ; \qquad\qquad -- (4-2a)$$

$$g_{AB}\dot{q}^A \dot{q}^B = L^2 \quad . \qquad\qquad -- (4-2b)$$

Using Synge's definition (3-78) of momentum it follows from (4-2) that

$$p_A \equiv \partial L/\partial \dot{q}^A = 1/L \cdot g_{AB}\dot{q}^B \quad , \qquad\qquad -- (4-3)$$

giving the zero degree homogeneity required of the momenta in Synge's theory.

Now let us look more closely at what is involved in the transformation (4-3). It is clearly singular, since the $p_A$ are

homogeneous of degree zero in the velocities (see equation
(3-8)). From (4-3) we can see why this is: given an arbitrary
velocity q̇, the effect of the transformation is first to divide q̇
by its Finsler magnitude L(q,q̇) to collapse it onto the unit
sphere $S_L$ in $T_q$ (see fig.(4.1)). This sphere is then mapped by
the Finsler space metric $g_{AB}$ across to the corresponding surface
$S_H$ in $T_q^*$. Thus _every_ contravariant vector lying on the same
half-ray as q̇ is mapped to the same momentum $p \epsilon T_q^*$, and p is def-
ined solely by the direction of q̇, not by its magnitude. This
means that the half-rays in $T_q$ form equivalence classes which are
the fibres of the mapping (4-3); the range, or image space, of
the mapping is the surface $S_H \subset T_q^*$. Synge calls the surfaces $S_L$,$S_H$
the Lagrangian and Hamiltonian surfaces respectively, while in
Caratheodory's (1935) terminology they are respectively called
the _indicatrix_ and _figuratrix_. These two surfaces are represent-
ed by the equations

$$S_L : \quad L(q,\dot{q}) = 1 \quad ; \quad \left.\begin{array}{c} \\ \\ \end{array}\right\}$$
$$S_H : \quad g^{AB} v_A v_B = 1 \quad , \qquad\qquad\qquad -- \ (4-4)$$

where $g^{AB}$ is the inverse of $g_{AB}$ and $v \epsilon T_q^*$.



fig.(4.1)
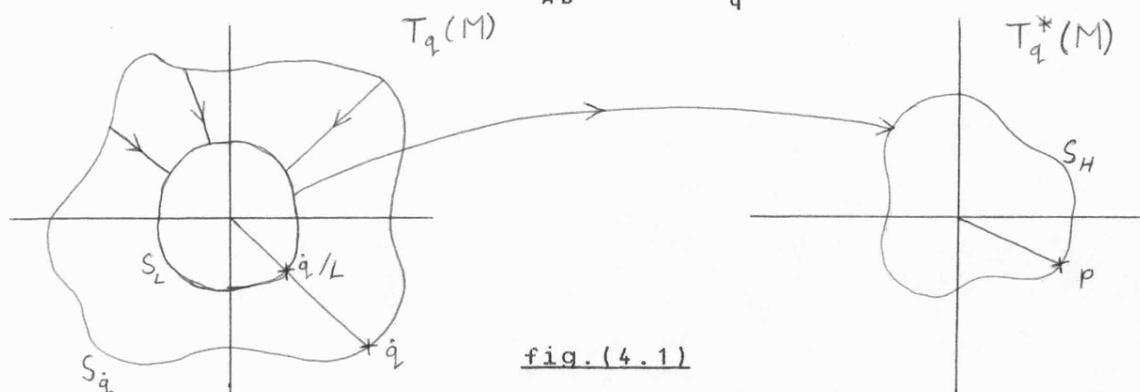
Now since (4-3) splits the tangent space naturally into
equivalence classes, we are essentially interested only in a rep-
resentative member from each class. Our aim will be to define a
bijection between the tangent and cotangent spaces which we can

justifiably call a Legendre transformation, so we wish to pick out a _unique_ member from each half-ray which is to be mapped to p by the transformation. In this way it will be possible to then invert the transformation, returning to the _same_ representative from which we started. Clearly one way of picking out these representatives is by means of the Lagrangian surface $S_L$. This surface cuts across all the equivalence classes and so picks out a member $q$ from each class which is such that $L(q,\dot{q})=1$. We can generalise this method of picking representatives by defining an arbitrary surface $S_{\dot{q}}$ which cuts across all the classes, thus identifying a unique member from each half-ray (see fig.(4.1)). The choice of the surface $S_{\dot{q}}$ is quite irrelevant to the system being described, since no one member of an equivalence class contains any more or less information about the system than any other. However, we shall see that the choice of $S_{\dot{q}}$ has a drastic effect on the way in which the system is to be described: indeed the choice of $S_{\dot{q}}$ _defines_ the language being employed to describe the system. The reason for this lies in the consequences of a particular choice of parameter. If differentiation of the coordinates with respect to a specific parameter $\tau$ leads to a specific velocity vector $\dot{q}$, then clearly a change to the parameter $\tau/2$ will yield the new velocity $2\dot{q}$. That is, a change of parameter moves us from one member of an equivalence class to another. Thus the choice of $S_{\dot{q}}$ for all q defines uniquely the choice of parameter to be used in describing the system. We shall assume the function $S_{\dot{q}}(q)$ to be twice differentiable with respect to both q and $\dot{q}$.

We now turn to a point glossed over to some extent by both Rund and Synge: the question of how to define an invertible Legendre transformation leading from the Lagrangian to the

Hamiltonian formulation. Rund abandons the classical Legendre transformation from L to H after noting equation (3-9), settling for the straightforward equivalence (3-20), while Synge makes no attempt to establish a transformation law between his two basic quantities L and $\Omega$. Yet in neither case is any reason given for this peremptory decision. Rund (1966) merely remarks that "we are forced to conclude that the left-hand side of [3-9] cannot serve as Hamiltonian function". But how justified is Rund in discarding out of hand a zero Hamiltonian? The Hamiltonian he uses to achieve his central results is identically equal to 1, while Synge makes no bones about basing his entire formalism on a function which is identically zero.

There is a further point to note. As Synge brings out so clearly, the specification of a mechanical problem does not just consist of choosing a Lagrangian, but of choosing a Lagrangian with the property that it is homogeneous of degree 1 in $\dot{q}$, and this property corresponds in some way to the fact that Synge's "Hamiltonian" obeys the equation $\Omega(q,p)=0$. But suppose we take a general function $L(q,\dot{q})$ (not necessarily homogeneous) and form from it the Hamiltonian

$$H \equiv \dot{q}^A . \partial L / \partial \dot{q}^A - L .$$

If we now apply the new energy equation $H(q,p)=0$ then we obtain precisely the condition that L be homogeneous of degree unity in $\dot{q}$, so we see that the conventional Legendre transformation

$$p_A = \partial L / \partial \dot{q}^A ; \qquad \qquad \text{-- (4-5a)}$$

$$H(q,p) = \dot{q}^A p_A - L(q,\dot{q}) \qquad \qquad \text{-- (4-5b)}$$

induces a direct correspondence between the energy equation in Hamiltonian mechanics and the homogeneity condition in Lagrangian mechanics. We must bear in mind, of course, that the simple $(q,p)$ dependence given in (4-5b) is reliant upon the existence of

an inversion procedure for (4-5a) which will give q as a function

of p: $q(p)$. This is a question which we will take up presently,

but we first note another property of the energy equation.

Like Synge's equation (3-87), our energy equation defines an

N-1 dimensional hypersurface (the _energy surface_) in each cotan-

gent space $T_q^*$ to which the momenta are confined. Now we have

already defined one surface, $S_H$, in $T_q^*$, and we would like to

relate this to the energy surface. To do this we now investigate

the corresponding surface in $T_q$ which results when the energy

surface is dragged back into T using the inverse $g^{AB}$ of the

Finsler metric (4-1). The equation of this surface will be

$$H(q,p) \equiv q^A(p)p_A - L(q,\dot{q}(p)) = 0 \quad,$$

but we are supposing that p is obtained from some vector $u \varepsilon T_q$ by

the equation $p_A = g_{AB} u^B$. In this case we can use (4-5a) and (4-3)

to obtain

$$g_{AB} u^B = p_A = \partial L(\dot{q}(p))/\partial \dot{q}^A$$

$$= g_{AB} \dot{q}^B / L(\dot{q})$$

$$=> \qquad L(\dot{q}) u^B = \dot{q}^B(p) \quad,$$

so the energy equation H=0 becomes

$$L(\dot{q}) u^A g_{AB} u^B - L(q, L(\dot{q}) u^A) = 0$$

$$=> \qquad L(\dot{q}).\{g_{AB} u^A u^B - L(q, u^A)\} = 0$$

$$=> \qquad g_{AB} u^A u^B - L(u^A) = 0$$

$$=> \qquad L^2(u^A) - L(u^A) = 0$$

$$=> \qquad L(q,u) = 1 \quad,$$

where we have in some places suppressed mention of the coord-

inates $q^A$ in the interests of lucidity. But this is simply the

equation of the Lagrangian surface defined in (4-4), and so we

find that the Hamiltonian surface $S_H(q)$ is just the restriction
to $T_q^*$ of the energy surface $H(q,p)=0$. The energy surface is thus
given by $U_{q \in M} S_H(q)$.

The question of the inversion of (4-5a) brings to light what
is in fact an essential difference between the theories of Synge
and Rund. This difference may be summarised as follows:

a. Synge's theory has the advantage that it retains the
   well-known form (4-5a) for the definition of momentum,
   but the disadvantage that it maps an entire half-ray in
   $T_q$ to a single value in $T_q^*$.

b. Rund's theory has the advantage that it prescribes a 1:1
   correspondence between points in $T_q$ and points in $T_q^*$,
   but the disadvantage that it does not yield the standard
   formula (4-5a) relating p and $\dot{q}$.

The singularity of (4-5a) means that any $v \in T_q$ lying in the same
direction as $\dot{q}$ is also mapped to the same $p \in S_H$. Hence we must be
sure that when we invert the transformation and return to $T_q$, we
recover the velocity vector $\dot{q}$ and not some arbitrary vector lying
on the same half-ray. Fortunately (4-5) ensures that whatever
vector is used to pass to $T_q^*$ will be recovered when we return to
$T_q$. To see this we must define exactly what steps are to be
taken in implementing (4-5). The procedure we shall adopt is as
follows:

i. First choose a surface $S_{\dot{q}}$ in $T_q$ which picks out a
   specific member $\dot{q}$ from the fibre of each momentum vector
   $p \in S_H$.

ii. Use this representative $\dot{q}$ to define the momentum via
    (4-5a). The mapping into $T_q^*$ of points in $T_q$ not on $S_{\dot{q}}$
    may now be prescribed arbitrarily, _provided_ the mapping
    $T_q \rightarrow T_q^*$ taken as a whole is injective and $C^1$. The mapping

can therefore now be inverted to give meaning to (4-5b)

and the equation $H(q,v)=0$ now defines an energy surface

$S_H$ in $T_q^*$, where v is an arbitrary element of $T_q^*$. Note

that from the point of view of mechanics we do not in

fact need to define the mapping all over the tangent and

cotangent spaces - it merely needs to be defined within

a finite shell around the surfaces $S_L$ and $S_H$.

iii. For each point q∈M we now carry out the following calc-

ulation <u>on the energy surface</u> $S_H(q)$: Let v again be an

arbitrary element of $T_q^*$, then, suppressing all mention

of the coordinates, we have

$$H(v) = v_A \dot{q}^A(v) - L(\dot{q}(v))$$

=>
$$\frac{\partial H}{\partial v_A}(v) = q^A(v) + v_B \cdot \frac{\partial \dot{q}^B}{\partial v_A}(v) - \frac{\partial L}{\partial \dot{q}^B} \frac{\partial \dot{q}^B}{\partial v_A}(v)$$

$$= q^A(v) + \frac{\partial \dot{q}^B}{\partial v_A} \{v_B - p_B\} \quad .$$

Thus when v=p, ie, when v lies on the energy surface,

this second term cancels and we find

$$\frac{\partial H}{\partial p_A}\bigg|_{H=0} = q^A \quad . \qquad\qquad -- (4-6)$$

iv. (4-6) now tells us that the inverse mapping of (4-5a) is

provided by

$$p \rightarrow \partial H/\partial p\big|_{H=0} \quad ,$$

and this delivers the <u>same</u> representative velocity $\dot{q}$

which was used in (4-5) to define H, ie, that element of

the half-ray which lies in $S_{\dot{q}}$. In other words (4-5) so

arranges things that no matter what parameter is used,

the Hamiltonian is such that $\lambda=1$ in Synge's equations

(3-81).

Thus we have now defined a Legendre transformation between the surfaces $S_{\dot{q}}$ and $S_H$ which we have shown to be invertible. We lose no information in the tangent space by confining our attention to $S_{\dot{q}}$, as we mentioned earlier, and covariant vectors other than those on the energy surface play no part in the mechanics of either Rund or Synge: such vectors cannot be considered as physical momentum vectors. Similarly, the choice of $S_{\dot{q}}$ immediately robs all other elements of $T_q$ of any physical significance (at least as far as the particular descriptive language defined by $S_{\dot{q}}$ is concerned). However, although these vectors play no part in mechanics they must still be defined, at least within a finite shell around the surfaces $S_{\dot{q}}$ and $S_H$, since otherwise the derivatives in the inverse transformation (4-6) will have no meaning.

The non-injective nature of (4-5) is directly related to the fact that the Lagrangian theory is specifically constructed to have the same form for all parameters τ, while the Hamiltonian theory, as we saw in Synge's work, seems to pick out a special parameter w in the canonical equations (3-82). This raises the question posed at the end of section 3.4 concerning the meaning of parameters other than w. It seems as though w is more appropriate to relativistic mechanics, and Rund makes no real use of the other parameters available. Synge, however, demonstrates that other parameters can also be useful - for example, the use of $q^N = t$ as the parameter. Looked at in this way the arguments in section 3.3 for an affine parameter are no longer valid, since the change of parameter w→t is by no means linear in general. We need the completely general choice of parameter τ to admit both cases, since τ now seems to be not a physical parameter, but rather a meta-physical one representing the transition from

classical to relativistic mechanics. Whereas the form of L determines the nature of the system under consideration, the choice of parameter determines the mode of description to be used. This placing of the action principle on a higher level than mechanics is quite in keeping with classical mechanical theory, since neither Lagrangian nor Hamiltonian mechanics actually makes use of the calculus of variations. Only the equations of motion are required for a dynamical theory - the calculus of variations merely supplies them in a particularly elegant way. The composite theory to be developed in this chapter, comprising an expression of both the mechanical system _and_ the language used to describe it, will be termed _homogeneous mechanics_. We shall now commence an exposition of the theory of homogeneous mechanics, starting with the dynamical viewpoint and then moving on to the geometrical viewpoint.

## a. The dynamical picture

From the homogeneous Lagrangian we obtain the equations of motion as before:

$$\frac{\partial L}{\partial q^A} - \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{q}^A}\right] = 0 \quad , \qquad \text{-- (4-7)}$$

and by applying the Legendre transformation (4-5) we obtain immediately the canonical equations:

$$\dot{q}^A = \partial H/\partial p_A \big|_{H=0} \quad ; \qquad \Bigg\}$$

$$\dot{p}_A = -\partial H/\partial q^A \big|_{H=0} \qquad \Bigg\} \qquad \text{-- (4-8)}$$

together with the energy equation

$$H(q,p) = 0 \quad . \qquad \text{-- (4-9)}$$

(4-9) is implicit throughout homogeneous mechanics, and indeed expresses the very homogeneity of the theory. Because of this we shall in future omit this stipulation in dynamical equations such as (4-8). We must also bear in mind that (4-8) and (4-9) come

with the proviso that a choice of $S_{\dot{q}}$ must still be specified to give them full meaning. Consequently we must eliminate this ambiguity by making a specific choice of parameter. As soon as this is done the velocities are unique and we have in effect chosen a particular surface $S_{\dot{q}}$ on which to construct our description of the system. One such choice is the one made by Rund, which corresponds to choosing the surface $S_L$ :

$$S_{\dot{q}} = S_L : \quad L(q,\dot{q}) \equiv 1 \quad . \qquad\qquad -- (4-10)$$

Note, however, the difference from Rund's theory that once we have chosen this particular descriptive system we apply (4-10) <u>throughout</u> dynamics as a supplementary condition in the same way that condition (3-95b) was applied in Synge's description of relativistic mechanics. Indeed the choice (4-10) <u>defines</u> Rund's theory of dynamics: another choice of parameter would define another descriptive system. One such alternative system is classical dynamics, following from the choice $\tau = q^N = t$ demonstrated in section 3.5. Here the equations of motion are

$$\frac{\partial L^*}{\partial q^\alpha} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial q^{\cdot\alpha}}\right] = 0 \quad , \qquad\qquad -- (4-11)$$

where $\qquad L^* = L(q^A, q^{\cdot\alpha}, 1)$

and we use the same notation as in section 3.5. Here the surface on which we construct our description is defined by

$$S_{\dot{q}} : \quad \dot{q}^N \equiv 1 \quad . \qquad\qquad -- (4-12)$$

In each of these two cases the specific choice of parameter "takes up the slack" of the redundancy implied by (3-72) by reducing the number of independent tangent space variables from N to N-1. As we saw in section 3.5, both of these choices lead to parallel Hamiltonian formulations with equations of motion (4-8) in the case of Rund's theory. In the classical case Synge's correspondence (3-84) continues to hold:

$$P_\alpha = p^*_\alpha \quad ; \quad P_N = -H^* \quad , \qquad\qquad \text{-- (4-13)}$$

together with the alternative (Pfaffian) form of the action principle:

$$\delta I = \int \{\delta p_A dq^A - \delta q^A dp_A\} = 0 \quad . \qquad\qquad \text{-- (4-14)}$$

Again we can in principle use (4-9) to solve $p_N$ as a function of the classical variables $q^\alpha, t, p_\alpha$:

$$H^* = \omega(q^\alpha, t, p_\alpha) \qquad\qquad \text{-- (4-15)}$$

and obtain the classical canonical equations

$$q^{\cdot\alpha} = \partial H^*/\partial p_\alpha \quad ; \quad p^{\cdot}_\alpha = -\partial H^*/\partial q^\alpha \quad ; \left.\begin{array}{c}\\[1em]\end{array}\right\} \qquad \text{-- (4-16)}$$
$$dH^*/dt = \partial H^*/\partial t \quad ;$$

Finally, by defining the 2-point characteristic function $S(Q,q)$ as before we obtain again the relations

$$\partial S/\partial q^A = p_A \quad ; \quad \partial S/\partial Q^A = -P_A \qquad\qquad \text{-- (4-17)}$$

which lead to the H-J equation

$$H(q, \partial S/\partial q) = 0 \qquad\qquad \text{-- (4-18)}$$

and its classical form

$$\partial S/\partial t + H^*(q^\alpha, \partial S/\partial q^\alpha, t) = 0 \quad . \qquad\qquad \text{-- (4-19)}$$

Thus we see that the considerations at the beginning of this chapter lead to a theory as general as that of Synge, with the advantage that we now have a transformation law (4-5) from the Lagrangian formulation to the Hamiltonian. It must be remembered that the key equations (4-7), (4-8) and (4-18) are all non-specific ones. That is, they hold independently of the type of dynamics chosen to describe a system. This choice is made afterwards by including a supplementary condition of the form (4-10) or (4-12).

Before moving on to the geometrical viewpoint of homogeneous mechanics it is worth mentioning briefly another class of existing relativistic Hamiltonian theories which we did not mention in chapter 3. These theories are exemplified by that proposed in

chapter 21 of Sudarshan & Mukunda (1974), and effectively consist
in treating the energy equation as a constraint on the full
2N-dimensional system represented in phase space.  The theory
thus arrived at is very similar to that of Synge, but now the use
of the constraint function as the Hamiltonian can be defended
more rigorously by means of Dirac's theory of systems with
constraints.   We shall not go into Dirac's work here since it
would merely be a reproduction of the somewhat lengthy exposition
in chapter 8 of Sudarshan & Mukunda (1974).  What Dirac does is
to show that Synge's choice of the energy function as the
fundamental quantity of Hamiltonian mechanics is essentially
unique in the sense that any such quantity must equal the energy
function on the energy surface, and also that its first partial
derivatives with respect to the canonical variables must equal
those of the energy function on the energy surface.  We see from
the above work that the energy equation is a constraint equation
which specifies the points in phase space which are relevant to a
given system.  If the system is now further constrained by, for
instance, confinement to a physical surface, then this is simply
one more constraint necessary to the specification of the prob-
lem.   Thus the prescription of forces implicit in the choice of
Hamiltonian is on a par with the prescription of physical
constraints; the distinction made in classical mechanics between
these two types of condition is an artificial one necessitated by
the clumsiness of the classical decomposition.

We now move on to see how homogeneous mechanics leads to a
geometrical picture in the same way as Rund's theory of section
3.2.

## b. The geometrical picture

Equations (4-1) to (4-4) summarise the way in which Rund's

work brings out the geometric nature of Lagrangian and Hamilton-
ian mechanics. We have seen that by using the transformation
(4-5a) we involve the complication that any given momentum
corresponds to an infinity of values for the velocity, which
causes difficulties for a geometric correspondence between $T_q$ and
$T_q^*$. However, in our study of dynamics we eliminated the problem
of the non-injective nature of (4-5a) by choosing a particular
surface $S_q$ in $T_q$ and arbitrarily requiring p to be mapped back to
the corresponding point on $S_q$ in the inverse transformation. The
obvious candidate for this surface is $S_L$ - the one used for
Rund's description of dynamics, and from (4-10) we see that on $S_L$
(4-3) reduces to

$$P_A = g_{AB} \dot{q}^B . \qquad\qquad -- (4-20)$$

Equation (4-20) is fundamental to Rund's geometrical picture of
mechanics. He required a bijection between the tangent and
cotangent spaces, and opted for (4-20), but it is important to
note that he obtains (4-20) only by giving up (4-5a). In
homogeneous mechanics we take the alternative route: in retain-
ing (4-5a) we sacrifice the neat geometrical picture offered by
(4-20). Yet we see that (4-20) still obtains on the Lagrangian
surface, exactly as Rund's dynamics yielded (4-5a) when τ was
chosen such that L(q,q̇)=1. On $S_L$ (4-20) gives a linear trans-
formation from $T_q$ to $T_q^*$ which exactly duplicates the action of
the Legendre substitution (4-5a). In addition we stated above
that the effect of the Legendre transformation on points not on
$S_q$ may be prescribed arbitrarily. Clearly in some cases one such
prescription will recommend itself more than another, and Rund's
dynamics is precisely such a case. (4-20) maps the whole of $T_q$
linearly and injectively onto $T_q^*$, but only on $S_L$ does it rep-
resent (4-5a). However, since Rund's dynamics only gives useful

results on $S_L$ this will be sufficient - (4-20) will provide a Finsler space model of dynamics exactly as in section 3.2. The relationship of this model to the dynamics of the system will only be apparent if Rund's descriptive language is used, but this does not invalidate the fact that dynamics contains an innate geometrical aspect. We shall now investigate this geometrical aspect further.

Using (4-20) we can define the natural parameter w on $S_L$ which is such that

$$dw \equiv L(q,dq) \; . \qquad\qquad -- \; (4-21)$$

This parameter is the one specified by condition (4-10), and so in fact defines Rund's dynamics. We assume that $(g_{AB})$ is non-singular, in which case we can form the contravariant metric tensor $g^{AB}$ satisfying

$$
\left.
\begin{aligned}
g^{AB} g_{BC} &= \delta^A_C \; ; \\
\dot{q}^A &= g^{AB} p_B \; ; \\
L^2(q,\dot{q}) &= g^{AB} p_A p_B \; .
\end{aligned}
\right\} \qquad -- \; (4-22)
$$

As before, these definitions ensure that the geodesics in the Finsler space model are precisely the extremals of the Rund dynamical problem, satisfying the following equations on $S_L$ :

$$\frac{\partial L}{\partial q^A} - \frac{d}{dw}\left[\frac{\partial L}{\partial \dot{q}^A}\right] \equiv -g_{AD}[\ddot{q}^D + \gamma^D_{BC}\dot{q}^B\dot{q}^C] = 0 \; ; \qquad -- \; (4-23)$$

$$\gamma^D_{BC} = \frac{1}{2} g^{AD}\left[\frac{\partial g_{AB}}{\partial q^C} + \frac{\partial g_{AC}}{\partial q^B} - \frac{\partial g_{BC}}{\partial q^A}\right] \; .$$

The corresponding equations in $T^*_q$ are the canonical equations (4-8) using the special parameter w. Again Rund's dynamics is constructed on the special surface $S_H$ in $T^*_q$, the equation of which is simply the energy equation (4-9). Since, however, we have said that p lies by definition on the energy surface, (4-9)

expresses a property of p rather than defining the energy

surface; to obtain the equation of $S_H$ we should properly sub-

stitute into this equation the general covariant vector $v \varepsilon T_q^*$ (not

necessarily on $S_H$):

$$H(q,v) = 0 \quad .$$

The energy equation then defines the energy surface $S_H$. In the

particular case of Rund's dynamics this equation reduces to

$$H = v_A \dot{q}^A - L$$

$$= v_A \dot{q}^A - 1 = 0$$

$$\Rightarrow \quad S_H : \quad v_A \dot{q}^A = 1 \ , \quad \text{or} \quad g^{AB} v_A v_B = 1. \qquad \qquad \text{-- } (4-24)$$

This is just the equation already given in (4-4).


## 4.2 Relativistic dynamics

We have now defined two different modes of description of a

dynamical system: Rund's dynamics defined by (4-10), and

classical dynamics defined by (4-12). Since our central aim is

to find a Hamiltonian description of relativistic dynamics, we

now seek a choice of parameter which will yield such a descrip-

tion. Clearly classical dynamics is unsuitable, since $\tau = q^N$ is

dependent on the coordinates chosen. On the other hand we know

from Rund's work that the choice $\tau = w$ will not be an easy one to

study, as a glance at the metric (3-67) shows. There is also a

second disadvantage to working in the full Finsler space of

Rund's dynamics. We wish to be able to describe at least all

particles moving with timelike velocity $u^a = \dot{z}^a$, yet we mentioned

(and passed over) in section 3.2 the fact that the barrier $L=0$

presents a problem. The transformation (4-3) only has meaning

when $L$ is nonzero, so Rund's dynamics is only of use with

particles for which $z$ satisfies

$$L(z,\dot{z}) = -m(-g_{ab}\dot{z}^a\dot{z}^b)^{1/2} + eA_a\dot{z}^a \neq 0 \quad . \qquad\qquad \text{-- (4-25)}$$

Because of the term $eA_a\dot{z}^a$ it is by no means clear whether this will encompass all timelike $\dot{z}$. Hence we are led to select a third special choice of parameter which will define <u>relativistic dynamics</u>. This choice is simply the proper time s calculated from the Riemannian metric $g_{ab}$ appearing in the Lagrangian (4-25). This parameter is clearly a natural choice in a relativistic Hamiltonian theory, yet Rund ignored it completely, and in Synge's work it arose almost by accident as the value of w for a relativistic charged particle. The definition of proper time now becomes an alternative supplementary condition which defines the surface $S_{\dot{z}}$ on which relativistic dynamics is constructed:

$$S_{\dot{z}}: \quad u^a u_a \equiv g_{ab}\dot{z}^a\dot{z}^b = -1 \quad . \qquad\qquad \text{-- (4-26)}$$

In this way we split the Finsler space on which Rund's dynamics is based into a (Riemannian) "geometric" part based on the metric $g_{ab}$, and a "field" part in $A_a$ representing the anisotropic properties of the Finsler space metric.

To see relativistic dynamics in action we now study the case of the single relativistic charged particle, using the Lagrangian (4-25) together with the supplementary condition (4-26). We assume the $g_{ab}$ to form a metric field on spacetime in the usual relativistic sense, in which case $g_{ab}(z)$ is independent of the velocities and we obtain

$$p_a = \partial L/\partial\dot{z}^a = m(-u^c u_c)^{-1/2} g_{ab} u^b + eA_a$$

$$= mg_{ab}u^b + eA_a \qquad\qquad \text{-- (4-27)}$$

by application of (4-26). (4-27) can easily be inverted to give u as a function of p:

$$u^a = 1/m.g^{ab}(p_b - eA_b) \quad , \qquad\qquad \text{-- (4-28)}$$

where $g^{ab}$ is the usual inverse of the metric $g_{ab}$. The Hamilton-

ian is given by

$$H = p_a u^a - L(z,u)$$

$$= 1/m \cdot p_a (p^a - eA^a) + \{-(p_a - eA_a)(p^a - eA^a)\}^{1/2}$$

$$- e/m \cdot A^a (p_a - eA_a)$$

$$= 1/m \cdot (p_a - eA_a)(p^a - eA^a) + \{-(p_a - eA_a)(p^a - eA^a)\}^{1/2}, \quad -- \quad (4-29)$$

from which we can calculate the canonical equations:

$$\frac{\partial H}{\partial p_a} = \frac{2}{m}(p^a - eA^a) - \frac{(p^a - eA^a)}{\{-(p_b - eA_b)(p^b - eA^b)\}^{1/2}}$$

$$= 2u^a - u^a$$

$$= u^a . \qquad\qquad -- \quad (4-30a)$$

$$\frac{\partial H}{\partial z^b} = \frac{-2e}{m}(p^a - eA^a)A_{a,b} + \frac{e(p^a - eA^a)A_{a,b}}{\{-(p_a - eA_a)(p^a - eA^a)\}^{1/2}}$$

$$+ 1/2m \cdot g^{cd}{}_{,b}(p_c - eA_c)(p_d - eA_d)$$

$$= -e/m \cdot (p^a - eA^a)A_{a,b} + 1/2m \cdot g^{cd}{}_{,b}(p_c - eA_c)(p_d - eA_d)$$

$$= -eA_{a,b}\dot{z}^a - m/2 \cdot g_{cd,b}\dot{z}^c \dot{z}^d$$

$$= -\dot{p}_b$$

$$= -mg_{bc,d}\dot{z}^c \dot{z}^d - mg_{bc}\dot{z}^c - eA_{b,a}\dot{z}^a$$

$$\Rightarrow \quad m\left[\ddot{z}^b + \left\{\begin{matrix} b \\ c\ d \end{matrix}\right\}\dot{z}^c \dot{z}^d\right] = eF^{ab}\dot{z}_a . \qquad\qquad -- \quad (4-30b)$$

Thus we obtain the Lorentz force law in Riemannian space exactly as required. The energy equation takes the form

$$H \equiv 1/m \cdot (p^a - eA^a)(p_a - eA_a) + \{-(p^a - eA^a)(p_a - eA_a)\}^{1/2} = 0$$

$$\Rightarrow \quad (p^a - eA^a)(p_a - eA_a) + m^2 = 0 , \qquad\qquad -- \quad (4-31)$$

which is precisely Synge's energy equation (3-99), with the advantage that it is now unique in the sense that the Legendre transformation gives an explicit recipe for transfering from the homogeneous Lagrangian (4-25) to the energy equation (4-31).

(4-31) also gives the same H-J equation (3-101) that we found in Synge's work:

$$g^{ab} \left[\frac{\partial S}{\partial z^a} - eA_a\right]\left[\frac{\partial S}{\partial z^b} - eA_b\right] + m^2 = 0 \quad . \qquad -- \quad (4-32)$$

**Example (4.2)** - Consider a special relativistic particle of mass m and charge e moving in the purely magnetic field $\underline{B} = (0,0,B)$, where we adopt the rectangular coordinates $(x,y,z,t)$ in space-time. For simplicity we have aligned the z-axis with the magnetic field. The Maxwell tensor for this field is

$$F_{ab} = \begin{bmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ,$$

so that a possible 4-potential is

$$A_a = B/2.(y,-x,0,0) \quad . \qquad -- \quad (4-33)$$

This leads to the following expressions for the Lagrangian and momentum of the particle:

$$\left.\begin{aligned} L(z,u) &= -m(-u^a u_a)^{1/2} + eB/2.(yu^1 - xu^2) \\ P_a &= (mu_1 + eBy/2, \ mu_2 - eBx/2, \ mu_3, \ mu_4) \end{aligned}\right\} \quad -- \quad (4-34)$$

and the canonical equations (4-30):

$$u^a = g^{ab}/m.(p_b - eA_b) \ ; \quad m\ddot{z}^b = e(-B\dot{y}, B\dot{x}, 0, 0) \quad . \qquad -- \quad (4-35)$$

Thus the relativistic case of the uniform magnetic field is essentially the same as the classical case, with the coordinates satisfying the following equations:

$$\left.\begin{aligned} \ddot{x} &= -eB\dot{y}/m \ ; \quad z = As + B \ ; \\ \ddot{y} &= eB\dot{x}/m \ ; \quad t = Cs + D \quad . \end{aligned}\right\} \quad -- \quad (4-36)$$

These are precisely the classical equations (with t replaced by s) and are satisfied by the solutions

$$\left.\begin{aligned} x &= r \cos \omega s \ ; \quad y = r \sin \omega s \ , \\ \text{where} \quad r &= \text{const. and } \omega = eB/m \quad . \end{aligned}\right\} \quad -- \quad (4-37)$$

This is the equation of motion of a particle following a circle

of radius r and proper angular frequency ω in the (x,y) plane. By a suitable choice of geometry we can reduce the constants of integration A,B,D to zero, and to evaluate C we apply the relativistic supplementary condition (4-26):

$$r^2\omega^2 - c^2 = -1$$

$$\Rightarrow \qquad c = \sqrt{(1 + r^2\omega^2)} \quad .$$

This is exactly what we should expect from a particle with constant (proper) speed ωr.

---

A major problem throughout the history of analytical mechanics has been the description of systems of two or more particles, and the problem becomes more complex in relativity because in the absence of absolute Newtonian time we need to find a parameter which will serve for two particles on completely separate worldlines. We shall now indicate briefly how homogeneous mechanics may be used to describe such systems, although we shall not develop this far because it brings to the fore another far-reaching problem, namely that of interaction. In the single-particle theory presented above we ignored the problem of the "kickback" caused by energy radiated by the particle (see Teitelboim,1970), but in the many-particle formalism far more is swept under the carpet. Not only is radiated energy ignored, but also the problem of interaction between distinct particles. It may be possible to encompass such interactions by allowing the potentials $A_a$ to depend on coordinate differences $(z_1 - z_2)$ (although not according to the "No-Interaction" theorem - see pp.535 *et seq.* of Sudarshan & Mukunda,1974), but we shall not follow up this possibility here. We are intent simply upon showing how a single parameter may be defined for all particles of a system on their separate worldlines.

During the following discussion we suspend a number of conventions used elsewhere in this thesis, but we shall revert to the former conventions from the end of this section. We are working with an N-particle system, so the indices A,B,C,... will now label the distinct particles. **All** summations over these indices will be made explicit by the use of a $\Sigma$ sign: the summation convention will be confined to the 4-indices a,b,c,... Using these conventions we can concatenate the 4-coordinates $z^a$ of the particles into a single 4N-component position vector $z^a_A \epsilon M^N$, where $M^N$ is the Cartesian product of N replicas of the spacetime manifold M. We then assume that some universal parameter $\tau$ has been chosen, so that the velocities $\dot{z}^a_A = dz^a_A/d\tau$ have meaning, and use the N-particle Lagrangian

$$L(z_A, \dot{z}^A) = -(\Sigma_A m_A)^{1/2} \{-\Sigma_B m_B g_{ab}(z_B) \dot{z}^a_B \dot{z}^b_B\}^{1/2}$$
$$+ \Sigma_A e_A A_a(z_A) \dot{z}^a_A \quad , \qquad\qquad -- \quad (4-38)$$

where $m_A, e_A$ are the mass and charge respectively of the A-th particle. From (4-38) we can calculate the 4N components of the momentum:

$$p_{Aa} = \frac{\partial L}{\partial \dot{z}^a_A} = (\Sigma_A m_A)^{1/2} m_A \dot{z}_{Aa} \{-\Sigma_B m_B \dot{z}^2_B\}^{-1/2} + e_A A_a(z_A). \quad -- \quad (4-39)$$

Now this was the crucial point in the exposition of relativistic particle mechanics above. There we were able to reduce (4-39) to the form

$$p_{Aa} = m_A \dot{z}_{Aa} + e_A A_a(z_A) \qquad\qquad -- \quad (4-40)$$

by choosing $\tau$ to be the proper time along the worldline of the particle. But this is still possible here if we choose $\tau$ such that

$$(\Sigma_A m_A)^{1/2} = \{-\Sigma_A m_A \dot{z}^a_A \dot{z}_{Aa}\}^{1/2} \quad , \qquad\Bigg\}$$
$$\text{ie:} \qquad d\tau^2 = d\sigma^2 \equiv \Sigma_A m_A ds^2_A / \Sigma_B m_B \quad , \qquad\Bigg\} \qquad -- \quad (4-41)$$

where $s_A$ is the proper time along the worldline of the A-th

particle. We see that this special choice of parameter element $d\sigma$ is a sort of RMS value of the proper time along each of the N particle worldlines, where the masses of the particles appear as frequencies, or weightings, of each particle. As in the single particle case, this then gives the Lorentz force on each particle:

$$\partial L/\partial z_A^b = e_A A_{a,b}(z_A)\dot{z}_A^a$$

$$\Rightarrow \quad \frac{\partial L}{\partial z_A^b} - \frac{d}{d\sigma}\left[\frac{\partial L}{\partial \dot{z}_A^b}\right] = e_A \dot{z}_A^a [A_{a,b}(z_A) - A_{b,a}(z_A)] - m_A z_{Ab} = 0$$

$$m_A \ddot{z}_A^b = e_A F^{ab}(z_A)\dot{z}_{Aa} \quad , \qquad\qquad -- \quad (4-42)$$

where for simplicity we have assumed the $g_{ab}$ to be constants.

Note that the choice (4-41) of parameter, like all such choices, is not necessary - it merely makes life easier. In the development of homogeneous mechanics carried out in the rest of this chapter we shall at all times assume that some specific choice of parameter has been made, defining the descriptive system to be used and giving the canonical equations (4-8) meaning. In general we shall not need to know the specific form of the supplementary condition involved in this choice, but in order to apply the formalism it is necessary to assume that some such condition holds. The above remarks on many-particle systems demonstrate how best to make the choice of parameter in practical situations: first derive the expression (4-39) for the momenta from the Lagrangian (4-38), and then choose a supplementary condition like (4-41) which reduces the momenta to the manageable form (4-40). Again, this form is _not_ in principle necessary, but in applications may turn out to be so from a pragmatic point of view.

## 4.3 Canonical transformations

As in classical dynamics, it is often convenient to consider the complete cotangent bundle $T^*(M)$ over the manifold M, rather than simply M itself. M is the N-dimensional __configuration space__ of events, coordinatised by the set $\{q^A\}$, while the cotangent bundle is referred to by Synge (1960) as the 2N-dimensional __space of states and energy__, coordinatised by the canonical variables $\{q^A, p_A\}$. Since, however, $T^*$ is the fundamental entity of homogeneous mechanics, we shall refer to it as the __phase space__ of the system under consideration, in line with the terminology of classical Hamiltonian theory. The energy equation (4-9) then represents the (2N-1)-dimensional __energy surface__ $U_{q \in M} S_H(q) c T^*(M)$, to which the physical states of the system are confined. In such a space it is natural to look at all transformations of the canonical variables

$$Q^A = Q^A(q,p) \quad ; \quad P_A = P_A(q,p) \qquad \text{-- (4-43)}$$

which leave the action integral invariant. Hence, using the homogeneity of L, we require that the integral

$$I = \int L d\tau = \int \dot{q}^A p_A d\tau \qquad \text{-- (4-44)}$$

remains invariant under the transformations (4-43): such transformations are called __canonical transformations__ (CT's). Note that the form (4-44) of the action integral is only true on the energy surface, but this does not matter, since we are not concerned with what happens to non-physical systems which do not obey the energy equation (or, equivalently, whose Lagrangian is not homogeneous). A simple way of ensuring the invariance of (4-44) is to demand that

$$\left. \begin{array}{l} \dot{Q}^A P_A = \dot{q}^A p_A + dF/d\tau \quad , \\ \text{where} \qquad F = F(q,p) \end{array} \right\} \qquad \text{-- (4-45)}$$

is the __generating function__ of the transformation.

Because we assume the new canonical variables to be independent we can imagine F,q and p to be expressed in terms of them, in which case (4-45) becomes

$$\dot{Q}^A P_A = P_A \left[ \frac{\partial q^A}{\partial Q^B} \dot{Q}^B + \frac{\partial q^A}{\partial P_B} \dot{P}_B \right] + \frac{\partial F}{\partial Q^A} \dot{Q}^A + \frac{\partial F}{\partial P_A} \dot{P}_A$$

$$\Rightarrow \quad P_A - P_B \frac{\partial q^B}{\partial Q^A} = \frac{\partial F}{\partial Q^A} \quad ; \quad -P_B \frac{\partial q^B}{\partial P_A} = \frac{\partial F}{\partial P_A} \qquad -- (4-46)$$

by equating coefficients. The necessary and sufficient conditions for the existence of the function F are the three integrability conditions:

$$\frac{\partial}{\partial Q^C} \left[ P_A - P_B \frac{\partial q^B}{\partial Q^A} \right] = \frac{\partial}{\partial Q^A} \left[ P_C - P_B \frac{\partial q^B}{\partial Q^C} \right] \quad ;$$

$$\frac{\partial}{\partial P_C} \left[ P_B \frac{\partial q^B}{\partial P_A} \right] = \frac{\partial}{\partial P_A} \left[ P_B \frac{\partial q^B}{\partial P_C} \right] \quad ;$$

$$\frac{\partial}{\partial P_C} \left[ P_A - P_B \frac{\partial q^B}{\partial Q^A} \right] = -\frac{\partial}{\partial Q^A} \left[ P_B \frac{\partial q^B}{\partial P_C} \right] \quad ,$$

which immediately simplify to

$$\left[ \frac{\partial q^B}{\partial Q^A} \frac{\partial p_B}{\partial Q^C} - \frac{\partial q^B}{\partial Q^C} \frac{\partial p_B}{\partial Q^A} \right] = 0 \quad ;$$

$$\left[ \frac{\partial q^B}{\partial Q^A} \frac{\partial p_B}{\partial P_C} - \frac{\partial q^B}{\partial P_C} \frac{\partial p_B}{\partial Q^A} \right] = \delta_A^C \quad ; \qquad \qquad -- (4-47)$$

$$\left[ \frac{\partial q^B}{\partial P_A} \frac{\partial p_B}{\partial P_C} - \frac{\partial q^B}{\partial P_C} \frac{\partial p_B}{\partial P_A} \right] = 0 \quad ;$$

As in section 2.3 we again have the characteristic structure in these equations which leads us to define the Lagrange bracket (LB). If $A_1(q,p),\ldots,A_{2N}(q,p)$ is a set of 2N functions such that the $q^A, p_A$ can be written as functions of the A's, the LB of two A's computed with respect to the canonical variables (q,p) is defined by

$$[A_K, A_L] \equiv \left[ \frac{\partial q^B}{\partial A_K} \frac{\partial p_B}{\partial A_L} - \frac{\partial q^B}{\partial A_L} \frac{\partial p_B}{\partial A_K} \right] = 0 \quad , \qquad -- (4-48)$$

where the block indices K,L,M will be reserved for the moment to run from 1 to 2N.

Similarly, we can define the Poisson bracket (PB) of two functions F and G:

$$\{F,G\} \equiv \left[\frac{\partial F}{\partial q^A}\frac{\partial G}{\partial p_A} - \frac{\partial G}{\partial q^A}\frac{\partial F}{\partial p_A}\right] \quad , \qquad \text{-- (4-49)}$$

permitting us to express the $\tau$-derivative of a function $F(q,p)$ along an extremal in the following way:

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial q^A}\dot{q}^A + \frac{\partial F}{\partial p_A}\dot{p}_A$$

$$= \left[\frac{\partial F}{\partial q^A}\frac{\partial H}{\partial p_A} - \frac{\partial H}{\partial q^A}\frac{\partial F}{\partial p_A}\right]$$

$$= \{F,H\} \quad . \qquad \text{-- (4-50)}$$

The <u>fundamental PB relations</u> between the canonical variables are

$$\{q^A,q^B\} = \{p_A,p_B\} = 0 \quad ; \quad \{q^A,p_B\} = \delta^A_B \quad , \qquad \text{-- (4-51)}$$

from which all further PB's are in principle obtainable by algebraic processes. In particular it is easily seen that

$$\{q^A,F\} = \partial F/\partial p_A \quad ; \quad \{p_A,F\} = -\partial F/\partial q^A \quad , \qquad \text{-- (4-52)}$$

which leads to the following concise form of the canonical equations (4-8):

$$\{q^A,H\} = \dot{q}^A \quad ; \quad \{p_A,H\} = \dot{p}_A \quad . \qquad \text{-- (4-53)}$$

It is clear that all the above work is essentially the same as that of section 2.3. As before, the LB's and PB's of a set of 2N independent functions $A_K$ (K=1,...,2N) form matrices which are inverses of one another:

$$\Sigma_K [A_K,A_L]\{A_K,A_M\} = \delta^L_M \quad . \qquad \text{-- (4-54)}$$

The proof of (4-54) is the same as that of theorem (2.2), although now simplified due to the absence of hypersurface integrations in particle mechanics. (4-54) enables us to write the condition (4-47) for a CT either in LB form:

$$[Q^A,Q^B] = [P_A,P_B] = 0 \quad ; \quad [Q^A,P_B] = \delta^B_A \qquad \text{-- (4-55)}$$

or in the equivalent PB form:

$$\{Q^A, Q^B\} = \{P_A, P_B\} = 0 \quad ; \quad \{Q^A, P_B\} = \delta^A_B \quad . \qquad -- \ (4\text{-}56)$$

In both cases the brackets are computed with respect to the original canonical variables $(q,p)$. As in section 2.3, we characterise CT's by saying that they preserve the values of the fundamental PB relations (4-51). Again $\underline{all}$ PB's between arbitrary quantities are in fact preserved under CT's.

In deriving the conditions (4-47) for a CT we considered the function F in (4-45) as a function of the new variables $(Q,P)$ and used this to derive a set of integrability conditions. An alternative way of viewing F is as a given generating function, from which a particular CT may be derived. F will in general be a function of the 4N variables $q^A, p_A, Q^A, P_A$, but in view of the relations (4-43) we can reduce this to any 2N of these. We now look at four different choices of independent variables.

## Case 1

Consider the case $F = -F_1(q,Q)$, where we have inserted the minus sign purely for convenience. Then

$$\dot{F}_1 = \dot{q}^A . \partial F_1 / \partial q^A + \dot{Q}^A . \partial F_1 / \partial Q^A$$

$$\Rightarrow \qquad \{p_A - \partial F_1 / \partial q^A\}\dot{q}^A - \{P_A + \partial F_1 / \partial Q^A\}\dot{Q}^A = 0$$

by substitution into (4-45). Since $q^A, Q^A$ may be regarded as independent variables we find

$$P_A = \partial F_1 / \partial q^A \quad ; \quad P_A = -\partial F_1 / \partial Q^A \quad . \qquad -- \ (4\text{-}57)$$

For a given function $F_1(q,Q)$ these equations completely define a CT. We must solve the first equation for $Q^A$ in terms of $(q,p)$ and then substitute this into the second equation to obtain $P_A$ in terms of $(q,p)$. In order to carry out the first step the necessary and sufficient requirement which $F_1$ must satisfy is that

$$\det \left| \frac{\partial^2 F_1}{\partial q^A \partial Q^A} \right| \neq 0 \quad ,$$

so provided $F_1$ satisfies this condition the equations (4-57) uniquely define the CT generated by $F_1$. Note that the energy equation does <u>not</u> invalidate this determinantal condition, since a CT is a transformation of the entire phase space, not just of the energy surface.

## Case 2

The work for the remaining cases has already been done in section 1.2, so we shall merely quote the results here.

$$F_2(q,P) = F_1(q,Q) + Q^A P_A \quad ;$$

=> $$P_A = \partial F_2/\partial q^A \quad ; \quad Q^A = \partial F_2/\partial P_A \quad . \qquad \text{-- (4-58)}$$

## Case 3

$$F_3(p,Q) = F_1(q,Q) - q^A P_A \quad ;$$

=> $$q^A = -\partial F_3/\partial P_A \quad ; \quad P_A = -\partial F_3/\partial Q^A \quad . \qquad \text{-- (4-59)}$$

## Case 4

$$F_4(p,P) = F_1(q,Q) + P_A Q^A - p_A q^A \quad ;$$

=> $$q^A = -\partial F_4/\partial p_A \quad ; \quad Q^A = \partial F_4/\partial P_A \quad . \qquad \text{-- (4-60)}$$

## Example (4.3)

i. $F = P_A q^A$ is clearly an instance of a case 2 generating function, for which we have

$$p_A = \partial F/\partial q^A = P_A \quad ; \quad Q^A = \partial F/\partial P_A = q^A \quad . \quad \text{-- (4-61)}$$

Hence this choice of F generates the identity transformation.

ii. $F = f^A(q) P_A$ is also case 2, where the $f^A$ are arbitrary independent functions, so

$$p_A = \partial F/\partial q^A = P_B \partial f^B/\partial q^A \quad ; \quad Q^A = \partial F/\partial P_A = f^A(q) \quad .$$

Thus F here generates the general point transformation

of the $q^A$ alone.

---

To close this section we now look briefly at the classical view of the above work. Letting $\tau = t = q^N$, $p_N = -H^*$ the general phase space function may be written $F(q^\alpha, t, p_\alpha, H^*)$, for which we have, from (4-62),

$$\dot{F} = \frac{\partial F}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial F}{\partial H^*}\frac{dH^*}{dt}$$

$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial H^*}\frac{\partial H^*}{\partial t} + \{F, H^*\}^* \quad , \qquad \text{-- (4-62)}$$

where
$$\{F, G\}^* \equiv \frac{\partial F}{\partial q^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q^\alpha} \qquad \text{-- (4-63)}$$

$$= \{F, G\} + \frac{\partial F}{\partial t}\frac{\partial G}{\partial H^*} - \frac{\partial F}{\partial H^*}\frac{\partial G}{\partial t}$$

is the classical PB. Since we can always use (4-15) to express $H^*$ in terms of the other phase space variables on the energy surface, we may rewrite the dependence of F as $F(q^\alpha, t, p_\alpha)$, in which case we arrive at the classical expression (1-27):

$$\dot{F} = \partial F/\partial t + \{F, H^*\}^* \quad .$$

Using (4-62) and (4-63) we can compare the fundamental PB's (4-51) with their classical counterparts:

$$\{q^\alpha, p_\beta\} = \{q^\alpha, p_\beta\}^* = \delta^\alpha_\beta \quad ;$$

$$\{q^\alpha, p_N\} = -\{q^\alpha, H^*\}^* = 0 \quad ; \quad \{q^N, p_\beta\} = \{t, p_\beta\}^* \quad ;$$

$$\{q^N, p_N\} = -\{t, H^*\}^* + 1 - \frac{\partial t}{\partial H^*}\frac{\partial H^*}{\partial t}$$

$$= -\left[\frac{dt}{dt} - \frac{\partial t}{\partial t} - \frac{\partial t}{\partial H^*}\frac{\partial H^*}{\partial t}\right] + 1 - \frac{\partial t}{\partial H^*}\frac{\partial H^*}{\partial t}$$

$$= 1 \quad .$$

The generated transformations (4-57) to (4-60) also have their classical counterparts, for example the case 1 transformation reads

$$F_1(q^\alpha, t, Q^\alpha, T): \quad p_\alpha = \partial F_1/\partial q^\alpha \quad ; \quad -H^* = \partial F_1/\partial t \quad ; \\ P_\alpha = -\partial F_1/\partial Q^\alpha \quad ; \quad -\overline{H}^* = -\partial F_1/\partial T \ . \quad \bigg\} \quad -- \quad (4-64)$$

Here the symbols $(T, \overline{H}^*)$ represent the transformed values of the pair $(t, H^*)$. To obtain the classical case, in which time is unaffected by transformations, we apply a Legendre transformation to the N-th argument of $F_1$ to obtain

$$f = -t\overline{H}^* + F_1(q^\alpha, t, Q^\alpha, T) \ , \qquad -- \quad (4-65)$$

where f is now a case 2 generating function in the N-th components of its arguments, but case 1 in all other components. In this case we obtain the classical equations (1-19a):

$$p_\alpha = \partial f/\partial q^\alpha = \partial F_1/\partial q^\alpha \quad ; \\ P_\alpha = -\partial f/\partial Q^\alpha = -\partial F_1/\partial Q^\alpha \quad ; \\ -H^* = \partial f/\partial t \quad => \quad \overline{H}^* - H^* = \partial F_1/\partial t \quad ; \\ T = -\partial f/\partial \overline{H}^* = t \ , \qquad \qquad \bigg\} \quad -- \quad (4-66)$$

where we have used (4-58) and (4-64).


## 4.4 Hamilton-Jacobi theory

In definition 3.5 we defined the 2-point characteristic function $S(Q,q)$ on a manifold M as the action measured along a trajectory $\Gamma$ from Q to q:

$$S(Q,q) \equiv \int_\Gamma L d\tau = \int_\Gamma P_A dq^A \ , \qquad -- \quad (4-67)$$

noting that S may well be many-valued and deriving the results

$$\partial S/\partial Q = -P_A \quad ; \quad \partial S/\partial q = p_A \quad ; \quad \delta S = p_A \delta q^A - P_A \delta Q^A \ . \qquad -- \quad (4-68)$$

In order to extend these ideas further it will be helpful to recall a series of definitions drawn from topology.

### Definition (4.4):

   i. A <u>circuit</u> is a closed curve in M.

   ii. A circuit is <u>reducible</u> if it can be reduced to a single point in M by continuous transformations within M; otherwise it is <u>irreducible</u>.

   iii. Two circuits are <u>reconcilable</u> if they can be transformed
        into one another by continuous transformations within M;
        otherwise they are <u>irreconcilable</u>.

    iv. Two irreducible circuits are <u>independent</u> if they are
        irreconcilable with each other.

     v. A manifold M is <u>n-tuply connected</u> if it possesses
        precisely (n-1) independent irreducible circuits; if it
        possesses none then it is <u>simply connected</u>.

    vi. A <u>congruence</u> of trajectories is an (N-1)-parameter
        family of trajectories covering a region R of M in such
        a way that precisely one passes through each point of R.

---

As an example of these definitions, a torus is triply connected,
since it contains the two independent irreducible circuits (a)
and (b) shown in fig.(4.5). The circuit (c) is reducible. A
possible congruence of trajectories on the torus would be the
1-parameter family of circuits of type (b), covering the entire
surface of the torus in such a way that none of the circuits
cross.

(a)                     (b)                     (c)



<u>fig.(4.5)</u>
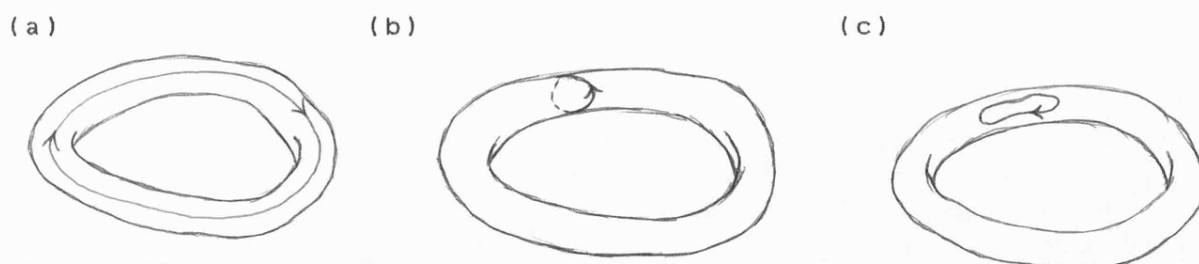
   Now consider a congruence of trajectories filling some
region R of M. Throughout R we can associate a covariant vector
field with each point by means of the rule

$$p_A \equiv \partial L(q,\dot{q})/\partial \dot{q}^A \quad , \qquad\qquad -- (4-69)$$

where $\dot{q}$ is the tangent vector at the point q to the trajectory
through q. We say that the congruence is <u>coherent</u> if for every

reducible circuit C in R we have

$$\int_C p_A \, dq^A = 0 \quad .$$   -- (4-70)

In particular we see from (4-68) that the family of all trajectories drawn from a given point Q form a coherent congrence. The importance of coherent congruences of trajectories arises from the fact that they enable us to define a path-independent action function which is single-valued.

**Definition (4.6):**   Let a coherent congruence of trajectories be defined on R as above.  Choose some _fixed_ point Q in R and let q be any other point in R.  Now join Q to q by an arbitrary curve C, then the 1-point characteristic function U(q) is defined by

$$U(q) = \int_C p_A \, dq^A \quad ,$$   -- (4-71)

where we must be careful to note that p is here the momentum (4-69) defined by the _congruence_, and not by the curve C (see fig.(4.7)).

---

fig.(4.7)



coherent congruence
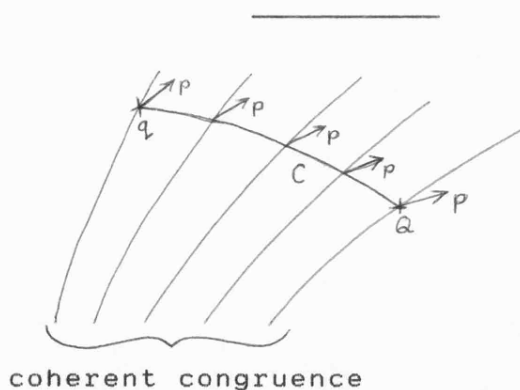
In the above definition we have defined U(q) in terms of some arbitrary curve C, but we see from (4-70) that the choice of C is irrelevant to the value of U _provided_ all such choices are reconcilable with one another.  Hence if R is simply connected then U(q) is single-valued, while if R is multiply connected then U(q) is multiple-valued.  Also note that from (4-71) we have

$$p_A = \partial U / \partial q^A \quad ,$$

so that using (4-9), U must satisfy the H-J equation:

$$H(q, \partial U/\partial q) = 0$$

or $$\partial U/\partial t + H^*(q^\alpha, t, \partial U/\partial q^\alpha) = 0 .$$

-- (4-72)

Given a coherent congruence of trajectories we can define the surfaces of constant action by the equation

$$U(q) = const.$$ -- (4-73)

These surfaces cut the trajectories orthogonally in the sense that for an arbitrary displacement $\delta q^A$ within one of the surfaces (4-73) we have

$$p_A \delta q^A = \delta U = 0 ,$$ -- (4-74)

where we have used (4-71). (4-74) is the closest we can get to an orthogonality relation between the trajectories and the action surfaces, since only on $S_L$ do we have a metric with which to test for orthogonality between the contravariant vectors $\dot{q}$ and $\delta q$. Note that due to the coherent nature of the congruence the change in action in going from one action surface $\overline{W}$ to another W is equal to the integral $\int p_A dq^A$ taken along any curve in M drawn from any point in $\overline{W}$ to any point in W. In particular the change in action may be measured along any trajectory from $\overline{W}$ to W. Bearing in mind the close analogy brought out in Rund's theory between geometrical and dynamical quantities, the surfaces (4-74) are therefore said to be geodesically equidistant.

We shall now see that the action surfaces possess a wavelike nature brought out by Huygen's construction of geometrical optics, in which the wavefronts are the envelopes of equidistant surfaces drawn from each point of a preceding wavefront. To carry this construction over into the realm of dynamics we suppose an action surface W to be generated from another surface $\overline{W}$, as in fig.(4.8).

Let Q be some point on $\overline{W}$. We construct trajectories Γ from Q in
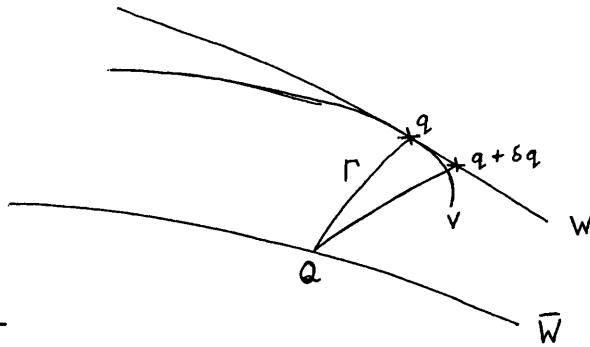
fig.(4.8)

all directions in M and measure off on them an action

$$A = U(q) - U(Q) \quad ,$$

where q is the point at which the trajectory through Q cuts W.

This construction gives us an (N-1)-dimensional subspace V with

equation

$$S(Q,q) = A$$

which is itself a wavefront with Q as source. Now it is clear

from the definition of V that q lies on it. Also, if we displace

q by an infinitesimal amount $\delta q$ along the surface W, then the

change in S is given by (4-68) as $p_A \delta q^A$: this is the amount by

which $S(Q,q+\delta q)$ exceeds A. But by (4-74) this is zero. Hence,

to first order, $q+\delta q$ lies within V, proving that V is tangential

to W at q. The subspaces V drawn from each point Q of $\overline{W}$ with

constant A therefore form an envelope which is the surface W;

these waves W are known as <u>Hamilton's waves</u>. This establishes

the connection between H-J theory and geometrical optics which

forms the basis of primitive quantisation. In chapter 6 we shall

take an alternative route to quantisation based on more recent

work in quantum theory, but primitive quantisation first

established the close link between quantum mechanics and the

Hamiltonian formulation (see addenda in Somerfeld,1928).

We now shed some light on the significance of the H-J

equation using the material developed in section 4.3 on generat-

ing functions. The canonical equations (4-8) define a set of

trajectories, precisely one of which passes in each direction

through each point of a region of M.  If, however, we extend our interest to the cotangent bundle $T^*(M)$, we find that the trajectories fill a region with a natural congruence: one trajectory through each point of phase space.  Given a general covariant vector $v \in T^*_q$ the canonical equations imply that $H(q,v)=$const. along a trajectory, so we are assured that a subset of these trajectories fill the energy surface

$$U_{q \in M} \{ S_H(q) : H(q,v) = 0 \} \quad .$$

This congruence on the energy surface presents a much simpler geometrical picture than is the case in M, since now a single point on the energy surface defines a unique trajectory passing through that point.  The effect of a canonical transformation is to change these curves by performing a point transformation in phase space, the PB conditions (4-56) ensuring that the resulting curves still represent the trajectories of the system.  It is therefore desirable to find a canonical transformation $(q,v) \to (Q,V)$ which transforms the natural congruence into a congruence of parallel straight lines.

Let $G(q,V)$ be any solution of the partial differential equation

$$H(q, \partial G / \partial q) = V_N \quad , \qquad\qquad -- \ (4-75)$$

this solution being such that

$$\det \left| \frac{\partial^2 G}{\partial q^A \partial V_B} \right| \neq 0 \quad .$$

Then from (4-58) the equations

$$V_A = \partial G / \partial q^A \quad ; \quad Q^A = \partial G / \partial V_A \qquad\qquad -- \ (4-76)$$

define a canonical transformation for which the new Hamiltonian $\overline{H}(Q,V)$ satisfies

$$\overline{H}(Q,V) = H(q,v) = H(q, \partial G / \partial q) = V_N \quad . \qquad -- \ (4-77)$$

The new canonical equations therefore read

$$\dot{Q}^\alpha = \partial\overline{H}/\partial V_\alpha = 0 \quad ; \quad \dot{Q}^N = \partial\overline{H}/\partial V_N = 1 \quad ;$$

$$\dot{V}_\alpha = -\partial\overline{H}/\partial Q^\alpha = 0 \quad ; \quad \dot{V}_N = -\partial\overline{H}/\partial Q^N = 0 \quad ,$$

which on integration give

$$Q^\alpha = a^\alpha \quad ; \quad V_A = b_A \quad ; \quad Q^N = \tau \quad , \qquad\qquad -- \quad (4-78)$$

where $a^\alpha, b_A$ are constants and we have neglected the constant of integration in $Q^N$. Thus we see that the solution of equation (4-75) leads to a Gaussian coordinate system in which the parameter $\tau$ becomes one of the coordinates and in which all other canonical variables are constant along the natural congruence of trajectories. The family of trajectories defined by the equation $V_N = 0$ cover the energy surface $H=0$, which has therefore been transformed to a plane. In this case (4-75) becomes

$$H(q, \partial G/\partial q) = 0 \quad , \qquad\qquad -- \quad (4-79)$$

which is clearly of identical form with the H-J equation (4-72), with the difference that (4-79) involves the function $G(q,V)$, while (4-72) involves the 1-point function $U(q)$.

We may easily show that (4-79) is in fact a general form of the H-J equation. We first note that since $V_A =$const., the effective dependence of $G(q,V)$ is only on the $q^A$. If we write (4-79) in its classical form then we find

$$\partial G/\partial t + H^*(q^\alpha, t, \partial G/\partial q^\alpha) = -\overline{H}^* \quad ,$$

and using the same trick as in (4-65) to write

$$G = -\overline{H}^* t + U(q^\alpha, t) \qquad\qquad -- \quad (4-80)$$

we see that U must satisfy

$$\partial U/\partial t + H^*(q^\alpha, t, \partial U/\partial q^\alpha) = 0 \quad ,$$

which is precisely the classical form of (4-72). Thus the 1-point characteristic function $U(q)$ may be used to generate a canonical transformation via (4-80) to a set of co-moving (Gaussian) coordinates for phase space of the form (4-78). The

physical momentum vectors - ie, those lying on $S_H$ - are those satisfying $P_N = V_N = 0$, and are obtained by the straightforward substitution $G \equiv U$.

## 4.5 Transformation generators

As we saw in chapter 2, the action principle not only offers us a compact statement of the equations of motion but can also be used to deduce conservation laws characteristic of the system. These are consequences of the symmetry properties of the Lagrangian, and follow from its functional form. The *sine qua non* of all such methods is contained in the following generalised form of Noether's theorem.

THEOREM (4.9) - Let $\Gamma$ be a trajectory of a system and define the following infinitesimal transformation on configuration space:

$$q \rightarrow q + \delta q \ ,$$
where $\quad \delta q^A = \varepsilon . \xi^A (q, \dot{q}) \qquad \qquad -- (4-81)$

is a specific, rather than arbitrary, variation of the coordinates. Here the $\xi^A$ are specific functions of $(q, \dot{q})$ specifying the transformation and $\varepsilon$ is a small parameter independent of both coordinates and velocities. The transformation (4-81) represents a mapping of configuration space onto itself, which when applied to the curve $\Gamma$ yields another curve differing infinitesimally from $\Gamma$. The variation of the velocity brought about by the transformation is given by the $\tau$-derivative of (4-81):

$$\delta \dot{q}^A = d(\delta q^A)/d\tau \ ,$$

which allows us to calculate the Lagrangian on both $\Gamma$ and its image curve. Suppose that the change in L under

this transformation can be expressed as the total deriv-
ative of some function:

$$\delta L = L(q+\delta q, q+\delta \dot{q}) - L(q, \dot{q})$$

$$= \varepsilon \, dF(q,\dot{q})/d\tau \quad , \qquad \qquad \text{-- (4-82)}$$

then L is said to be _quasi-invariant_ under (4-81) and
the quantity $\xi^A p_A - F(q,\dot{q})$ _is a constant of the motion_
along $\Gamma$, where p is the momentum corresponding to $\dot{q}$.


PROOF:         We know from (3-90) that the variation of the
action integral under (4-81) is given by

$$\delta I(\Gamma) = [p_A \delta q^A(\tau)]_1^2$$

provided $\Gamma$ is a trajectory. On the other hand $\delta I$ is
also equal to the integral along $\Gamma$ of the variation $\delta L$
of the Lagrangian, and we suppose that this is given by
(4-82):

$$\delta I(\Gamma) \equiv \int_\Gamma \delta L(q,\dot{q})d\tau \equiv [\varepsilon F(q,\dot{q})]_1^2$$

equating these two ways of computing $\delta I(\Gamma)$ we obtain an
identity valid for all paths $\Gamma$:

$$[p_A \delta q^A - \varepsilon F]_1^2 = 0 \quad ,$$

expressing the quasi-invariant nature of L.  Thus the
bracketed factor has the same values at both endpoints
of $\Gamma$ and  so also at all points of a given state of
motion.  Using (4-81) we therefore obtain

$$\xi^A(q,\dot{q})p_A - F(q,\dot{q}) = \text{constant of motion.} \qquad \text{-- (4-83)}$$

                                                    QED

COROLLARY  (4.10) (Noether's theorem) - If F=0 in theorem (4.9)
then L is said to be simply _invariant_ under (4-81) and
we obtain

$$\xi^A(q,\dot{q})p_A = \text{constant of motion} \quad . \qquad \text{-- (4-84)}$$

If the $\xi^A$ are chosen to depend only on the coordinates

$q^A$ then (4-81) describes a point transformation, and the resulting constant of the motion will be linear in the conjugate momenta $p_A$ .

---

One immediate application of Noether's theorem is in looking at _infinitesimal generators_, but in order to do this we shall first have to backtrack slightly to the work of section 4.3. In (4-61) we saw that the choice of the case 2 generating function $F_2 = q^A p_A$ generates the identity transformation. Hence we can generate infinitesimal transformations by means of the infinitesimal parameter $\varepsilon$ and the arbitrary function G:

$$F_2 = q^A P_A + \varepsilon G(q,P) \quad .$$

From this we obtain

$$Q^A = \partial F_2 / \partial P_A = q^A + \varepsilon \partial G / \partial P_A \quad ;$$

$$p_A = \partial F_2 / \partial q^A = P_A + \varepsilon \partial G / \partial q^A$$

and hence to first order in $\varepsilon$

$$\delta q^A = \varepsilon \partial G / \partial p_A \quad ; \quad \delta p_A = -\varepsilon \partial G / \partial q^A \quad , \qquad \text{-- (4-85)}$$

or in an alternative notation, using (4-52),

$$\delta q^A = \varepsilon \{q^A, G\} \quad ; \quad \delta p_A = \varepsilon \{p_A, G\} \quad . \qquad \text{-- (4-86)}$$

Equations (4-85) and (4-86) describe an infinitesimal contact transformation, and G is the _generator_ of the transformation.

We shall now look at the generators of a number of infinitesimal transformations arising from possible symmetry properties of the Lagrangian. We consider three distinct symmetries: these are associated with the requirements that there be no preferred origin of the coordinates, no preferred orientation of the axes, and no preferred zero of the parameter $\tau$. The first two of these correspond to the symmetry transformations of translation and rotation, which constitute the inhomogeneous Lorentz group. In order to make this

correspondence clear we shall imagine the coordinates q to be the concatenation of the spacetime coordinates $q^a$ of a system of N particles in the manner described in the discussion of many-particle systems at the end of section 4.2. In this case the coordinates will be labelled $q^{Aa}$, the block index labelling the particle and the 4-index labelling the spacetime coordinates of each particle; again we shall make all summations over block indices explicit. We now study in turn the effect of each of the above mentioned transformations on this system.

## i. Translations

We assume the Lagrangian to be unaffected by a translation of the coordinates which is the same for all particles:

$$L(q^{Aa}+\varepsilon^a,\dot{q}) = L(q,\dot{q}) \ . \qquad\qquad -- \ (4-87)$$

The transformation $q^{Aa} \to q^{Aa}+\varepsilon^a$ is an example of a geometric, or point, transformation in configuration space, and thus leads to a constant of the motion according to Noether's theorem which is linear in the momenta. Since $\varepsilon^a$ is assumed arbitrary we obtain from (4-84)

$$P_a \equiv \Sigma_A P_{Aa} = \Sigma_A \partial L/\partial \dot{q}^{Aa} = const. \ , \qquad -- \ (4-88)$$

where $P_a$ is the <u>total</u> <u>momentum</u> of the system. On the other hand for an arbitrary function F(q,p) we have

$$\varepsilon^a\{F,P_a\} = \Sigma_A \varepsilon^a \partial F/\partial q^{Ab} . \partial P_a/\partial p_{Ab}$$
$$= \varepsilon^a \Sigma_A \partial F/\partial q^{Aa} \ , \qquad\qquad -- \ (4-89)$$

which is the total change brought about in F by the spacetime translation $\varepsilon^a$, ie, precisely the transformation in (4-87). Thus $P_a$ is the infinitesimal generator of the translations (4-87).

## ii. Rotations

An infinitesimal rotation of the spacetime coordinates is represented by the infinitesimal antisymmetric tensor $\omega^{ab}=-\omega^{ba}$, so we accordingly assume

$$L(q^{Aa} + \omega^a{}_b q^{Ab}, \dot{q}^{Aa} + \omega^a{}_b \dot{q}^{Ab}) = L(q, \dot{q}) \quad . \qquad \text{-- (4-90)}$$

Once again this is a point transformation, and so by (4-84) leads to a constant of the motion linear in the momenta. Since the $\omega^{ab}$ are arbitrary this constant of the motion may be written as $\omega^{ba} M_{ab}$, where

$$M_{ab} = \Sigma_A q^A_{[a} p^A_{b]} = \Sigma_A q^A_{[a} \frac{\partial L}{\partial \dot{q}^{Ab]}} = \text{const.} \qquad \text{-- (4-91)}$$

$M_{ab}$ is the total angular momentum of the system, and is the infinitesimal generator of the spacetime rotations (4-90):

$$\omega^{ba} \{F, M_{ab}\} = \omega^{ba} \Sigma_A \left[ \frac{\partial F}{\partial q^{Ac}} \frac{\partial M_{ab}}{\partial p_{Ac}} - \frac{\partial M_{ab}}{\partial q^{Ac}} \frac{\partial F}{\partial p_{Ac}} \right]$$

$$= \omega^{ba} \Sigma_A \left[ \frac{\partial F}{\partial q^A}{}_{[b} q^A_{a]} - \frac{\partial F}{\partial p_A}{}_{[a} p^A_{b]} \right]$$

$$= \omega^{ba} \Sigma_A \left[ \frac{\partial F}{\partial q^A}{}_{[b} q^A_{a]} + \frac{\partial F}{\partial p_A}{}_{[b} p^A_{a]} \right] , \qquad \text{-- (4-92)}$$

which is the total change brought about in F by the spacetime rotation (4-90). From the above considerations we see the importance of the infinitesimal generators $P_a$, $M_{ab}$, since they represent the Lorentz nature of the spacetime manifold. The system of particles $q^{Aa}$ forms a 4N-dimensional manifold which has no inherent relationship to spacetime: it is the generators which "group the coordinates into 4's" and yield the transformation properties required of spacetime.

### iii. Parameter transformations

The final transformation we shall look at is generated by the Hamiltonian H. To obtain this transformation we substitute $\delta\tau$ and H for $\varepsilon$ and G respectively in (4-85):

$$\delta q^{Aa} = \delta\tau \partial H / \partial p_{Aa} = \dot{q}^{Aa} \delta\tau \quad ;$$

$$\delta p_{Aa} = \delta\tau \partial H / \partial q^{Aa} = \dot{p}_{Aa} \delta\tau$$

or alternatively

$$\delta q^{Aa} = \delta\tau \{q^{Aa}, H\} \quad ; \quad \left.\rule{0pt}{3em}\right\}$$
$$\delta p_{Aa} = \delta\tau \{p_{Aa}, H\} \quad , \quad$$

$$-- (4-93)$$

where we have made use of the canonical equations. Thus we see that H is the infinitesimal generator of the dynamical development of the system with respect to $\tau$; the evolution of a system may, as in the classical case, be considered as Hamilton's "continuous unfolding" of a contact transformation whose generator is the Hamiltonian. Now suppose the Lagrangian is parameter-invariant:

$$L(q+\dot{q}\delta\tau, \dot{q}+\ddot{q}\delta\tau) = L(q, \dot{q}) \quad ,$$

then Noether's theorem says

$$\Sigma_A \dot{q}^{Aa} p_{Aa} = \text{const.}$$

and hence
$$H = \Sigma_A \dot{q}^{Aa} p_{Aa} - L = \text{const.} \quad ,$$

which is in any case clear from (4-50).


Note that a form of Noether's theorem may be expressed within the Hamiltonian formalism in a very straightforward way. Suppose the Hamiltonian is unchanged by some transformation generated by the quantity G, then $\delta H$ is proportional to $\{H,G\}$, which is therefore zero. But this is precisely equivalent to saying that G is a constant of the motion:

$$dG/d\tau = \{G, H\} = 0 \quad .$$

Thus the relationship between symmetries and conservation laws is immediate in Hamiltonian mechanics. We shall look at this in more depth in chapter 5.

Up to now we have looked at the infinitesimal contact transformations defined by (4-86), but each of the examples considered above also possesses a finite form: the finite translations and rotations and the development of a system along a finite parameter interval. Suppose we are given an initial

state of a system $(q_0, p_0)$ and its infinitesimal change in terms of an infinitesimal contact transformation

$$\delta q^A = \delta\varepsilon\{q^A, G\} \quad ; \quad \delta p_A = \delta\varepsilon\{p_A, G\} \quad ,$$

and we wish to calculate this change for some finite value of the parameter $\varepsilon$. We can do this by means of a Taylor expansion in $\varepsilon$:

$$\delta q^A = \varepsilon\{q^A, G\} + \frac{\varepsilon^2}{2!}\{\{q^A, G\}, G\} + \ldots \quad ,$$

but there is a problem to be overcome here. Both the generator G and the PB's themselves are evaluated at the point $(q, p)$ in phase space, which will change as we move away from the initial values $(q_0, p_0)$. However, we know that the PB is invariant under canonical transformations, so we can equally well evaluate it using the $(q_0, p_0)$ system. In addition the generator G is a constant of the transformation, since from (4-85)

$$\frac{dG(q, p)}{d\varepsilon} = \frac{\partial G}{\partial q^A}\frac{dq^A}{d\varepsilon} + \frac{\partial G}{\partial p_A}\frac{dp_A}{d\varepsilon}$$

$$= \frac{\partial G}{\partial q^A}\frac{\partial G}{\partial p_A} - \frac{\partial G}{\partial p_A}\frac{\partial G}{\partial q^A}$$

$$= 0 \quad ,$$

and so can also be evaluated at $(q_0, p_0)$. In this way we can view the values $(q, p)$ as being generated from $(q_0, p_0)$ by a finite transformation built up from a succession of infinitesimal contact transformations:

$$q = q_0 + \varepsilon\{q_0, G(q_0, p_0)\} + \frac{\varepsilon^2}{2!}\{\{q_0, G(q_0, p_0)\}, G(q_0, p_0)\} + \ldots \quad ;$$

$$p = p_0 + \varepsilon\{p_0, G(q_0, p_0)\} + \frac{\varepsilon^2}{2!}\{\{p_0, G(q_0, p_0)\}, G(q_0, p_0)\} + \ldots \quad ,$$

all PB's being evaluated at $(q_0, p_0)$. We can also develop a similar expansion to represent functions on phase space. Let $F(q, p)$ be such a function with a specified functional form and define $F_0 \equiv F(q_0, p_0)$. Then we have

$$\delta F(q, p) = \left[\frac{\partial F}{\partial q^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial q^A}\right]\delta\varepsilon$$

$$= \delta\varepsilon \; \{F,G\} \qquad\qquad -- \; (4-94)$$

$$\Rightarrow \quad F(q,p) = F_o + \varepsilon\{F_o,G\} + \frac{\varepsilon^2}{2!}\{\{F_o,G\},G\} + \ldots \;\; , \quad -- \; (4-95)$$

all functions and brackets on the RHS being evaluated at $(q_o,p_o)$. The finite transformation of the canonical variables $(q,p)$ is clearly a special case of (4-95), obtained by taking $F(q,p)=q$ and $F(q,p)=p$ respectively.

Sudarshan and Mukunda (1974) give a more compact notation for expressing these results. Given the generator $G(q_o,p_o)$ we define the linear partial differential operator $\hat{G}(q_o,p_o)$ associated with it in terms of its action on an arbitrary phase space function $F(q_o,p_o)$:

$$\hat{G}(q_o,p_o)F(q_o,p_o) \equiv \{G,F\}_{(q_o,p_o)} \;\; . \qquad -- \; (4-96)$$

The operator $\hat{G}\equiv\hat{G}(q_o,p_o)$ is designed to act on functions of $(q_o,p_o)$, in which case it yields the PB of $G_o\equiv G(q_o,p_o)$ with these functions. The canonical transformation generated by $G_o$ takes $F_o$ into $F(q,p)$ according to (4-95), which using (4-96) can be written

$$F(q,p) = F_o - \varepsilon\hat{G}_o F_o + \frac{\varepsilon^2}{2!}\hat{G}_o\hat{G}_o F_o - \ldots$$

$$= \exp\{-\varepsilon\hat{G}_o\} \; F(q_o,p_o) \;\; , \qquad -- \; (4-97)$$

with the special cases

$$q = \exp\{-\varepsilon\hat{G}_o\} \; q_o \;\; ; \;\; p = \exp\{-\varepsilon\hat{G}_o\} \;\; . \qquad -- \; (4-98)$$

Combining (4-97) with (4-98) we can write the simple equation

$$F[\exp\{-\varepsilon\hat{G}_o\}(q_o,p_o)] = \exp\{-\varepsilon\hat{G}_o\}F[q_o,p_o] \;\; . \qquad -- \; (4-99)$$

Using this notation it is clear that the set of all canonical transformations generated by some generator $G(q,p)$ form a continuous group. One important example of such a group is provided by the dynamical development of a system. We saw in (4-93) that the Hamiltonian is the infinitesimal generator of

this development, and we now see that the equations of motion

describe a continuous 1-parameter group of transformations in

phase space such that

$$(q,p) = \exp\{-\tau \hat{H}_o\}(q_o, p_o) \quad . \qquad \text{-- (4-100)}$$

As $(q_o, p_o)$ is allowed to vary over all of phase space, so

$\exp\{-\tau \hat{H}_o\}$ acts as a $\tau$-dependent canonical mapping of phase space

onto itself. If $F(q,p)$ is a dynamical variable with a given form

then its parameter dependence is given by

$$F[q(\tau), p(\tau)] = \exp\{-\tau \hat{H}_o\}F(q_o, p_o) \quad . \qquad \text{-- (4-101)}$$

This formalism leads to two major subject areas which have

developed out of Hamiltonian mechanics. In the first place it

forms the basis of a form of perturbation theory developed by

Kilmister and Reeve (1966), although their notation is slightly

different from ours. In the second place, if the function F is

such that its integral over phase space is unity, then it

describes the state of the system at a given $\tau$-instant in a

probabilistic manner. (4-101) then describes the $\tau$-evolution of

this state in a way which could lend itself to a fully relativ-

istic treatment of statistical thermodynamics.

CHAPTER 5

CONSTANTS AND SYMMETRIES


Having developed the general theory of homogeneous mechanics in the previous chapter we now look in this chapter at a major application of analytical mechanics in the area of symmetries and conservation laws. The fundamentals of the subject were studied in section 4.5 and we shall develop these ideas later in this chapter, but first we look at an alternative route to the symmetries of a system which arises from the study of the Hamilton-Jacobi equation.


## 5.1 The Hamilton-Jacobi method

We saw in section (4.4) how a solution of the H-J equation may be used to transform to a new set of coordinates and momenta forming a Gaussian system in phase space, in which the energy equation becomes simply $P_N = 0$. One such solution was seen to be the 1-point characteristic function $U(q)$, but the 2-point function $S(Q,q)$ is also a possible solution. This is clear from the properties summarised in (4-67) and (4-68); indeed from (4-68) we see that S satisfies the two equations:

$$H(q, \partial S/\partial q) = H(Q, -\partial S/\partial Q) = 0 .$$

(Here we have dispensed with the bars of section 3.5, since we assume $(q,p)$ and $(Q,P)$ to lie within the same coordinate patch.) We can obtain a single-point function from $S(Q,q)$ if we imagine the Q's to be a set of constant initial values $a^A$ labelling the trajectory $\Gamma$ along which S is measured. At first sight this seems to be a much less general solution than U, which may be defined in terms of _any_ congruence of trajectories (not merely those converging at the point a). However, the solution $S(a,q)$,

where the $a^A$ are arbitrary constants, may also be regarded as any solution of the H-J equation if we take as independent variables the 2N quantities $(a^A, q^A)$, of which the first set do not appear explicitly in the equation.

The **Hamilton-Jacobi** (or **H-J**) **method** is a technique for getting round the integration of the ordinary differential equations of motion by working instead with the H-J equation. Thus the problem of motion is reduced to finding a complete integral of the H-J equation. The basis of the method lies in the following theorem.

**THEOREM (5.1) (Hamilton-Jacobi)** - Let $S(a,q)$ be any complete integral of the equation

$$H(q, \partial S/\partial q) = 0 , \qquad -- (5-1)$$

the quantities $a^A$ being arbitrary constants. Now let $b_A$ be a further set of constants, then the equations

$$b_B = -\partial S/\partial a^B ; \quad p_B = \partial S/\partial q^B \qquad -- (5-2)$$

define a congruence of curves on phase space. If we choose a suitable parameter $\tau$ along these curves then the curves of the congruence are the trajectories of the canonical equations

$$\dot{q}^A = \partial H/\partial p_A ; \quad \dot{p}_A = -\partial H/\partial q^A . \qquad -- (5-3)$$

PROOF: The first point to note is that we necessarily have the two determinantal conditions

$$\det \left| \frac{\partial^2 S}{\partial q^A \partial q^B} \right| = 0 ; \quad \det \left| \frac{\partial^2 S}{\partial q^A \partial a^B} \right| = 0 . \qquad -- (5-4)$$

The first follows immediately from the relation (5-1). However (5-1) holds at every point of the energy surface, so if we choose the particular point $q^A = a^A$ we can be sure that an algebraic relation also exists between the $a^A$ and $p_A$. Hence the $p_A$ cannot all be

independent of the $a^A$ and the second condition (5-4) follows. We shall assume both matrices of coefficients in (5-4) to be of rank N-1, since this is the case of a system with no constraints other than the energy equation.

Given the equations (5-2), the solution of them consists in inverting the first equation to obtain q as a function of (a,b), then substituting this into the second equation to do the same for p to obtain

$$q = Q(a,b) \quad ; \quad p = P(a,b) = \partial S/\partial q \big|_{Q(a,b)} \quad .$$

Thus for a constant set of a's, the first of (5-2) defines a mapping from the space of q's to the space of b's, and the solution procedure consists in inverting this mapping. However the second condition (5-4) denies the possibility of carrying out this procedure, so the most we can hope for is to split the phase space into fibres of the mapping. Since we assume the rank of the matrices in (5-4) to be N-1, the "rank theorem" of analysis ensures that for an arbitrary constant choice of $a^A$ these fibres comprise an (N-1) parameter congruence of curves in the space of coordinates (see, for example, chapter X of Dieudonné,1960 for a discussion of the analytic provisos contingent to this theorem). Each of these curves in q-space is mapped to a single point in b-space. Along each curve we can now choose a parameter $\tau$ in some smooth way, so that any position q is uniquely defined by the N+1 quantities $(b_A, \tau)$. The statement of the Hamilton-Jacobi theorem is now that with an appropriate choice of $\tau$ the fibre curves of this

congruence satisfy the canonical equations (5-3).[1]

Now let $(q(\tau), p(\tau))$ be some solution of (5-2). Differentiating (5-1) with respect to a and q respectively we find

$$\frac{\partial H}{\partial p_A} \frac{\partial^2 S}{\partial q^A \partial a^B} = 0 \;\; ; \;\; \frac{\partial H}{\partial q^B} + \frac{\partial H}{\partial p_A} \frac{\partial^2 S}{\partial q^A \partial q^B} = 0 \; , \;\; -- \;(5\text{-}5)$$

and differentiation of (5-2) with respect to $\tau$ gives

$$0 = \frac{\partial^2 S}{\partial q^A \partial a^B} \dot{q}^A \;\; ; \;\; \dot{p}_B = \frac{\partial^2 S}{\partial q^A \partial q^B} \dot{q}^A \; . \;\;\;\;\;\; -- \;(5\text{-}6)$$

Combining the second of each of these two sets of equations we have

$$\dot{p}_B + \frac{\partial H}{\partial q^B} = \frac{\partial^2 S}{\partial q^A \partial q^B} \left[ \dot{q}^A - \frac{\partial H}{\partial p_A} \right] \; . \;\;\;\;\;\; -- \;(5\text{-}7)$$

Now from the first of (5-5) and (5-6) we see that both $\dot{q}$ and $\partial H / \partial p$ lie in the null space of the mapping defined by (5-2), which from the above considerations has dimension 1. Thus we must have

$$\partial H / \partial p = \lambda \dot{q}$$

for some real number $\lambda$. Substituting into (5-7) we find in addition that

$$\partial H / \partial q = -\lambda \dot{p} \; .$$

As in Synge's theory, we can always change parameter in these equations in such a way that $\lambda$ transforms to 1. Thus the canonical equations (5-3) hold, but _only_ for the specific parameter $\tau$ which sets $\lambda = 1$. The mapping (5-2) defines the trajectories, but _not_ the parameter - we must choose the particular $\tau$ defined by the Hamiltonian in (5-1) to ensure $\lambda = 1$. This in turn will be

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[1] I am grateful to my supervisor Dr.Dampier for clarifying this aspect of the H-J theorem.

specified by the particular dynamical problem with which we are concerned.

The Hamilton-Jacobi theorem shows that in order to solve the Hamiltonian problem (5-3) all we have to do is to find _any_ complete integral of the H-J equation (5-1), which then gives the solution via the equations (5-2). This solution will involve 2N constants $(a^A, b_A)$, but not all of these are independent. First, since (5-1) only contains the derivatives of S, S can only be determined up to a purely additive constant which has no effect on the equations (5-2). If we select this constant to be $a^M$ (1⩽M⩽N) then it may be ignored. In addition the substitution of this constant into the first of (5-2) yields $b_M = 1$, so the conjugate quantity $b_M$ may also be ignored. It may be convenient (as in classical mechanics) to choose the set $(a^\alpha, b_\alpha)$ as the independent constants, or alternatively the problem may suggest some other choice - as in the example (5.2) given later in this section.

Before tackling an example using the H-J method it will be convenient to look at two ways in which the technique can be simplified for certain classes of systems. The first of these is the method of _ignorable_ _coordinates_. One of the $q^A$ is said to be ignorable if H does not contain that coordinate explicitly. Suppose the particular coordinate $q^M$ (1⩽M⩽N) is ignorable in a given problem for which the H-J equation reads

$$H(q, \partial S/\partial q) = 0 .$$

To determine a complete integral we write

$$S \equiv a^M q^M + K , \text{ (no summation)} \qquad\qquad -- (5-8)$$

where K is a function of the constants $a^A$ and of all coordinates $q^A$ _except_ the particular coordinate $q^M$. K is then a complete

integral of the equation

$$H(q^1,...,q^{M-1},q^{M+1},...,q^N,\partial K/\partial q^1,...,a^M,...,\partial K/\partial q^N) = 0 \; ,$$

and the integrals of the canonical equations are given by

$$-b_M = q^M + \partial K/\partial a_M \quad ; \quad p_M = a^M$$

and

$$-b_A = \partial K/\partial a^A \quad ; \quad p_A = \partial K/\partial q^A \; , \quad \text{for all } A \neq M \; .$$

Notice that the top pair of equations appears somewhat similar to the trivial transformation (1-16), with coordinates transformed into momenta and *vice versa*. This arises because of our use of the 2-point function as the generating function in (5-1). In the classical case (1-21) we chose a case 2 generating function $S(q,P)$, which maintained the position/momentum distinction. Since this distinction is largely nominal we prefer to use the case 1 function $S(Q,q)$, which illustrates more effectively the physical setup behind the H-J method.

The second simplification we can sometimes make to the H-J method occurs if we can express S as the sum of functions of separate coordinates, each function involving just one of the $q^A$ (plus the constants $a^A$). In this case the system is said to be <u>separable</u> in the particular coordinates chosen. Many important classical systems admit separable solutions, and it is always worth trying for this simplification, especially as separability is not merely a property of the system, but also of the particular coordinates used to describe it.

## Example (5.2) - The relativistic Kepler problem.

Consider a particle of mass m and charge e in the central electrostatic field of a stationary charge e' at the origin in Minkowski spacetime. The H-J equation for this problem is obtained immediately from (4-32):

$$g^{ab} \left[ \frac{\partial S}{\partial z^a} - eA_a \right] \left[ \frac{\partial S}{\partial z^b} - eA_b \right] + m^2 = 0 \; , \qquad \text{-- (5-9)}$$

where
$$A_a = (0,0,0,e'/r) \ .$$

Using spherical polar coordinates we obtain

$$\left[\frac{\partial S}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial S}{\partial \theta}\right]^2 + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial S}{\partial \varphi}\right]^2 - \left[\frac{\partial S}{\partial t} - \frac{ee'}{r}\right]^2 + m^2 = 0 .\text{--} \quad (5\text{-}10)$$

Here $\varphi$ and $t$ are ignorable coordinates, so if we assume a separable solution it will take the form

$$S = a_1 + S_1(r) + S_2(\theta) + a_3\varphi + a_4 t \qquad \text{--} \quad (5\text{-}11)$$

and substituting this into (5-10) we find

$$\left[\frac{\partial S_1}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial S_2}{\partial \theta}\right]^2 + \frac{(a_3)^2}{r^2\sin^2\theta} - \left[a_4 - \frac{ee'}{r}\right]^2 + m^2 = 0$$

$$\Rightarrow \quad r^2\left\{\left[\frac{\partial S_1}{\partial r}\right]^2 - \left[a_4 - \frac{ee'}{r}\right]^2 + m^2\right\} = -\left[\frac{\partial S_2}{\partial \theta}\right]^2 - \frac{(a_3)^2}{\sin^2\theta}$$

$$= (a_2)^2 \ ,$$

where $a_2$ is a separation constant. Hence

$$\left.\begin{aligned}\left[\frac{\partial S_1}{\partial r}\right]^2 - \left[a_4 - \frac{ee'}{r}\right]^2 + m^2 &= \left[\frac{a_2}{r^2}\right]^2 \ ; \\[2mm] \left[\frac{\partial S_2}{\partial \theta}\right]^2 + \frac{(a_3)^2}{\sin^2\theta} &= -(a_2)^2 \ ,\end{aligned}\right\} \quad \text{--} \quad (5\text{-}12)$$

where we therefore have

$$S_1 = S_1(r,a_2,a_4) \ ; \quad S_2 = S_2(\theta,a_2,a_3) \ .$$

Using the above results it is possible to draw certain conclusions which enable us to simplify the situation greatly. From (5-12) we see that

$$p_2 = \partial S_2/\partial \theta = \sqrt{\{-[(a_2)^2 + (a_3)^2/\sin^2\theta]\}} \ ,$$

but on the other hand $p_2 = mr^2\dot\theta$, so

$$\dot p_2 = \frac{(a_3)^2\cos\theta.\dot\theta}{p_2\sin^3\theta} = 2mr\dot r\dot\theta + mr^2\ddot\theta$$

$$\Rightarrow \quad \ddot\theta = \frac{1}{mr^2}\left[\frac{(a_3)^2\cos\theta}{mr^2\sin^3\theta} - 2mr\dot r\dot\theta\right] \ .$$

Thus when the simultaneous conditions $\theta = \pi/2$, $\dot\theta = 0$ are fulfilled $\ddot\theta$ is necessarily zero. If the particle is at any instant moving in

the (x,y) plane then it will continue to do so for all time. We can therefore simplify our problem by setting $\theta = \pi/2$ and eliminating it from our calculations. In this case $p_2 = 0$ and we obtain the relation

$$(a_3)^2 = -(a_2)^2 ,$$

so the assumed solution (5-11) becomes

$$S = a_1 + S_1(r, a_3, a_4) + a_3\varphi + a_4 t . \qquad -- (5-13)$$

Substitution into the H-J equation then yields

$$\left[\frac{\partial S_1}{\partial r}\right]^2 - \left[a_4 - \frac{ee'}{r}\right]^2 + m^2 = -\left[\frac{a_3}{r}\right]^2 , \qquad -- (5-14)$$

which on integration gives

$$\left. \begin{array}{l} S_1 = \int \sqrt{f(r)} . dr , \\[2mm] \text{where} \qquad f(r) = \left[a_4 - \frac{ee'}{r}\right]^2 - \left[\frac{a_3}{r}\right]^2 - m^2 . \end{array} \right\} \qquad -- (5-15)$$

The solution of the Hamiltonian problem is then given by

$$\left. \begin{array}{l} -b_3 = \varphi + \dfrac{\partial S_1}{\partial a_3} = \varphi + \displaystyle\int \frac{a_3 \, du}{\sqrt{f(r)}} \; ; \\[4mm] -b_4 = t + \dfrac{\partial S_1}{\partial a_4} = t + \displaystyle\int \frac{(ee'u - a_4) \, du}{u^2 \sqrt{f(r)}} \; ; \\[4mm] p_3 = a_3 \; ; \quad p_4 = a_4 \; ; \\[2mm] p_1 = \sqrt{f(r)} \\[2mm] \quad = \sqrt{\{[(ee')^2 - (a_3)^2]u^2 - 2ee'a_4 u + (a_4)^2 - m^2\}} \; , \\[2mm] \text{where} \qquad u = 1/r \end{array} \right\} \qquad -- (5-16)$$

is the usual substitution taken from celestial mechanics.

At this point we define certain constants which will simplify the coming work. These are:

$$\left. \begin{array}{l} E = -a_4 \; ; \quad h = a_3 \; ; \\[2mm] a = (ee')^2 - h^2 \; ; \quad b = 2ee'E \; ; \quad c = E^2 - m^2 \; ; \\[2mm] \omega^2 = -a/h^2 = 1 - (ee'/h)^2 \; ; \quad \beta = \omega b_3 . \end{array} \right\} \qquad -- (5-17)$$

The rationale behind this choice of symbols becomes clear on a closer examination of (5-16), where we see that E is the total energy of the orbiting particle and h is the angular momentum.

In this notation the equations (5-16) now take the form

$$-b_3 = \varphi + \int \frac{h\,du}{\sqrt{(au^2 + bu + c)}} \; ;$$

$$-b_4 = t + \int \frac{(ee'u + E)\,du}{u^2\sqrt{(au^2 + bu + c)}} \; ; \qquad \text{-- (5-18)}$$

$$p_3 = h \; ; \quad p_4 = -E \; ; \quad p_1 = \sqrt{(au^2 + bu + c)}$$

and a lengthy integration leads to

$$-b_3 = \varphi - \frac{h}{\sqrt{(-a)}} \sin^{-1}\left[\frac{2au + b}{\sqrt{(b^2 - 4ac)}}\right] \; ; \qquad \text{-- (5-19a)}$$

$$-b_4 = t - \frac{\sqrt{(au^2 + bu + c)}}{u} - \frac{m^2 ee'}{(-c)^{3/2}} \sin^{-1}\left[\frac{bu + c}{u\sqrt{(b^2 - 4ac)}}\right] . \qquad \text{-- (5-19b)}$$

The second of these will clearly not admit of an immediately illuminating solution, and indeed the Newtonian Kepler problem does not possess a closed solution for r in terms of t, so we concentrate our attention on the equation (5-19a) of the orbit. Turned "inside-out" this gives

$$\frac{2au + b}{\sqrt{(b^2 - 4ac)}} = \sin\left[\frac{\sqrt{(-a)}}{h}(\varphi + b_3)\right] \; ,$$

and using the constants (5-17) we find

$$u = A \sin(\omega\varphi + \beta) + \frac{ee'E}{h^2\omega^2} \; ,$$

$$\text{where} \qquad A^2 = \frac{b^2 - 4ac}{4a^2} = \frac{E^2 - m^2\omega^2}{h^2\omega^4} \; . \qquad \text{-- (5-20)}$$

The solution (5-20) is in many ways similar to the solution of the classical Kepler problem. It consists of a constant term involving the energy E and angular momentum h plus a sine term. Here, however, a major difference occurs, since the orbit will only be closed if ω is a rational multiple of 1/π. In the classical case ω≈1, since a factor $c^{-2}$ is involved in its definition, and so the orbit is aways closed when finite. In order

to compare this with the relativistic case we now seek the finite orbits of (5-20).

First of all, we clearly require $\omega^2 \geqslant 0$. If this were not so then $\omega$ would be imaginary and the solution would be hyperbolic - either spiralling into the origin or spiralling out to infinity; neither case has a classical counterpart. In the case $\omega = 0$ the sine term vanishes and the orbit is a circle of radius R given by

$$\frac{1}{R} = \frac{ee'E}{h^2\omega^2} . \qquad\qquad -- (5-21)$$

Now for the orbit to be finite we require the radius r to oscillate between two extreme points at each of which $p_1 = 0$ (or, more strictly, $\dot{r} = 0$). These extreme values are therefore given by the expression

$$au^2 + bu + c = 0$$

$$=> \qquad u = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

and substituting into this from (5-17) we find

$$u = \frac{ee'E}{h^2\omega^2} \pm A .$$

Clearly only those solutions corresponding to positive values of r and u will be meaningful, in which case the condition for a finite orbit is

$$ee'E/(h^2\omega^2) > A$$

$$=> \qquad E^2(ee')^2 > h^2\{E^2 - m^2\omega^2\}$$

$$=> \qquad E < m , \qquad\qquad -- (5-22)$$

where we have used (5-17). Note that we could have obtained (5-22) directly from (5-20); we use the above approach merely to illustrate a more general method. When condition (5-22) is fulfilled the orbit will be finite, but in general not closed. r is a function in $\varphi$ of period $2\pi/\omega$, and $\omega < 1$; the values of r do

not begin to repeat until slightly <u>after</u> the radius vector has made a complete revolution. The advance of perihelion in one revolution is

$$\Delta\varphi = 2\pi/\omega - 2\pi = 2\pi(\omega^{-1} - 1)$$
$$\approx \pi (ee'/h)^2 ,$$

provided this last fraction is small.

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## 5.2 Lie groups

Many of the ideas related to Hamiltonian mechanics can be expressed compactly in the language of <u>Lie groups</u>. A Lie group is basically a manifold with a group structure defined on it; put more formally, it is a topological group in which there exists some neighbourhood of the identity which is homeomorphic to an open, bounded subset of $\underline{R}^N$. There are many types of representation of Lie groups, but what we shall be concerned with here is the Lie group of transformations of some N-dimensional manifold M. In fact, in any particular discussion we shall restrict our attention to the 1-parameter family of point transformations indexed by the real numbers:

$$\overline{q} = q_\varepsilon(q) , \qquad\qquad -- (5-23)$$

where we assume that the family $G_1 = \{q_\varepsilon : \varepsilon$ is a real number$\}$ forms a continuous Lie subgroup of transformations. Finally, we assume the existence of a suitable number of derivatives of $q_\varepsilon$ in the neighbourhood of the identity.

The analytic dependence of $q_\varepsilon$ upon $\varepsilon$ implies the existence of the infinitesimal transformations: Let $q_0$ be the identity transformation, then for a small change $d\varepsilon$ in the parameter we have

$$\overline{q} = q + \frac{dq_{\varepsilon}(q)}{d\varepsilon}\Big|_{o} \cdot d\varepsilon \quad .$$

Now let $\quad \xi(q) \equiv \frac{dq_{\varepsilon}(q)}{d\varepsilon}\Big|_{o} \quad , \qquad\qquad$ -- (5-24)

then the infinitesimal transformation can be written

$$\overline{q}^A = q^A + \xi^A d\varepsilon \quad , \qquad\qquad \text{-- (5-25)}$$

where the functions $\xi^A(q)$ define the transformation locally. We can construct the global (finite) transformation (5-23) from the $\xi^A$ in the following way: The $\xi^A$ attach to each point q in M a direction field defined by

$$d\varepsilon = \frac{dq^1}{\xi^1(q)} = \frac{dq^2}{\xi^2(q)} = \ldots = \frac{dq^N}{\xi^N(q)} \quad , \qquad \text{-- (5-26)}$$

which is equivalent to the following set of first-order differential equations (DE's):

$$dq^A/d\varepsilon = \xi^A(q) \quad . \qquad\qquad \text{-- (5-27)}$$

Integration of these equations generates a unique curve $q(\varepsilon)$ through each initial point, along which the higher derivatives corresponding to (5-27) are

$$\frac{d^2q^A}{d\varepsilon^2} = \frac{\partial\xi^A}{\partial q^B}\frac{dq^B}{d\varepsilon} = \xi^A_{,B}\,\xi^B \quad . \qquad\qquad \text{-- (5-28)}$$

Therefore, for $\overline{q}$ close to the initial point q, we have

$$\overline{q}^A = q^A + \varepsilon\xi^A(q) + \frac{\varepsilon^2}{2}\xi^A_{,B}\,\xi^B + \ldots \qquad \text{-- (5-29)}$$

This power series will in general converge in some neighbourhood of $\varepsilon=0$ and represents the global transformation within the neighbourhood $N_o$ of convergence. Proof of the group properties of (5-29) within $N_o$ is straightforward.

The integral curves of (5-27) are referred to as the _trajectories_ of that equation, and form a congruence on $N_o$. We can label the trajectories using the N-1 functions $u^{\alpha}$:

$$u^{\alpha}(q) = c^{\alpha} = \text{const.} \quad , \qquad\qquad \text{-- (5-30)}$$

where each collection of N-1 values $c^{\alpha}$ defines the curve which is

the intersection of the corresponding N-1 hypersurfaces. The $u^\alpha(q)$ are called _integrals_ of (5-26) and (5-27); they have the property, following directly from (5-30), that they are _invariants_ of the transformation (5-25). An invariant is any function $\varphi$ with the property that it is constant along all trajectories of (5-27), ie:

$$\varphi(\overline{q}) = \varphi(q) \quad .$$

Since for the infinitesimal transformation

$$\varphi(\overline{q}) = \varphi(q) + d\varepsilon\xi^A \partial\varphi/\partial q^A$$

a necessary condition for $\varphi$ to be invariant under (5-25) is that it satisfies the associated partial differential equation (PDE)

$$\hat{U}\varphi \equiv \xi^A \partial\varphi/\partial q^A = 0 \quad . \qquad\qquad \text{-- (5-31)}$$

This may also be shown to be a sufficient condition (see for example Bluman & Cole,1974). $\hat{U}$ is called the _infinitesimal generator_ of the group $G_1$. Under a change of coordinates $q^A \rightarrow q'^A$ (5-31) transforms into the new form

$$\hat{U}'\varphi = (\hat{U}q'^A) \; \partial\varphi/\partial q'^A \qquad\qquad \text{-- (5-32)}$$

provided $\varphi$ is a scalar quantity (see Bluman & Cole,1974).

The importance of invariant functions lies in the fact that they enable us to test for a variety of types of symmetry of a system. A symmetry in this sense is a family of points which is mapped to itself under the group $G_1$. As an example, consider a specific curve $c^\alpha$ defined by (5-30). Under a transformation (5-23) the curve is "shifted" a distance $\varepsilon$ along itself, so although each point of the curve is affected by the transformation, the curve as a whole is not. In the same way any independent collection of invariant functions $u^\alpha$ defines an invariant domain on M, where $\alpha$ varies from 1 to some number n$\leq$N. If n=N then (5-30) defines an invariant point; if n=N-1 then (5-30) defines an invariant curve; and if n=N-2 then (5-30)

defines an invariant 2-surface. Analytically, we can obtain the condition for a collection $u^\alpha$ ($1 \leqslant \alpha \leqslant n$) to represent an invariant hypersurface from (5-31):

$$u^\alpha(q) = c^\alpha \text{ is an invariant hypersurface iff}$$
$$\hat{U}u^\alpha = 0 \text{ when } u^\alpha = c^\alpha \text{ (not all } u^\alpha_{,A} = \text{zero).}$$
$$-- \text{(5-33)}$$

Note that the case of the invariant point (n=N) is a special one, depending on the particular group considered, whereas we are assured of the existence of N-1 integrals $u^\alpha$ since each member of the congruence of integral curves is an invariant curve. The general invariant is then written

$$I = I(u^\alpha) . \qquad\qquad -- \text{(5-34)}$$

In view of the obvious relevance of the above to dynamical systems it was natural in the context of classical mechanics to attempt to incorporate time symmetrically into the transformations of the group $G_1$; this led to the development of the <u>extended theory</u> of Lie groups. Consider the situation where we adopt one of the coordinates - $q^N$=t, say - as the dynamical parameter. In this case the equations (5-23):

$$\bar{q} = q_\varepsilon(q) = \bar{q}(q^\alpha, t) \qquad\qquad -- \text{(5-35)}$$

directly affect the dynamical parameter, and the resulting changes in $\dot{q}$ must be taken into account. Let

$$q^\alpha = q^\alpha(t)$$

be an arbitrary curve in M, then under (5-35) a new curve

$$\bar{q}^\alpha = \bar{q}^\alpha(\bar{t})$$

is generated (note how Lie's theory contained implicit reference to arbitrary transformations of time more than thirty years before the advent of relativity). The tangential directions to this curve transform in the following way:

$$\bar{\dot{q}}^\alpha \equiv \frac{d\bar{q}^\alpha}{d\bar{t}} = q^\alpha_{,A}\frac{dq^A}{t_{,B}dq^B} = \frac{q^\alpha_{,\beta}\dot{q}^\beta + q^\alpha_{,N}}{t_{,\gamma}\dot{q}^\gamma + t_{,N}} \quad .$$

Thus we obtain the extended transformation defined in terms of (5-35):

$$\bar{q}^\alpha = \bar{q}^\alpha(q^\alpha,t) \quad ; \quad \bar{t} = \bar{t}(q^\alpha,t) \quad ; \quad \left.\begin{array}{c} \\ \\ \end{array}\right\}$$

$$\dot{\bar{q}}^\alpha = \frac{q^\alpha_{,\beta}\dot{q}^\beta + q^\alpha_{,N}}{t_{,\gamma}\dot{q}^\gamma + t_{,N}} \quad . \qquad \qquad \text{-- (5-36)}$$

These transformations form a group of necessity, since (5-36) simply expresses a special formulation (the "classical decomposition") of the general Lie group transformation (5-23). The point about (5-36) is that it extends the effect of the N transformations (5-35) to a transformation on the extended space M' of the (2N-1) variables $(q^A,\dot{q}^\alpha)$.

For the investigation of invariance properties in M' it is again useful to look at the effect of infinitesimal transformations (5-25) on M'. The infinitesimal form of (5-36) arises from the local transformation

$$\bar{q}^\alpha = q^\alpha + \xi^\alpha d\epsilon \quad ; \quad \bar{t} = t + \xi^N d\epsilon$$

with infinitesimal generator

$$\hat{U}f \equiv \xi^\alpha \partial f/\partial q^\alpha + \xi^N \partial f/\partial t \quad .$$

From this the transformation of $\dot{q}^\alpha$ follows:

$$\dot{\bar{q}}^\alpha = \frac{dq^\alpha + d\epsilon d\xi^\alpha}{dt + d\epsilon d\xi^N}$$

$$= \frac{\dot{q}^\alpha + d\epsilon\dot{\xi}^\alpha}{1 + d\epsilon\dot{\xi}^N}$$

$$= \dot{q}^\alpha + d\epsilon \{\dot{\xi}^\alpha - \dot{q}^\alpha\dot{\xi}^N\} \quad ,$$

using which we can write down the form of the infinitesimal generator Û' in M':

$$\hat{U}'f \equiv \xi^A \partial f/\partial q^A + \eta^\alpha \partial f/\partial \dot{q}^\alpha \quad , \quad \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}$$

$$\text{where} \qquad \eta^\alpha \equiv \{\dot{\xi}^\alpha - \dot{q}^\alpha\dot{\xi}^N\}$$

$$= \frac{\partial\xi^\alpha}{\partial t} + \dot{q}^\beta\left[\xi^\alpha_{,\beta} - \frac{\partial\xi^N}{\partial t}\delta^\alpha_\beta\right] \qquad \text{-- (5-37)}$$

$$- (\dot{q}^\alpha\dot{q}^\beta)\,\xi^N_{,\beta} \quad .$$

$\overset{\wedge}{U}{}'$ is called the <u>first</u> <u>extension</u> of the generator $\overset{\bullet}{U}$. This definition may easily be extended to higher derivatives, but (5-37) will suffice for our purposes.

Using the extended infinitesimal generator we can now formulate the requirement that a system of DE's must fulfill if it admits a group of symmetries. The most general system of first-order DE's needed to define a congruence on M may be written

$$u^{\alpha}(q^{\alpha}, t, \dot{q}^{\alpha}) = c^{\alpha} = \text{const.} \qquad \text{-- (5-38)}$$

In M (5-38) defines a congruence of integral curves, but in M' it defines an N-dimensional hypersurface generated by the integral curves. The system (5-38) is said to <u>admit</u> a given group $G_1$ of transformations if the effect of all elements of $G_1$ is simply to transform the integral curves of (5-38) into one another. A necessary condition for this is that the hypersurface (5-38) is transformed into itself under all members of $G_1$. But this is also a sufficient condition, since (5-38) may be regarded as defining the tangent direction of the integral curves at all points of M. Thus by using the condition (5-33) for the invariance of a hypersurface we see that (5-38) admits the group generated by $\xi^A$ iff

$$\left. \begin{array}{c} \overset{\wedge}{U}{}' u^{\alpha} \equiv \xi^A \dfrac{\partial u^{\alpha}}{\partial q^A} + \eta^{\beta} \dfrac{\partial u^{\alpha}}{\partial \dot{q}^{\beta}} = 0 \\[2ex] \text{whenever} \qquad u^{\alpha}(q^A, \dot{q}^{\beta}) = c^{\alpha}. \end{array} \right\} \qquad \text{-- (5-39)}$$

## 5.3 Application to symmetries in mechanics

The extended theory of Lie groups has recently been applied (Prince & Leach,1980; Leach,1981) to a number of problems in classical mechanics in order to calculate symmetries. This method has the advantage that it yields a number of symmetries

which are ignored by Noether's theorem - notably the Runge-Lenz vector for the Kepler problem. Before attempting to recast the Lie method in the notation of homogeneous mechanics we shall summarise in this section the findings in the two papers mentioned above.

Suppose we have a (classical) Lagrangian $L(q^\alpha, \dot{q}^\alpha, t)$ for a system and a Hamiltonian related to the Lagrangian by the Legendre transformation

$$\left.\begin{array}{ll}
& H(q^\alpha, p_\alpha, t) = \dot{q}^\alpha p_\alpha - L(q^\alpha, \dot{q}^\alpha, t) \ , \\
\text{where} & p_\alpha = \partial L / \partial \dot{q}^\alpha \\
\text{and for which} & \dot{q}^\alpha - \partial H / \partial p_\alpha = 0 \ ; \\
& \dot{p}_\alpha + \partial H / \partial q^\alpha = 0 \ .
\end{array}\right\} \quad -- (5\text{-}40)$$

The general infinitesimal generator for the Lagrangian formulation may be obtained directly from (5-37):

$$\xi^A(q^\alpha, t) \partial/\partial q^A + \{\dot{\xi}^\alpha - \dot{q}^\alpha \dot{\xi}^N\} \partial/\partial \dot{q}^\alpha \ ,$$

which by application of (5-32) may be transformed to the canonical coordinates $(q^A, p_\alpha)$:

$$\left.\begin{array}{l}
\tilde{U}(q^\alpha, p_\alpha, t) = \xi^A \dfrac{\partial}{\partial q^A} + \pi_\alpha \dfrac{\partial}{\partial p_\alpha} \ , \\[2mm]
\text{where } \pi_\alpha(q^\alpha, p_\alpha, t) = \xi^A \dfrac{\partial^2 L}{\partial q^A \partial \dot{q}^\alpha} + \{\dot{\xi}^\beta - \dot{q}^\beta \dot{\xi}^N\} \dfrac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\alpha} .
\end{array}\right\} \quad -- (5\text{-}41)$$

(5-41) may in turn be extended to include the variables $\dot{q}^\alpha, \dot{p}_\alpha$ by repeated application of (5-37):

$$\left.\begin{array}{l}
\tilde{U}'(q^A, p_\alpha) = \xi^A \dfrac{\partial}{\partial q^A} + \pi_\alpha \dfrac{\partial}{\partial p_\alpha} + \xi'^\alpha \dfrac{\partial}{\partial \dot{q}^\alpha} + \pi'_\alpha \dfrac{\partial}{\partial \dot{p}_\alpha} \ , \\[3mm]
\text{where } \xi'^\alpha = \{\dot{\xi}^\beta - \dot{q}^\beta \dot{\xi}^N\} \\[2mm]
\qquad = \dfrac{\partial \xi^\alpha}{\partial t} + \dfrac{\partial H}{\partial p_\beta} \dfrac{\partial \xi^\alpha}{\partial q^\beta} - \dfrac{\partial H}{\partial p_\alpha}\left[\dfrac{\partial \xi^N}{\partial t} + \dfrac{\partial H}{\partial p_\beta} \dfrac{\partial \xi^N}{\partial q^\beta}\right] \ ; \\[3mm]
\quad \pi'_\alpha = \dot{\pi}_\alpha - \dot{p}_\alpha \dot{\xi}^N \\[2mm]
\qquad = \dfrac{\partial \pi_\alpha}{\partial t} + \dfrac{\partial H}{\partial p_\beta} \dfrac{\partial \pi_\alpha}{\partial q^\beta} + \dfrac{\partial H}{\partial q^\alpha}\left[\dfrac{\partial \xi^N}{\partial t} + \dfrac{\partial H}{\partial p_\beta} \dfrac{\partial \xi^N}{\partial q^\beta}\right] \\[3mm]
\qquad\quad - \dfrac{\partial H}{\partial q^\beta} \dfrac{\partial \pi_\alpha}{\partial p_\beta} \ ,
\end{array}\right\} \quad -- (5\text{-}42)$$

and where we have substituted from the last two equations (5-40).

Having calculated the extended generators (5-41) and (5-42) we

now seek those point transformations in phase $(q^A, p_\alpha)$ space which

will leave the form of the canonical equations in (5-40) invar-

iant. The infinitesimal condition for this is given by

$$\hat{U} \cdot (\dot{q}^\alpha - \partial H/\partial p_\alpha) = 0 \quad \text{whenever} \quad \dot{q}^\alpha - \partial H/\partial p_\alpha = 0 \; ; \left.\right\}$$
$$\hat{U} \cdot (\dot{p}_\alpha + \partial H/\partial q^\alpha) = 0 \quad \text{whenever} \quad \dot{p}_\alpha + \partial H/\partial q^\alpha = 0 \; , \left.\right\} \quad \text{-- (5-43a)}$$

or alternatively

$$\xi^{\cdot \alpha} - \hat{U} \, \partial H/\partial p_\alpha = 0 \quad \text{whenever} \quad \dot{q}^\alpha - \partial H/\partial p_\alpha = 0 \; ; \left.\right\}$$
$$\pi'_\alpha + \hat{U} \, \partial H/\partial q^\alpha = 0 \quad \text{whenever} \quad \dot{p}_\alpha + \partial H/\partial q^\alpha = 0 \; . \left.\right\} \quad \text{-- (5-43b)}$$

We are now in a position to apply the above method to the

special case of the classical Kepler problem (see Leach,1981).

Here the Hamiltonian is

$$H = p^2/2m - ee'/r \qquad \text{-- (5-44)}$$

and the canonical equations take the form

$$\dot{q}^\alpha - p_\alpha/m = 0 \quad ; \quad \dot{p}_\alpha + ee'q^\alpha/r^3 = 0 \; , \left.\right\}$$
$$\text{where} \quad r^2 = q^\alpha q^\alpha = x^2 + y^2 + z^2 \; . \left.\right\} \quad \text{-- (5-45)}$$

Here $\alpha$ takes the range 1,2,3, and N=4. Leach makes the unusual

choice of Cartesian coordinates for the Kepler problem because of

certain simplifications thus incurred in the following calcula-

tions (see section 5.4). Applying (5-42) to (5-45) we have

$$\xi^{\cdot \alpha} - \hat{U} p_\alpha/m = 0 \quad ; \quad \pi'_\alpha + ee' \hat{U} (q^\alpha/r^3) = 0$$
$$\Rightarrow \quad \xi^{\cdot \alpha} - \pi_\alpha/m = 0 \; ; \; \pi'_\alpha + \frac{ee' \xi^\beta}{r^3} \left[ \delta_{\alpha\beta} - \frac{3q^\alpha q^\beta}{r^2} \right] = 0 . \quad \text{-- (5-46)}$$

Since the coming calculation is a little tedious, we had

best summarise what we are doing beforehand. The first of

(5-46), together with (5-42), gives us an expression for $\pi_\alpha$ in

terms of $\xi^a$, which may then be substituted into the last of

(5-42) to obtain $\pi'_\alpha$ in terms of $\xi^a$. But the second of (5-46) is

also such a relation, so we can equate these two expressions for

$\pi'_\alpha$ in (5-47), and by equating powers of q and p obtain the final solution (5-54) for $\xi^a$.

$$\pi_\alpha = m\xi^{\cdot\alpha} = m\frac{\partial\xi^\alpha}{\partial t} + p_\beta\frac{\partial\xi^\alpha}{\partial q^\beta} - p_\alpha\frac{\partial\xi^4}{\partial t} - \frac{p_\alpha p_\beta}{m}\frac{\partial\xi^4}{\partial q^\beta}$$

$$\Rightarrow \quad \pi'_\alpha = \frac{ee'\xi^\beta}{r^5}(3q^\alpha q^\beta - \delta_{\alpha\beta}q^\gamma q_\gamma)$$

$$= m\frac{\partial^2\xi^\alpha}{\partial t^2} + p_\beta\frac{\partial^2\xi^\alpha}{\partial q^\beta\partial t} - p_\alpha\frac{\partial^2\xi^4}{\partial t^2} - \frac{p_\alpha p_\beta}{m}\frac{\partial^2\xi^4}{\partial q^\beta\partial t}$$

$$+ \frac{p_\beta}{m}\left[m\frac{\partial^2\xi^\alpha}{\partial t\partial q^\beta} + p_\gamma\frac{\partial^2\xi^\alpha}{\partial q^\beta\partial q^\gamma} - p_\alpha\frac{\partial^2\xi^4}{\partial t\partial q^\beta} - \frac{p_\alpha p_\gamma}{m}\frac{\partial^2\xi^4}{\partial q^\gamma\partial q^\beta}\right]$$

$$- \frac{ee'q^\beta}{r^3}\left[\frac{\partial\xi^\alpha}{\partial q^\beta} - \delta_{\alpha\beta}\frac{\partial\xi^4}{\partial t} - p_\alpha\frac{\partial\xi^4}{m\partial q^\beta} - \delta_{\alpha\beta}\frac{p_\gamma}{m}\frac{\partial\xi^4}{\partial q^\gamma}\right]$$

$$+ \frac{ee'q^\alpha}{r^3}\left[\frac{\partial\xi^4}{\partial t} + \frac{p_\beta}{m}\frac{\partial\xi^4}{\partial q^\beta}\right] \qquad\qquad -- (5-47)$$

Equating terms in $p^3$ and $p^2$ in (5-47) now yields the basic forms of $\xi^\alpha$, $\xi^4$:

$(p^3)$: $\qquad 0 = -\frac{p_\alpha p_\beta p_\gamma}{m^2}\frac{\partial^2\xi^4}{\partial q^\beta\partial q^\gamma}$ ;

$(p^2)$: $\qquad 0 = -\frac{2p_\alpha p_\beta}{m}\frac{\partial^2\xi^4}{\partial q^\beta\partial t} + \frac{p_\beta p_\gamma}{m}\frac{\partial^2\xi^\alpha}{\partial q^\beta\partial q^\gamma}$ .

This comparison is possible because we know that $\xi$ contains no p dependence. $(p^3)$ implies that $\xi^4$ is at most linear in the co-ordinates $q^\beta$, given which fact it follows from $(p^2)$ that $\xi^\alpha$ is at most quadratic in $q^\beta$. It may easily be shown from these equations that the leading coefficients in $\xi^\alpha$ are the time derivatives of the linear coefficients in $\xi^4$:

$$\xi^4 = a(t) + b_\beta(t)q^\beta ;$$
$$\xi^\alpha = \dot{b}_\beta q^\alpha q^\beta + c_{\alpha\beta}(t)q^\beta + d_\alpha(t) . \left.\begin{array}{c}\\\\\end{array}\right\} \quad -- (5-48)$$

Up to now equating coefficients has posed no problems because the $\xi^a$ have been assumed independent of p, permitting the equations $(p^3)$ and $(p^2)$. Linear and lower terms in p, however, contain explicit reference to the $q^a$, so care must now be taken.

$(p^1)$:
$$0 = p_\beta \frac{\partial^2 \xi^\alpha}{\partial q^\beta \partial t} - p_\alpha \frac{\partial^2 \xi^4}{\partial t^2} + p_\beta \frac{\partial^2 \xi^\alpha}{\partial t \partial q^\beta}$$
$$+ \frac{ee'q^\beta}{mr^3} \left[ p_\alpha \frac{\partial \xi^4}{\partial q^\beta} + 2\delta_{\alpha\beta} p_\gamma \frac{\partial \xi^4}{\partial q^\gamma} \right]$$

By substituting from (5-48) into this equation we obtain two relations corresponding to terms in $q^0$ and $q^1$ respectively:

$$a \, \delta_{\alpha\beta} = 2\dot{c}_{\alpha\beta} \; ; \qquad\qquad -- \; (5-49)$$

$$(q^\beta p_\alpha + 2q^\alpha p_\beta) \, b_\beta = 0 \quad \Rightarrow \quad b_\beta = 0 \; . \qquad -- \; (5-50)$$

Using this in the remainder of (5-47) we obtain the terms independent of p:

$(p^0)$: $\quad m\frac{\partial^2 \xi^\alpha}{\partial t^2} - \frac{ee'q^\beta}{r^3}\frac{\partial \xi^\alpha}{\partial q^\beta} + \frac{2ee'q^\alpha}{r^3}\frac{\partial \xi^4}{\partial t} = \frac{ee'\xi^\beta}{r^5}(3q^\alpha q^\beta - \delta_{\alpha\beta}q^\gamma q_\gamma)$

$\Rightarrow \qquad\qquad c_{\alpha\beta}q^\beta + d_\alpha + \frac{ee'q^\beta}{r^3}(2\dot{a}\delta_{\alpha\beta} - c_{\alpha\beta})$

$$= \frac{ee'}{r^5}(c_{\beta\gamma}q^\gamma + d_\beta)(3q^\alpha q^\beta - \delta_{\alpha\beta}q^\gamma q_\gamma)$$

The first two terms here are independently zero, and by cancelling them and multiplying through by $r^5 = (q^\gamma q_\gamma)^{5/2}$ we obtain the following two independent relations in $q^2$ and $q^3$:

$$(3q^\alpha q^\beta - \delta_{\alpha\beta}q^\gamma q_\gamma)d_\beta = 0 \quad \Rightarrow \quad d_\beta = 0 \qquad -- \; (5-51)$$

$$2\dot{a}q^\alpha q^\gamma q_\gamma = 3c_{\beta\gamma}q^\alpha q^\beta q^\gamma$$

$\Rightarrow \qquad\qquad c_{(\beta\gamma)} = 2/3 . \dot{a}\delta_{\beta\gamma} \; . \qquad -- \; (5-52)$

Now (5-49) tells us that $\dot{c}_{\alpha\beta}$ is symmetric, so it follows that the antisymmetric part $c_{[\alpha\beta]}$ is constant. On the other hand, substituting (5-52) into (5-49) gives

$$a \, \delta_{\alpha\beta} = 4a/3 . \delta_{\alpha\beta} \; ,$$

so a is linear in t:

$$a = At + B$$
$$\text{and} \qquad c_{(\alpha\beta)} = 2/3 . A\delta_{\alpha\beta} \; . \qquad\qquad -- \; (5-53)$$

Collecting together the results (5-48) through to (5-53) we have finally

$$\xi^{\alpha} = 2/3.Aq^{\alpha} + C_{\alpha\beta}q^{\beta} \quad ;$$

$$\xi^{4} = At + B \quad ,$$

$$-- \quad (5-54)$$

where $A, B, C_{\alpha\beta}$ are constants and $C_{\alpha\beta}$ is antisymmetric.

Given the results (5-54) we can now obtain the coefficients $\pi_{\alpha}$ from (5-46) and (5-42):

$$\pi_{\alpha} = (C_{\alpha\beta} - 1/3.A\delta_{\alpha\beta}) \, p_{\beta} \qquad -- \quad (5-55)$$

and the independent generators (5-41) of the symmetry group are found by setting B, A, C equal to 1 in turn and the other constants to zero:

$$\hat{U}_{1} = \partial/\partial t \quad ;$$

$$\hat{U}_{2} = t\partial/\partial t + 2/3.q^{\beta}\partial/\partial q^{\beta} - 1/3.p_{\beta}\partial/\partial p_{\beta} \quad ;$$

$$\hat{U}_{3\alpha\beta} = q^{\alpha}\partial/\partial q^{\beta} - q^{\beta}\partial/\partial q^{\alpha} + p_{\alpha}\partial/\partial p_{\beta} - p_{\beta}\partial/\partial p_{\alpha} \quad .$$

$$-- \quad (5-56)$$

The generators $\hat{U}_{1}, \hat{U}_{3}$ are clearly related to conservation of energy and angular momentum, so as an example of the calculation of an invariant we shall consider the generator $\hat{U}_{2}$. If the first integral of this generator is $I(q,p,t)$ then the equation

$$\hat{U}_{2}I = 0 \qquad -- \quad (5-57)$$

has the associated system

$$\frac{dt}{t} = \frac{3dq^{\beta}}{2q^{\beta}} = -\frac{3dp_{\beta}}{p_{\beta}} \quad . \qquad -- \quad (5-58)$$

where no summation occurs over $\beta$. When (5-58) is integrated we find the following set of possible functions for the integral I:

$$I = u^{\alpha} \equiv q^{\alpha}t^{-2/3} \quad ; \quad I = v_{\alpha} \equiv p_{\alpha}t^{1/3} \quad . \qquad -- \quad (5-59)$$

It may be worthwhile to look back now at what we have done so far in this section. In (5-56) we have found the most general set of generators which leave the equations of motion (5-45) invariant. Taking the particular generator $\hat{U}_{2}$ we then found in (5-59) a complete set of 2N-2 integrals $u^{\alpha}, v_{\alpha}$ for this generator, so that the most general integral of $\hat{U}_{2}$ is

$$I = I(u^{\alpha}, v_{\alpha}) \quad . \qquad -- \quad (5-60)$$

We now come to the question: What is a symmetry of a system? The answer we shall use here is the following:

**Definition (5.3)**: Let $\{\dot{U}_i\}$ be some complete collection of 2N-2 generators which leave the equations of motion of a certain system invariant. An **invariant** of this system is a function $I(q,p,t)$ which has the following two properties:

i. I is an integral of some linear combination of the $\dot{U}_i$ ;

ii. I is a constant of the motion of the system:

$$\dot{I} = 0 .$$

If such an I is found, then the 1-parameter group of transformations generated by the linear combination in condition (i) is a **symmetry** of the system.

---

According to this definition, then, we must check all the integrals defined by (5-59) and (5-60) to find one which is a constant of the motion. The condition for this to be true is

$$0 = \dot{I} = \frac{\partial I}{\partial u^{\alpha}} \dot{u}^{\alpha}(q,p,t) + \frac{\partial I}{\partial v_{\alpha}} \dot{v}_{\alpha}(q,p,t) ,$$

or the associated conditions

$$\frac{du^{\alpha}}{\dot{u}^{\alpha}} = \frac{dv_{\alpha}}{\dot{v}_{\alpha}} . \qquad\qquad \text{-- (5-61)}$$

Using (5-59) and the equations of motion (5-45), (5-61) becomes

$$\left. \begin{array}{c} \dfrac{du^{\alpha}}{du^{\beta}} = \dfrac{3v_{\alpha} - 2mu^{\alpha}}{3v_{\beta} - 2mu^{\beta}} \quad ; \quad \dfrac{dv_{\alpha}}{dv_{\beta}} = \dfrac{v_{\alpha} - 3ee'u^{\alpha}p^{-3}}{v_{\beta} - 3ee'u^{\beta}p^{-3}} \quad ; \\[3mm] \dfrac{dv_{\alpha}}{du^{\beta}} = \dfrac{v_{\alpha} - 3ee'u^{\alpha}p^{-3}}{3v_{\beta} - 2mu^{\beta}} , \\[3mm] \text{where} \qquad p^2 = u^{\alpha}u_{\alpha} . \end{array} \right\} \qquad \text{-- (5-62)}$$
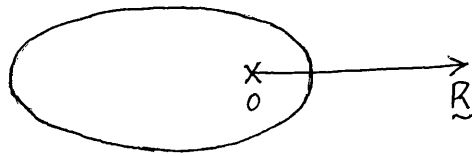
These equations may (laboriously) be manipulated to obtain

$$d \ (u^\alpha v^\beta v_\beta - u^\beta v_\alpha v_\beta - ee'u^\alpha/P) = 0 \ ,$$

that is,
$$R_\alpha \equiv u^\alpha v^\beta v_\beta - u^\beta v_\alpha v_\beta - ee'u^\alpha/P$$

$$= q^\alpha p^\beta p_\beta - q^\beta p_\alpha p_\beta - ee'q^\alpha/r = const. \ -- \ (5\text{-}63)$$

The components $R_\alpha$ form the <u>Runge-Lenz</u> vector, which is an invariant of the (classical) Kepler problem not given by Noether's theorem. It is a vector pointing from the origin in the direction of the point of nearest approach on the orbit:



The conservation of $R_\alpha$ represents the fact that the classical Kepler orbit is closed, ie, the periods in r and φ are the same. The fact that this symmetry is not given by Noether's theorem shows that the Lie theory of extended groups is in some way more powerful than Noether's theorem. We should perhaps mention here that all the work of this section is based upon Leach's (1981) paper, although considerable work was required to bring it into the somewhat more lucid form presented here.


## 5.4 Symmetries in homogeneous mechanics

From the complicated nature of equations (5-36) and (5-37) it might be guessed that the extended theory of Lie groups finds a more natural expression in the notation of homogeneous mechanics, but unfortunately the very generality of homogeneous mechanics proves a stumbling block to the calculation of symmetries. In this section we shall first attempt to carry out the programme described in sections 5.2 and 5.3 by considering the most general 1-parameter family of transformations

$$q \to q_\varepsilon(q,p) \quad ; \quad p \to p_\varepsilon(q,p) \qquad -- \ (5\text{-}64)$$

in phase space. This transformation is clearly the natural extension to phase space of the purely geometric transformations considered by Leach, but it should be noted here that we have also tried, unsuccessfully, to carry Leach's point transformations over into homogeneous mechanics. We noted in the last section the unusual choice of Cartesian coordinates for the Kepler problem. The reason for this choice was that Leach's solution procedure only works for a very limited set of Hamiltonians - Leach himself restricts his attention to cases in which the canonical equations take the form

$$\dot{q}^\alpha - f^{\alpha\beta}(t)p_\beta = 0 \quad ;$$

$$\dot{p}_\alpha + g_\alpha(q^\beta,t) = 0 \quad .$$

While this form covers a number of important classical systems, it is still severely limited, and a glance at the equations of motion (4-30) shows that it can have little applicability in homogeneous mechanics. Leach's method certainly carries over, but as a calculation procedure for symmetries it fails abysmally. The transformation (5-64) represents our first, more general, attempt at applying Lie group theory to homogeneous mechanics.

Again consideration of the infinitesimal transformation

$$q \to q + \xi(q,p)d\varepsilon \quad ; \quad p \to p + \pi(q,p)d\varepsilon \qquad \qquad \text{-- (5-65)}$$

leads to the infinitesimal generator for (5-65):

$$\dot{U}f \equiv \xi^A \partial f/\partial q^A + \pi_A \partial f/\partial p_A \qquad \qquad \text{-- (5-66)}$$

and the invariance equation $\dot{U}f=0$ for f has the associated system of DE's

$$d\varepsilon = dq^A/\xi^A = dp_B/\pi_B \quad \text{(no sum)} .$$

As before, in order to study transformations which leave the canonical equations invariant it is necessary to look at the first extension of (5-66), to which end the transformation law for $\dot{q}$ is calculated exactly as before:

$$\bar{\dot{q}} \equiv \frac{d\bar{q}^A}{d\tau} = \frac{dq^A + d\varepsilon d\xi^A}{d\tau}$$

$$= \dot{q}^A + \dot{\xi}^A d\varepsilon \ .$$

Since a similar result holds for $\bar{p}$ we can immediately write down the form of the extended generator:

$$\dot{U}' \equiv \xi^A \frac{\partial}{\partial q^A} + \pi_A \frac{\partial}{\partial p_A} + \dot{\xi}^A \frac{\partial}{\partial \dot{q}^A} + \dot{\pi}_A \frac{\partial}{\partial \dot{p}_A}$$

If this operator is applied to the canonical equations we obtain

$$\dot{\pi}_A + \dot{U}\partial H/\partial q^A = 0 \ ; \quad \dot{\xi}^A - \dot{U}\partial H/\partial p_A = 0 \ . \qquad -- \ (5\text{-}67)$$

We will now try to apply Leach's method to the case of the homogeneous Kepler problem. From the work of sections 4.2 and 5.1 we know that in spherical polar coordinates, omitting the variable $\theta$,

$$q^1 = r \ , \ q^3 = \varphi \ , \ q^4 = t \ ;$$

$$A_{a,b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -e'/r^2 & 0 & 0 \end{bmatrix} \ ; \quad g^{cd}{}_{,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2/r^3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

so we arrive at the following expressions:

$$\frac{\partial H}{\partial p_a} = \frac{1}{m}(p^a - eA^a) = \frac{1}{m}\left[p_1 \ , \ \frac{p_3}{r^3} \ , \ \frac{ee'}{r} - p_4\right] \qquad -- \ (5\text{-}68a)$$

$$\frac{\partial H}{\partial q^a} = \frac{-e}{m}A_{b,a}(p^b - eA^b) + \frac{1}{2m}g^{cd}{}_{,a}(p_c - eA_c)(p_d - eA_d)$$

$$= \begin{bmatrix} \frac{1}{mr^2}\left[\frac{(ee')^2}{r} - \frac{(p_3)^2}{r^3} - ee'p_4\right] \\ 0 \\ 0 \end{bmatrix} \ . \qquad -- \ (5\text{-}68b)$$

Substituting these results into (5-67) we find

$$\left. \begin{array}{l} \dot{\xi}^1 - \pi_1/m = 0 \ ; \ \dot{\xi}^3 - 1/mr^3 . \{\pi_3 r - 2p_3\xi^1\} = 0 \ ; \\[2mm] \dot{\xi}^4 + 1/mr^2 . \{ee'\xi^1 + \pi_4 r^2\} = 0 \ ; \ \dot{\pi}_3 = \dot{\pi}_4 = 0 \ ; \\[2mm] \dot{\pi}_1 + \xi^1/mr^4 . \{2ee'p_4 r + 3(p_3)^2 - 3(ee')^2\} \\[2mm] \qquad\qquad - 2p_3\pi_3/mr^3 - ee'\pi_4/mr^2 = 0 \ . \end{array} \right\} \qquad -- \ (5\text{-}69)$$

Using these results we again carry out the procedure of section

5.3:    we express $\dot{\xi}$ in terms of $\xi$ with the help of (5-68) and substitute this into (5-69), then take the $\tau$-derivative and equate to the final expression in (5-69).  In this way we arrive at the following result for $\dot{\pi}_1$ :

$$r^6 p^b p^c \partial^2 \xi^1 /\partial q^b \partial q^c \; + \; ee'r^5 p^c \partial^2 \xi^1 /\partial q^c \partial t \; - \; ee'r^4 p^1 \partial \xi^1 /\partial t$$

$$+ \; \{3(ee')^2 r^2 \; - \; 3(p_3)^2 r^2 \; - \; 2ee'r^3 p_4\} \; p^1 \partial \xi^1 /\partial p_1$$

$$+ \; \{ee'r^4 p_4 \; + \; (p_3)^2 r^3 \; - \; (ee')^2 r^3 \} \; p^c \partial^2 \xi^1 /\partial q^c \partial p_1$$

$$+ \; ee'\{r^5 p^b \partial^2 \xi^1 /\partial t \partial q^b \; + \; ee'r^4 \partial^2 \xi^1 /\partial t^2$$

$$+ \; 2[ee'r^3 p_4 \; + \; (p_3)^2 r^2 \; - \; (ee')^2 r^2 ] \; \partial^2 \xi^1 /\partial t \partial p_1\}$$

$$+ \; \{ee'rp_4 \; + \; (p_3)^2 \; - \; (ee')^2 \}\{r^3 \partial \xi^1 /\partial r \; + \; r^3 p_b \partial^2 \xi^1 /\partial q^b \partial p_1$$

$$+ \; [ee'rp_4 \; + \; (p_3)^2 \; - \; (ee')^2 ] \; \partial^2 \xi^1 /\partial p_1{}^2\}$$

$$= \; 3(ee')^2 r^2 \xi^1 \; - \; 3(p_3)^2 \; - \; 2ee'r^3 p_4 \xi^1 \; + \; 2p_3 r^3 \pi_3 \; + \; ee'r^4 \pi_4 \quad .$$

This equation presents a hopeless case for solution.  In Leach's work it was only possible to solve (5-47) because the lack of dependence of $\xi$ on p enabled us to obtain the two initial expressions (5-48) for $\xi$, but here the single equation above is insufficient to specify the coefficients of the generator, yielding a multiplicity of solutions.

As mentioned above, we have tried a number of different ways, both plausible and implausible, of carrying the Leach method over into homogeneous mechanics.  Our lack of success is certainly due in part to the fact that homogeneous mechanics necessarily involves more complicated expressions and dependences than its classical counterpart, but it is also due to the fact that Leach's work is in no way a coherent theory of symmetries. It is rather a way of calculating the symmetries without giving any account of how they arise.  The problem of a full description of the relationship between symmetries and conservation laws seems as yet to remain unsolved.  The approach of Noether's theorem is to define a symmetry as any transformation (4-81)

which leaves the Lagrangian invariant and to develop the conservation law (4-83) as a consequence of this symmetry. The version which we presented in chapter 4 has the advantage that it includes coordinate time into the transformation, but the very nature of the Lagrangian description in which the theorem is formulated means that it can never incorporate the full diversity of the canonical transformations.

Leach's work goes further in one way, in that he aims the symmetry requirement at a more relevant entity than the Lagrangian: according to definition (5.3) a symmetry is a transformation which leaves the equations of motion invariant. Unfortunately, however, Leach does not develop a relationship between the generators and the conserved quantities, even suggesting that no such relationship exists in the case of the Runge-Lenz vector. While this claim is clearly not contradicted by his work, it is also certainly not proven. At any rate, we can be sure that any possible generator of the Runge-Lenz vector cannot be of the form $\xi^A(q)\partial/\partial q$, since we know from corollary (4.10) that such a generator corresponds necessarily to constants of motion linear in p, which R certainly is not. It may, however, be possible to develop a generator from R which has a p dependence not allowed for either in Noether's or Leach's work.

We have so far met three basic methods concerned with conservation and symmetry, and their aims are essentially different. They are:

    i.   The H-J method calculates constants of the motion.

    ii.   The extended Lie group method calculates the generators of symmetries.

    iii.   Noether's theorem gives a relationship between the generators and the constants.

Of these three the Lie group method seems to be of limited applicability, since the calculation procedure breaks down for at least some important systems. Noether's theorem as it stands is also limited, since given a conserved quantity there is no guarantee that we can find a corresponding symmetry. On the other hand the H-J method guarantees a complete set of constants for the system (although, as in (5-19b) of the Kepler problem, these may not be in a very enlightening form). It therefore seems possible that if we could find a more general form of Noether's theorem valid for general canonical transformations, then we might be able to find symmetries by applying it to the constants produced by the H-J method. This, then, is our next objective.

Noether's theorem can be extended slightly by simply translating the argument of theorem (4.9) into the canonical formalism. Let us suppose again that the Lagrangian is unaffected by a certain variation of the variables on which it depends, but this time we shall permit a variation of all of the canonical variables:

$$\delta \ \{p_A \dot{q}^A - H\} = 0$$

under the transformation induced by

$$\delta q = \xi \delta\epsilon \ ; \ \delta p = \pi \delta\epsilon. \qquad\qquad -- \ (5\text{-}70)$$

From the work of the previous section we know that this involves a transformation

$$\delta \dot{q} = \dot{\xi}\delta\epsilon \ ; \ \delta \dot{p} = \dot{\pi}\delta\epsilon$$

of the first derivatives of the canonical variables. Using these equations we now calculate the corresponding variation of the action:

$$\delta S = \delta \int \{ p_A \dot{q}^A - H \} d\tau$$

$$= \delta\varepsilon . \int \{ \pi_A \dot{q}^A + p_A \dot{\xi}^A - \frac{\partial H}{\partial q^A} \xi^A - \frac{\partial H}{\partial p_A} \pi_A \} \, d\tau$$

$$= \delta\varepsilon . \int \{ \dot{q}^A - \frac{\partial H}{\partial p_A} \} \pi_A . d\tau - \delta\varepsilon . \int \{ \dot{p}_A + \frac{\partial H}{\partial q^A} \} \xi^A . d\tau$$

$$+ \delta\varepsilon . \int d\{ p_A \xi^A \} \ .$$

This expression gives the total variation in the action. If we confine our attention to a given trajectory then we know that the first two integrals vanish, and we arrive at the conclusion that

$$p_A \xi^A (q,p) = const. \qquad\qquad -- \ (5\text{-}71)$$

is a constant of the motion. Note that the variation $\pi$ assumed in (5-70) plays no part in the final result, yet it is still more powerful than the form of Noether's theorem proven in chapter 4, since the coefficients $\xi^A$ may depend arbitrarily on the momenta. Nevertheless, (5-71) does not lead in any obvious way to a generator corresponding to the Runge-Lenz vector, since it still requires a quasi-linear form for the conserved quantity.

An alternative way in which the results of Noether's theorem may be carried over into Hamiltonian mechanics is the manner mentioned at the end of chapter 4: the equation

$$\{H,U\} = 0$$

expresses simultaneously the invariance of H under a transformation generated by U and the conservation of U under the development of the system. In this form Noether's theorem states that a quantity U is conserved under the motion of the system iff the Hamiltonian H is unchanged by the (canonical) transformation generated by U:

$$\dot{U}f \equiv \{ f,U \} \ . \qquad\qquad -- \ (5\text{-}72a)$$

We must be a little careful with the definitions here, since multiplication of U by a quantity does <u>not</u> in general correspond to multiplication of the PB in (5-72a). Because of this we make

the following definition: given a set of generators $\hat{U}_i$, the linear combination $\xi^i \hat{U}_i$ is defined by

$$\xi^i \hat{U}_i f \equiv \{f, \xi^i U_i\} \quad . \qquad \qquad \text{-- (5-72b)}$$

Using these definitions we can always develop a symmetry from any given conserved quantity U, which is what we were searching for.

One simple application of the PB form of Noether's theorem again arises from Leach's (1981) work. Apart from considering the Kepler problem, Leach also looked at the problem of the classical harmonic oscillator, whose Hamiltonian is given by

$$H = q^2 + p^2 \qquad \qquad \text{-- (5-73)}$$

in rationalised units. In addition to the conservation of energy and angular momentum Leach arrives at the following conserved quantity:

$$A_{\alpha\beta} \equiv q_\alpha q_\beta + p_\alpha p_\beta \quad . \qquad \qquad \text{-- (5-74)}$$

This (symmetric) 3-tensor is called the <u>Jauch-Hill-Fradkin</u> tensor, and, like the Runge-Lenz vector, cannot be obtained by the application of Noether's theorem. If we substitute $A_{\alpha\beta}$ into (5-72a) then we can calculate the corresponding generator:

$$\hat{A}_{\alpha\beta} = \frac{\partial A_{\alpha\beta}}{\partial p_\gamma} \frac{\partial}{\partial q_\gamma} + \frac{\partial A_{\alpha\beta}}{\partial q_\gamma} \frac{\partial}{\partial p_\gamma}$$

$$= p_\beta \partial/\partial q_\alpha + p_\alpha \partial/\partial q_\beta - q_\beta \partial/\partial p_\alpha - q_\alpha \partial/\partial p_\beta, \quad \text{-- (5-75)}$$

and hence the infinitesimal transformation:

$$q_\alpha \to q_\alpha + \varepsilon_{\beta\gamma}(\delta_{\alpha\beta} p_\gamma + \delta_{\alpha\gamma} p_\beta) \quad ;$$

$$p_\alpha \to p_\alpha + \varepsilon_{\beta\gamma}(\delta_{\alpha\beta} q_\gamma + \delta_{\alpha\gamma} q_\beta) \quad .$$

This transformation has the form of a rotation in phase space involving both the coordinates <u>and</u> the momenta[2], and indeed reference to (5-56) shows that (5-75) looks very much like some kind

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[2] I am indebted to my supervisor, Dr.M.Dampier, for noticing this property of the Jauch-Hill-Fradkin symmetry.

of "mixed" rotation generator. This transformation seems very natural if we look at the Hamiltonian (5-73). This has the form of a metric funtion on a Euclidean phase space, and is clearly invariant not only under point rotations, but also under arbitrary rotations of the phase space coordinates. Note, however, that only certain of these rotations will also be canonical - it is not, for instance, permissible to rotate the $(q_1, p_1)$ plane without also performing a rotation of the $(q_2, p_2)$ plane.

It would be nice to be able to say that a similar intuitive significance can be attached to the Runge-Lenz vector, but unfortunately we have been unable to carry out an analogous analysis to the above for the Kepler problem. A generator may indeed be obtained from the vector, but it does not possess any obvious significance. It is our personal opinion that the Runge-Lenz vector corresponds to the constant $b_3$ in (5-19a), which essentially establishes a zero angle to which the variable $\varphi$ is referred. In the classical Kepler problem we find

$$-b_3 = \varphi - \sin^{-1} \left[ \frac{mee' - h^2 u}{\sqrt{\{m^2 (ee')^2 + 2mEh^2\}}} \right] ,$$

but even after changing to the Cartesian coordinates used in Leach's work we have been unable to establish any connection between this expression and (5-63). In summary, we must say that although our work has shed some light on the subject of conservation and symmetries, a coherent theory of the subject is still lacking.

CHAPTER 6

APPLICATIONS

In this final chapter we shall look at two major applic-
ations of the theory of homogeneous mechanics developed in this
thesis. The first of these applications is in perturbation
theory, and stems mainly from the work of section 5.1, while the
second explores the possibility of moving to a quantum formalism
based on the general theory of chapter 4.

## 6.1 Perturbation theory

In mechanics it frequently occurs that we wish to study a
system which is almost, but not quite, the same as a second
system which we have already solved. Since a system is charac-
terised by its Hamiltonian, it follows that we are talking about
two energy surfaces in phase space which are "close" together in
some sense. The difficulty here is that in order to compare
Hamiltonians on two different surfaces we need to be able to say
something about the value of the Hamiltonian <u>off</u> the energy
surface. Now the H-J method of section 5.1 makes no assumptions
about the value off the surface, but the related theory at the
end of section 4.4 copes with the problem by arranging that the
N-th component of the new momentum is the Hamiltonian - off the
energy surface as well as on. We therefore now need to relate
this situation more closely to the work of section 5.1.

We again start with any complete integral $S(Q,q)$ of the
equation

$$H(q, \partial S/\partial q) = Q^M \quad \text{(M a constant)}, \qquad \text{-- (6-1)}$$

but we note a number of differences between this equation and
(4-75). First, we are now looking at a case 1 generating

function $S(Q,q)$ as opposed to the case 2 function of section 4.4. Second, we allow for _any_ specific new coordinate $Q^M$ to be the Hamiltonian: we saw in the Kepler problem that it is not always convenient to pick out the N-th coordinate as the odd one out. Note that the corresponding constant $a^M$ of the H-J theory will still be a constant along the trajectories, since $\dot{Q}^M = 0$ on the energy surface. We assume S to be such that

$$\det \left| \frac{\partial^2 S}{\partial q^A \partial Q^B} \right| \neq 0 \qquad \text{-- (6-2)}$$

to ensure its validity as a generating function off the energy surface - this is in direct contrast to the H-J method.

The case 1 transformation equations (4-57) now define a canonical transformation generated by S:

$$v_A = \partial S / \partial q^A \quad ; \quad V_A = -\partial S / \partial Q^A , \qquad \text{-- (6-3)}$$

for which the new Hamiltonian satisfies

$$\overline{H}(Q,V) = H(q, \partial S / \partial q) = Q^M . \qquad \text{-- (6-4)}$$

Following a similar line of reasoning to that of section 4.4 we find

$$-V_M = \tau \quad ; \quad V_A = b_A \quad \text{(for all } A \neq M\text{)}; \left.\begin{array}{c} \\ \\ Q^A = a^A , \end{array}\right\} \qquad \text{-- (6-5)}$$

where a,b are constants and the physical position vectors are those whose M-th coordinate is zero. Notice that on the energy surface S satisfies the H-J equation, so all the constants $a^A, b_A$ other than the N-th components have the same significance (indeed the same value) that they did in the H-J method. The difference is that we are now making use of the two "surplus" components in a way which is useful for the construction of a _perturbation theory_.

Let $H_0$ now be the Hamiltonian of some soluble, or unperturbed, system - that is, one whose evolution is already known.

We now imagine the energy surface $H_O=0$ of this problem to be perturbed slightly into another surface which is "close" to the original. We achieve this by means of the perturbed Hamiltonian H, given by

$$H(q,p) = H_O(q,p) + \Delta H(q,p) \quad , \qquad \text{-- (6-6)}$$

where the term $\Delta H$ is in some sense small. Now suppose we have solved the unperturbed system by the modified H-J method outlined above, obtaining the Gaussian system (6-5). On the unperturbed energy surface $S(a,q)$ generates a set of a's constant along the trajectories, with $a^M=0$ all over the surface $H_O=0$; if we change to the perturbed surface, however, the quantities $(a,b)$ will no longer be constants and we will not have $H_O=0$. But because of (6-2) S still generates a perfectly valid canonical transform-ation to a system in which the Hamiltonian is given by (6-6) and (6-4):

$$H(a,b) = H_O(a,b) + \Delta H(a,b)$$
$$= a^M + \Delta H(a,b) \quad . \qquad \text{-- (6-7)}$$

Note that $a^M$ here is not necessarily a constant, since the perturbed surface will in general cut across the natural congruence of trajectories defined by $H_O$. From (6-7) we can now obtain the equations of motion satisfied by the transformed variables:

$$\left.\begin{array}{l} \dot{a}^A = \partial H/\partial b_A = \partial \Delta H/\partial b_A \quad ; \\[2mm] -\dot{b}_A = \partial H/\partial a^A = \partial \Delta H/\partial a^A + \delta_A^M \quad . \end{array}\right\} \qquad \text{-- (6-8)}$$

The equations (6-8) are rigorous, and in general offer no simplification of the problem of solving the perturbed system (6-6), but if we take advantage of the fact that $\Delta H$ is small then we obtain a first-order approximation to the solution. Since a and b will not change rapidly for small perturbations we can obtain the first approximation by replacing their occurrence on

the RHS of (6-8) by their constant unperturbed values:

$$\dot{a}_1^A = \left.\frac{\partial \Delta H}{\partial b_A}\right|_o \quad ; \quad -\dot{b}_{1A} = \delta_A^M + \left.\frac{\partial \Delta H}{\partial a^A}\right|_o \quad . \qquad \text{-- (6-9)}$$

Here $(a_1, b_1)$ are the first-order perturbation solutions for $(a, b)$, and the subscript 0 indicates substitution of the un-perturbed coordinates (denoted $a_o, b_o$) after differentiation. Similarly the second-order perturbation is obtained by substit-ution of the first-order perturbation into the RHS of (6-8):

$$\dot{a}_2^A = \left.\frac{\partial \Delta H}{\partial b_A}\right|_1 \quad ; \quad -\dot{b}_{2A} = \delta_A^M + \left.\frac{\partial \Delta H}{\partial a^A}\right|_1 \qquad \text{-- (6-10)}$$

and so on.

**Example (6.1)** - As an example of the above work consider the Kepler problem of example (5.2) and perturb it by superposing a pure (weak) magnetic field of the type looked at in example (4.2). Since both the Kepler and magnetic problems essentially involve motion in one plane only, we adopt cylindrical coord-inates $(r, \varphi, z, t)$, in which case the potentials for the perturbed problem become

$$A_a = (0, \, Br^2/2, \, 0, \, e'/r) \quad .$$

Using the expression (4-29) we have

$$H = \frac{1}{m}\left[p^2 - eBp_2 + \frac{2ee'p_4}{r} + \frac{e^2B^2r^2}{4} - \left[\frac{ee'}{r}\right]^2\right]$$

$$+ \left[-p^2 + eBp_2 - \frac{2ee'p_4}{r} - \frac{e^2B^2r^2}{4} + \left[\frac{ee'}{r}\right]^2\right]^{1/2}$$

$$\approx \frac{1}{m}\left[p^2 + \frac{2ee'p_4}{r} - \left[\frac{ee'}{r}\right]^2\right] - \frac{eBp_2}{m}$$

$$+ Z\left[1 + \frac{1}{Z^2}\left[eBp_2 - \frac{e^2B^2r^2}{4}\right]\right]^{1/2} \quad ,$$

where $\qquad -Z^2 = p^2 + \frac{2ee'p_4}{r} - \left[\frac{ee'}{r}\right]^2 \quad .$

Here we have omitted a second-order term in B and by the same weak field approximation we can expand the final term to obtain

$$H \approx -Z^2/m - eBp_2/m + Z + eBp_2/2Z \quad ,$$

whereupon we have

$$\Delta H = \frac{eBp_2}{2m} \left[ \frac{m}{Z} - 2 \right] \quad . \qquad\qquad \text{-- (6-11)}$$

In order to calculate the first-order perturbation we have, according to (6-9), to substitute into (6-11) the constants of the unperturbed (Kepler) problem, given in (5-17) and (5-18):

$$(p_1)^2 = (E + ee'/r)^2 - (h/r)^2 - m^2 \quad ;$$

$$p_2 = h \quad ; \quad p_3 = 0 \quad ; \quad p_4 = -E \quad .$$

The constants $a_1$, $b_1$ of the Kepler problem were redundant, so we choose $M=1$ in (6-9) and arbitrarily set $z=0$. In this way we obtain the following final expression for $\Delta H$:

$$\Delta H = \frac{eBh}{2m} [mZ^{-1} - 2]$$

$$= \frac{eBh}{2m} \left\{ m \left[ E^2 + m^2 + \left[ \frac{h}{r} \right]^2 - \left[ E + \frac{ee'}{r} \right]^2 - \left[ \frac{h}{r} \right]^2 + \frac{2ee'E}{r} \right. \right.$$

$$\left. \left. + \left[ \frac{ee'}{r} \right]^2 \right]^{-1/2} - 2 \right\}$$

$$= \frac{eBh}{2m} \{ 1 - 2 \}$$

$$= -eBh/2m \quad . \qquad\qquad \text{-- (6-12)}$$

By substituting (6-12) into (6-9) we see that the only first-order change induce by the magnetic field is the change

$$\dot{b}_2 = -\partial \Delta H/\partial h = eB/2m \quad . \qquad\qquad \text{-- (6-13)}$$

The expression (6-13) is known in classical mechanics as the Larmor frequency. Equation (5-19a) shows that $-b_2$ ($b_3$ in the language of section 5.1) represents the initial value of $\varphi$ in the unperturbed problem:

$$\varphi = \omega_o s - b_2 \quad ,$$

where $\omega_o$ is the angular frequency of the unperturbed particle. Thus from (6-13) we see that in the presence of a weak magnetic field B the new angular frequency is

$$\Omega = \omega_0 \pm eB/2m \quad , \qquad \text{-- (6-14)}$$

the choice of sign depending on the sense of the magnetic field ∓B. This splitting of the perturbed angular frequency into two values in the presence of a weak magnetic field is called the <u>normal</u> <u>Zeeman</u> <u>effect</u>. The result (6-14) is exactly that of the classical theory (see Menzel,1961), except that coordinate time t has been replaced by the proper time parameter s.

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In the interests of completeness we now present briefly an alternative perturbation theory which attempts to eliminate certain deficiencies of the above approach, but which, however, does not seem to work. The theory presented above suffers from the aesthetic defect that the addition of extra terms to the Hamiltonian does not appear to be a natural thing to do in homogeneous mechanics. We have seen, for example, that the inclusion of a perturbing electromagnetic field adds exact terms to the Lagrangian and momentum, but produces a change (6-11) in the Hamiltonian which is far from "natural". Consequently we now try to develop a perturbation theory based upon perturbations of the Lagrangian.

Let $L_0(q,\dot{q})$ be the unperturbed Lagrangian and define the perturbed Lagrangian as

$$L(q,\dot{q}) = L_0(q,\dot{q}) + \lambda(q,\dot{q}) \quad ,$$

where $\lambda$ is in some sense small. Then we find the following corresponding changes:

$$p \equiv \partial L/\partial \dot{q} = p_0 + \Delta \quad ,$$

where $\quad \Delta \equiv \partial \lambda/\partial \dot{q} \quad ; \quad p_0 \equiv \partial L_0/\partial \dot{q} \quad .$

=> $\quad H \equiv \dot{q}^A p_A - L = H_0(q,p_0) + h(q,\Delta) \quad ,$

where $\quad H_0 \equiv \dot{q}^A p_{0A} - L_0 \quad ; \quad h \equiv \dot{q}^A \Delta_A - \lambda \quad .$

In this way we have split the problem into two systems $H_0(q,p_0)$

and $h(q,\Delta)$. The equations of motion for the perturbed system are now given by

$$\dot{q}^A = \frac{\partial H}{\partial p_A} = \frac{\partial H_0}{\partial p_{0B}}\frac{\partial p_{0B}}{\partial p_A} + \frac{\partial h}{\partial \Delta_B}\frac{\partial \Delta_B}{\partial p_A} \quad ;$$

$$-\dot{p}_A = \frac{\partial H}{\partial q^A} = \frac{\partial H_0}{\partial q^A} + \frac{\partial H_0}{\partial p_{0B}}\frac{\partial p_{0B}}{\partial q^A} + \frac{\partial h}{\partial q^A} + \frac{\partial h}{\partial \Delta_B}\frac{\partial \Delta_B}{\partial q^A} \quad . \tag{6-15}$$

As a check on these equations we take the general electromagnetic case, where

$$\lambda = eA_a\dot{q}^a \quad ; \quad \Delta_a = eA_a \quad ; \quad H_0 = (p_0)^2/m + [-(p_0)^2]^{1/2} \quad .$$

In this case h is identically zero and we are looking solely at the change in $H_0$ produced by the transformation

$$q = q_0 \quad ; \quad p = p_0 + eA(q_0) \quad . \tag{6-16}$$

From (6-15) we have

$$\dot{q} = \partial H_0/\partial p_0 = p_0/m \quad ; \tag{6-17a}$$

$$-\dot{p}_a = -\dot{p}_{0a} - eA_{a,b}\dot{q}^b$$

$$= \partial H_0/\partial p_{0b} . \partial p_{0b}/\partial q^a$$

$$= -\dot{q}^b . eA_{b,a}$$

$$=> \quad \dot{p}_{0a} = eF_{ba}\dot{q}^b \quad . \tag{6-17b}$$

Thus we produce the correct equations of motion, but we are unable to develop the theory further. For the perturbation of constants relies upon applying the H-J transformation of the unperturbed system, but a glance at the equations (6-17) shows that under such a transformation all constants of the unperturbed system appear to remain constants in the perturbed system, which cannot be correct. What has gone wrong is that (6-16) represents a non-canonical transformation of the variables $(q_0,p_0)$, so that the H-J method cannot be carried over straightforwardly from the unperturbed to the perturbed system. While possessing certain promising features, this theory of perturbations is not viable as it stands.

## 6.2 Relativistic quantum theory

Our second application, and the original motivation for this work, is the transition from homogeneous mechanics to quantum theory. In the remainder of this chapter the word "classical" will continue to have the meaning "non-relativistic", rather than the common meaning of "non-quantum"; we shall use "homogeneous" to describe the unquantised theory of homogeneous mechanics discussed so far, in contradistinction to the fully relativistic quantum theory we shall now try to develop. This terminology is slightly misleading, since our formulation of quantum theory is also in a sense homogeneous, but it will suffice.

It would be inappropriate here to go deeply into the plethora of notations and conventions used in quantum theory. Our personal preference is for the lattice structure approach advocated by von Neumann (1955), which we feel brings the philosophical foundations of quantum theory beautifully into focus, but rather than get too involved in such basics we shall instead base our exposition heavily upon the approach of Dirac (1958) in his standard text on the subject. We shall assume all results in that book up to chapter 3 and commence our work at the beginning of chapter 4; our references to Dirac's equations will be of the form (D23.47), which will denote equation (47) appearing in section 23 of the book. Up to this chapter the only difference between Dirac's results and our assumptions is that the proposition "The time of an event is ≤ t" is to be considered a valid element of the lattice of propositions concerning a physical system, the totality of which form an infinite-dimensional Hilbert space $\underline{H}$. This means, of course, that the resulting quantum theory will describe <u>events</u>, rather than

particles.   For simplicity we shall assume the system to be in flat, Minkowski space.

If we are to phrase homogeneous mechanics in the language of quantum theory then our first priority is to find a set of operators $\hat{q}^A, \hat{p}_A$ to represent the position and momentum variables. To do this we choose a representation in which the basis kets are the eigenvectors of position, indexed by the position variables $q^A$ :

$$\langle q^A | \psi \rangle = \psi(q)$$

for an arbitrary ket $|\psi\rangle$ - this is the _position representation_. Now suppose $|\psi\rangle$ represents the state of a system at a given $\tau$-instant, and has modulus unity. The probability that a given event will have coordinates within the volume element

$$(q^1, Q^1) \times (q^2, Q^2) \times \ldots \times (q^N, Q^N)$$

is then given by

$$\int_q^Q \bar{\psi}(q')\psi(q')d^N q' \ . \qquad\qquad -- (6-18)$$

The eigenvectors $|q^A\rangle$ of position will therefore represent the situation where the event is known to occur at the _definite_ position $q^A = q'^A$, say. It follows that the eigenket $|q'^A\rangle = q'^A(q)$ vanishes for all values of q except those at the point $q^A = q'^A$. Yet the probability density for the eigenstate $|q'^A\rangle$ is given by (6-18), which must equal 1 if q,Q are extended to infinity. In the eigenstate $|q'^A\rangle$, therefore,

$$|q'(q)|^2 = \delta(q-q') \ ,$$

so          $$|q'^A\rangle = \delta^{1/2}(q^A - q'^A) \ , \qquad\qquad -- (6-19)$$

where $\delta$ is the N-fold Dirac $\delta$-function. The kets (6-19) are the eigenvectors of the position operator, defined by

$$\hat{q}^A \psi(q) \equiv q^A \psi(q) \qquad\qquad -- (6-20)$$

on the general vector $\psi$.

Our route to the operators $\hat{p}_A$ of momentum is via the

fundamental PB relations (4-51). To express these in quantum theory we require a bracket between operators which has the same algebraic properties as the PB. Dirac shows that the most general such bracket between two operators $\hat{a}, \hat{b}$ is the commutator

$$[\hat{a}, \hat{b}] \equiv 1/i\hbar.(\hat{a}\hat{b} - \hat{b}\hat{a}) \quad , \qquad -- (6-21)$$

where $\hbar$ is the reduced Planck's constant $h/2\pi$; this is accordingly taken as the quantum analogue of the PB. Note that the square brackets [,] now denote the commutator, and not the Lagrange bracket. The choice of momentum operator is then constrained by the <u>fundamental quantum conditions</u>

$$[\hat{q}^A, \hat{q}^B] = [\hat{p}_A, \hat{p}_B] = 0 \quad ; \quad [\hat{q}^A, \hat{p}_B] = \delta^A_B \quad . \qquad -- (6-22)$$

Again Dirac shows that the most general momentum operator satisfying these conditions can always be reduced to the form

$$\hat{p}_A \psi = \hbar/i.\nabla_A \psi \quad , \qquad -- (6-23)$$

where $\nabla$ may be expressed in rectangular coordinates as

$$\nabla_A = \partial/\partial q^A \quad , \qquad -- (6-24)$$

although this is <u>not</u> the general form of $\nabla$ in an arbitrary coordinate system. Unless otherwise specified (as in the following section) all the coming work will assume the convenient rectangular form (6-24) for the momentum operator. The operators (6-20) and (6-23) are the fundamental quantum variables in the position representation.

THEOREM (6.2) - Let two observables a, b be such that

$$[\hat{a}, \hat{b}] = c \quad ,$$

where c is a complex number, then the uncertainties $\Delta a$, $\Delta b$ satisfy

$$\Delta a \Delta b \geqslant \hbar/2.|c| \quad . \qquad -- (6-25)$$

PROOF: Taking the adjoint of the commutator we have

$$\overline{[\hat{a}, \hat{b}]} = -1/i\hbar.(\hat{b}\hat{a} - \hat{a}\hat{b}) = [\hat{a}, \hat{b}] = \overline{c} \quad ,$$

so $c = \overline{c}$, and c must therefore be real. Now write

$$\hat{\alpha} \equiv \hat{a} - \langle a \rangle \quad ; \quad \hat{\beta} \equiv \hat{b} - \langle b \rangle \quad ,$$

where $\langle \rangle$ denotes the expected value of an observable,

then we again have

$$[\hat{\alpha}, \hat{\beta}] = c \quad .$$

Consider now the imaginary part of the inner product $\langle \psi | \hat{\alpha}\hat{\beta} | \psi \rangle$ for some arbitrary vector $\psi$:

$$2 \, \text{Im}\langle \psi | \hat{\alpha}\hat{\beta} | \psi \rangle = \langle \psi | \hat{\alpha}\hat{\beta} | \psi \rangle - \langle \psi | \hat{\beta}\hat{\alpha} | \psi \rangle$$

$$= \langle \psi | (\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}) | \psi \rangle$$

$$= i\hbar c \quad .$$

Taking the modulus of this result we finally have

$$\hbar/2 . |c| = \text{Im}\langle \psi | \hat{\alpha}\hat{\beta} | \psi \rangle$$

$$\leqslant |\langle \psi | \hat{\alpha}\hat{\beta} | \psi \rangle|$$

$$\leqslant \| \hat{\alpha}\psi \| \, \| \hat{\beta}\psi \| = \Delta a \Delta b \quad ,$$

using the Schwarz inequality and the usual expressions

$$\langle o \rangle = \langle \psi | \hat{o} | \psi \rangle \quad ; \quad (\Delta o)^2 = \langle o^2 \rangle - \langle o \rangle^2$$

for the expected value and uncertainty of an observable $\hat{o}$.

QED

As a result of theorem (6.2) we can use the quantum conditions (6-22) to obtain the Heisenberg <u>uncertainty</u> <u>relations</u>

$$\Delta q^A \Delta p_B \geqslant \hbar/2 . \delta^A_B \quad , \qquad \qquad \text{-- (6-26)}$$

interpreted as meaning that conjugate components of momentum and position can never be simultaneously assigned exact values. Note that in the classical decomposition, where $\tau = q^N$, we have

$$\Delta t \Delta H^* \geqslant \hbar/2 \quad , \qquad \qquad \text{-- (6-27)}$$

so that the time-energy uncertainty principle appears on the same footing as the position-momentum relations. This is a desirable feature not occurring in existing theories. The time-energy relation is notorious for its difficulty of interpretation, but in the context of the present theory a possible interpretation is

available. Remembering that the observables here describe physical properties of events rather than particles, it seems to us that (6-27) carries the following interpretation:

At a given τ-instant the values of energy and coordinate time of an event on a particle's worldline can only be measured within uncertainties $\Delta H^*$, $\Delta t$ respectively, and these uncertainties will always be in the relation (6-27). Thus if the measurement of a particle's energy yields a precise value then we cannot have any information on when the measurement was made; conversely, if we make a measurement at a specified time then it will yield no information as to the particle's energy. This results, amongst other things, in the mono-chromatic form of wave functions in specified energy states which is familiar from classical quantum theory.

In homogeneous mechanics we saw that CT's are characterised by the invariance of the fundamental PB relations. Accordingly we shall define a CT in quantum mechanics as a transformation which leaves the fundamental quantum conditions (6-22) invariant. In order to see the consequences of this definition we now consider a general transformation of the dynamical variables:

$$\hat{Q}^A = \hat{Q}^A(\hat{q},\hat{p}) \quad ; \quad \hat{P}_A = \hat{P}_A(\hat{q},\hat{p}) \quad , \quad \left. \begin{array}{c} \\ \\ \end{array} \right\} \quad -- \; (6\text{-}28)$$

such that $[\hat{Q}^A,\hat{Q}^B] = [\hat{P}_A,\hat{P}_B] = 0 \; ; \; [\hat{Q}^A,\hat{P}_B] = \delta^A_B \; . \left. \begin{array}{c} \\ \end{array} \right.$

It follows from the remarks leading up to (6-23) that we can set up a position representation with respect to the new variables in which

$$\hat{Q}^A = Q^A \quad ; \quad \hat{P}_A = -i\hbar \; \partial/\partial Q^A$$

and the basis kets are denoted $|Q^A\rangle$. Now consider the linear operator $\hat{U}$ defined by

$$\langle Q|\hat{U}|q\rangle = \delta(Q-q) \quad ,$$

with adjoint $\bar{U}$ satisfying

$$\langle q|\bar{U}|Q\rangle = \delta(Q-q) \quad .$$

Hence for an arbitrary pair of eigenkets of $\hat{q}$ we have from (D17.47)

$$\langle q|\bar{U}\hat{U}|q'\rangle = \int\langle q|\bar{U}|Q\rangle d^N Q\langle Q|\hat{U}|q'\rangle$$

$$= \int\delta(Q-q)d^N Q\delta(Q-q')$$

$$= \langle q|q'\rangle \quad ,$$

which necessarily implies

$$\bar{U}\hat{U} = 1 \quad . \qquad\qquad -- (6-29)$$

In addition, for any pair of eigenvalues $Q,q$ we have

$$\langle Q|\hat{Q}^A\hat{U}|q\rangle = Q^A\delta(Q-q)$$

$$= q^A\delta(Q-q) = \langle Q|\hat{U}\hat{q}^A|q^A\rangle$$

$$\Rightarrow \qquad\qquad \hat{Q}^A\hat{U} = \hat{U}\hat{q}^A$$

$$\text{or} \qquad\qquad \hat{Q}^A = \hat{U}\hat{q}^A\hat{U}^{-1} \quad . \qquad\qquad -- (6-30a)$$

To obtain the corresponding transformation of the momentum operators we note that for an arbitrary ket $\psi$

$$\langle q'|\partial/\partial q^A|\psi\rangle = \langle q'|\partial\psi/\partial q^A\rangle = \partial\psi(q')/\partial q'^A$$

$$= \partial/\partial q'^A.\langle q'|\psi\rangle$$

$$\Rightarrow \qquad\qquad \langle q'|\hat{p}_A = -i\hbar\partial/\partial q'^A.\langle q'| \quad .$$

Taking the complex conjugate of both sides of this equation we also obtain

$$\hat{p}_A|q'\rangle = i\hbar\partial/\partial q'^A.|q'\rangle \quad .$$

Thus $\qquad \langle Q|\hat{P}_A\hat{U}|q\rangle = -i\hbar\partial/\partial Q^A.\delta(Q-q)$

$$= i\hbar\partial/\partial q^A.\delta(Q-q)$$

$$= \langle Q|\hat{U}\hat{p}_A|q\rangle$$

$$\Rightarrow \qquad\qquad \hat{P}_A\hat{U} = \hat{U}\hat{p}_A \quad ,$$

$$\text{or} \qquad\qquad \hat{P}_A = \hat{U}\hat{p}_A\hat{U}^{-1} \quad . \qquad\qquad -- (6-30b)$$

Thus we see that any canonical transformation (6-28) can be represented by a transformation (6-30), such that the operator $\hat{U}$ satisfies (6-29). Conversely, any transformation of the type

(6-30), where $\hat{U}$ satisfies (6-29), must clearly leave the quantum conditions unaltered.

**Definition (6.3):**      A <u>unitary operator</u> is a linear operator $\hat{U}$ satisfying any one of the three equivalent properties:

   i. $\hat{U}\overline{U} = 1$ .

   ii. $\overline{U} = \hat{U}^{-1}$ .

   iii. For any pair of vectors $\psi, \chi \in \underline{H}$, $\langle \hat{U}\psi | \hat{U}\chi \rangle = \langle \psi | \chi \rangle$ .

   A <u>unitary transformation</u> is a transformation mapping each linear operator $\hat{a}$ to the image $\hat{A}$ such that

$$\hat{A} = \hat{U}\hat{a}\overline{U} ,$$

where $\hat{U}$ is a unitary operator.

----

**Example (6.4)**

   i. Let $d^A$ be a fixed element of $\underline{R}^N$, then the <u>shift operator</u> $\hat{U}_d$ is defined by

$$(\hat{U}_d \psi)(q) = \psi(q+d) . \qquad\qquad -- (6-31)$$

$\hat{U}$ is unitary, since

$$\langle \hat{U}_d \psi | \hat{U}_d \chi \rangle = \int \overline{\psi}(q+d) \chi(q+d) \, d^N q$$

$$= \int \overline{\psi}(q') \chi(q') d^N q' \qquad (q' \equiv q+d)$$

$$= \langle \psi | \chi \rangle ,$$

and clearly $\hat{U}_d^{-1} = \hat{U}_{-d}$. The effect of the shift operators on the fundamental variables is easily calculated:

$$\hat{U}_d \hat{q} \hat{U}_d^{-1} \psi(q) = \hat{U}_d \hat{q}\psi(q-d)$$

$$= \hat{U}_d q\psi(q-d)$$

$$= (q+d)\psi(q)$$

$$\Rightarrow \qquad \hat{Q} = \hat{U}_d \hat{q} \hat{U}_d^{-1} = \hat{q} + d ,$$

and similarly $\qquad \hat{P} = \hat{U}_d \hat{p} \hat{U}_d^{-1} = \hat{p} .$

$$\left.\begin{array}{c} \\ \\ \end{array}\right\} \quad -- (6-32)$$

   ii.   For an arbitrary $\psi \in \underline{H}$ the <u>Fourier transform</u> is defined by

$$\tilde{F}\psi(q) = h^{-N/2} \int \exp\{ir_A q^A /\hbar\}\psi(r)d^N r \quad . \quad -- \ (6\text{-}33)$$

$\tilde{F}$ may be shown to be unitary (see Riesz,1955), and the action of $\tilde{F}$ is calculated as follows:

$$-i\hbar\tilde{F}\frac{\partial\psi(q)}{\partial q^B} = \frac{-i\hbar}{h^{N/2}} \exp\{ir_A q^A /\hbar\}\frac{\partial\psi(r)}{\partial r^B} d^N r$$

$$= \frac{-i\hbar}{h^{N/2}} \left\{ \begin{array}{c} \text{boundary terms} \\ \text{in } \psi(r) \end{array} - \frac{iq_B}{\hbar} \exp\{ir_A q^A /\hbar\}\psi(r)d^N r \right\}$$

Provided $\psi$ vanishes over the boundary of integration we therefore have

$$\tilde{F}\hat{p}_B \psi = -\hat{q}_B \tilde{F}\psi \quad ,$$

where $\hat{q}_B$ is defined analogously to $\hat{q}^B$. Hence

$$\left. \begin{array}{l} \tilde{P}_A = \tilde{F}\hat{p}_A \tilde{F}^{-1} = -\hat{q}_A \\[2mm] \text{and similarly} \qquad \tilde{Q}^A = \tilde{F}\hat{q}^A \tilde{F}^{-1} = \hat{p}^A \quad . \end{array} \right\} \qquad -- \ (6\text{-}34)$$

---

The above examples show that unitary transformations possess the same sort of generality as the CT's of homogeneous mechanics. Example (i) is very similar to the contact transformation (4-85), while example (ii) represents a mere change of viewpoint to the <u>momentum</u> <u>representation</u> (see Dirac,1958). This corresponds to the trivial transformation (1-16). The dynamical transformations (6-32) provide us with the means for studying infinitesimal contact transformations via Stone's theorem (for proof see Stone, 1932).

<u>STONE'S</u> <u>THEOREM</u> <u>(6.5)</u> - Let $\{\hat{U}_g : g\epsilon\underline{R}\}$ be a family of unitary operators on $\underline{H}$ such that

i. $\hat{U}_g \hat{U}_h = \hat{U}_{g+h}$ ; $\hat{U}_o = 1$ and

ii. for all $\psi,\chi \ \epsilon \ \underline{H}$, the map $g\rightarrow\langle\psi|\hat{U}_g|\chi\rangle$ is continuous,

then there exists a self-adjoint operator $\hat{G}$ with appropriate domain in $\underline{H}$, such that

a. $\hat{G}\psi \equiv \lim_{g\to 0} i\hbar/g \cdot (\hat{U}_g \psi - \psi)$ .     -- (6-35)

b. If $\hat{G}\psi$ exists, then for all $g \varepsilon \underline{R}$, $\psi_g \equiv \hat{U}_g \psi$ exists and

$$\hat{G}\psi_g = i\hbar \, d\psi_g/dg \quad ,$$

or alternatively     $\hat{G}\hat{U}_g = i\hbar \, d\hat{U}_g/dg$ .     -- (6-36)

---

$\hat{G}$ is called the _infinitesimal generator_ of the family $\{\hat{U}_g\}$, which forms a _continuous 1-parameter unitary group_. Formally we have, in view of (6-36),

$$\hat{U}_g = \exp\{g\hat{G}/i\hbar\} \quad . \qquad -- (6-37)$$

**Example (6.6)** - The family of shifts $\hat{U}_d$ (although continuity is _not_ straightforward). The infinitesimal generator is given by

$\hat{U}_d \psi(q) = \psi(q+d)$

$= \Sigma_n \{1/n! \cdot (d^A \cdot \partial/\partial q^A)^n\} \, \psi(q)$

$= \exp\{d^A \cdot \partial/\partial q^A\} \, \psi(q)$

$= \exp\{(i\hbar d^A \cdot \partial/\partial q^A)/i\hbar\} \, \psi(q)$ .     -- (6-38)

Since this is true for arbitrary $d^A$ we see that the operator $i\hbar \partial/\partial q^A$ generates the shift in the direction of the coordinate $q^A$.

---

The situation described in Stone's theorem, where state vectors are transformed by unitary operators, is called the _Schroedinger picture_ of quantum dynamics. However, since the physically significant quantities of quantum mechanics are inner products of the form $\langle\psi|\hat{a}|\chi\rangle$, we can ascribe the transformation of these products to either a transformation of the state or a transformation of the operator $\hat{a}$. This second viewpoint is called the _Heisenberg picture_. Denoting the transformed operator by $\hat{a}_g$, these two pictures of dynamics will agree if

$$\langle\psi|\hat{a}_g|\chi\rangle = \langle\psi_g|\hat{a}|\chi_g\rangle$$

$$= \langle \hat{U}_g \psi | \hat{a} | \hat{U}_g \chi \rangle$$

$$= \langle \psi | \overline{U}_g \hat{a} \hat{U}_g | \chi \rangle \quad ,$$

ie, the transformed operator in the Heisenberg picture is defined

by

$$\hat{a}_g \equiv \overline{U}_g \hat{a} \hat{U}_g \quad . \quad (\overline{U}_g \hat{U}_g = 1) \qquad \qquad -- \ (6\text{-}39)$$

Hence we see that dynamical transformations in quantum mechanics

are simply a certain species of CT, as was the case in homo-

geneous mechanics.    From (6-36) we obtain the differential

equation satisfied by $\hat{a}_g$ :

$$i\hbar \frac{d\hat{a}_g}{dg} = i\hbar \frac{d\overline{U}_g}{dg} \hat{a} \hat{U}_g + i\hbar \overline{U}_g \hat{a} \frac{d\hat{U}_g}{dg}$$

$$= -\hat{G}\overline{U}_g \hat{a} \hat{U}_g + \overline{U}_g \hat{a} \hat{G} \hat{U}_g \quad .$$

Assuming that $\hat{U}_g$ may be taken inside the limit in the definition

(6-35),   it follows that $\hat{U}_g$ commutes with $\hat{G}$ and we may write the

above expression in the form

$$\delta \hat{a}_g = \frac{\delta g}{i\hbar} \{ \hat{a}_g \hat{G} - \hat{G} \hat{a}_g \}$$

$$= \delta g . [\hat{a}_g , \hat{G}] \quad . \qquad \qquad -- \ (6\text{-}40)$$

Equation (6-40) is the quantum analogue of the infinitesimal

contact   transformation   equation (4-86), justifying the title

"infinitesimal generator" for $\hat{G}$. The analogy between (6-37) and

(4-97)   is also clear.   In the case of the shift operators $\hat{U}_d$ we

saw in (6-38) that the infinitesimal generator is given by

$$\hat{G} = i\hbar . \partial / \partial q^A = -\hat{p}_A \quad ,$$

so   the change in any observable due to application of the shift

transformation for small d is

$$\delta \hat{a}_d = -d^A [\hat{a}_d , \hat{p}_A] \quad . \qquad \qquad -- \ (6\text{-}41)$$

The   difference in sign between (6-41) and (4-89) is due to the

fact that $\hat{U}_d$ represents a coordinate transformation, while (4-89)

represents   a   point   transformation:   the displacement of an

observable through a shift +d is actually represented by the unitary operator $\hat{U}_{-d}$, in which case (6-41) adopts the form of (4-89). Using the Schroedinger picture of these events we find

$$\hat{p}_A \psi = -i\hbar \partial\psi/\partial q^A , \qquad\qquad -- (6-42)$$

and by applying the classical decomposition of section 3.5 we obtain the non-relativistic <u>Schroedinger equation</u> as the N-th component of (6-42):

$$\hat{H}^* \psi = i\hbar \partial\psi/\partial t . \qquad\qquad -- (6-43)$$

It at first seems reasonable to take (6-42) as the relativistic generalisation of the Schroedinger equation, in view of (6-43), but there is another contender for this rôle. Suppose we wish to consider the dynamical evolution of a system with respect to the parameter τ, then assuming this may be described by a CT, (6-36) gives us

$$i\hbar\frac{d\psi(q)}{d\tau} = \hat{H}\psi(q) \qquad\qquad -- (6-44)$$

for some self-adjoint operator $\hat{H}$. The analogy with standard quantum theory clearly prompts us to take $\hat{H}$ to be the Hamiltonian operator, which is supported by a closer examination of equation (6-44). It must be remembered that the argument q of ψ is purely an indexing variable, comprising the eigenvalues of position, which does not depend in any way upon τ. Thus the LHS of (6-44) is zero and

$$\hat{H}\psi = 0 , \quad \text{or} \quad \hat{H}(\hat{q},\hat{p}) = 0 . \qquad\qquad -- (6-45)$$

This will be called the relativistic Schroedinger equation; it is the quantum analogue of (4-9), which appears as a consequence of the parameter-independence of the theory. It asserts the existence of an algebraic relation between the position and momentum operators. However, looking back at the Hamiltonian (4-29) we see that its form is not of the easiest to solve if substituted

into (6-45). Consequently we shall assume that in the case of a single charged relativistic particle in Minkowski space the operator equation (6-45) can be manipulated to the form of the energy equation (4-31):

$$\{[-i\hbar\nabla^a - eA^a(q)][-i\hbar\nabla_a - eA_a(q)] + m^2\}\psi = 0 \ . \qquad -- \ (6-46)$$

We have absolutely no justification for such a manipulation of an operator equation, but for the purposes of the investigations in the following section we shall nevertheless assume (6-46) to be valid.


## 6.3 The hydrogen atom

As an example of the above exposition we now look at the application of it to the case of the hydrogen atom. Since this is a well-known result in classical quantum theory, it will provide useful information on the viability of the theory presented in the last section. The problem consists of a massive nucleus of charge +e, supposed to remain at rest at the origin, and an electron of charge -e and mass m moving in the resulting central electrostatic field. Using these values in (6-46) we obtain the following form of the Schroedinger equation in Minkowski space:

$$g^{ab}\hat{p}_a\hat{p}_b - 2e^2\hat{p}_4/r - e^4/r + m^2 = 0 \ .$$

We can conveniently split the space components away from the other terms in this equation by making use of the classical del operator

$$\underline{\nabla}_\alpha \equiv \partial/\partial q^\alpha$$

in Cartesian coordinates, in which case the Schroedinger equation becomes

$$\left[-\hbar^2 \underline{\nabla}^2 + \hbar^2 \frac{\partial^2}{\partial t^2} - \frac{2e^2\hbar}{ir}\frac{\partial}{\partial t} - \frac{e^4}{r^2} + m^2\right]\psi = 0 \quad . \quad -- \quad (6-47)$$

Here $\psi$ is a wave-function representing the electron in an eigen-state of the Hamiltonian. For convenience we shall work in spherical polar coordinates and make use of the corresponding form of the Laplacian operator

$$\underline{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$=> \qquad -\hbar^2\underline{\nabla}^2 = \frac{-\hbar^2}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\hat{L}^2 \quad ,$$

where $\qquad \hat{L}^2 = -\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} \quad -- \quad (6-48)$

is the square of the magnitude of the usual angular momentum operators resulting from the classical theory. The eigen-functions of $\hat{L}^2$ are well-known (see Lawden,1967), and are given by

$$\hat{L}^2 Y_{lk} = l(l+1)\hbar^2 Y_{lk} \quad ; \qquad \left.\begin{array}{c} \\ \end{array}\right\}$$
$$Y_{lk}(\theta,\varphi) = A_{lk}P_l^k(\cos\theta)\exp\{ik\varphi\} \quad , \quad \left.\begin{array}{c}\\\end{array}\right\} \quad -- \quad (6-49)$$

where $k\epsilon\underline{Z}$, $l=0,1,2,\ldots$, $A_{lk}$ is a normalising constant and $P_l^k$ is the associated <u>Legendre function</u> of degree l and order k. Now suppose our system is in an angular momentum eigenstate, then we can separate variables in $\psi$ by writing

$$\psi(r,\theta,\varphi,t) = Z(r,t).Y_{lk}(\theta,\varphi) \quad , \qquad -- \quad (6-50)$$

in which case

$$-\hbar^2\underline{\nabla}^2\psi = Y_{lk}\left[\frac{-2\hbar^2}{r}\frac{\partial Z}{\partial r} - \hbar^2\frac{\partial^2 Z}{\partial r^2} + \frac{\hbar^2 l(l+1)Z}{r^2}\right]$$

and (6-47) reduces to

$$-\hbar^2\left[\frac{\partial^2 Z}{\partial r^2} + \frac{2}{r}\frac{\partial Z}{\partial r} - \frac{l(l+1)Z}{r^2}\right] + \hbar^2\left[\frac{\partial^2 Z}{\partial t^2} + \frac{2e^2 i}{\hbar r}\frac{\partial Z}{\partial t}\right] - \left[\frac{e^4}{r} - m^2\right]Z = 0 \quad .$$

we now separate variables again and let

$$Z(r,t) = R(r).T(t) \quad , \qquad -- \quad (6-51)$$

whence we have

$$\frac{R''}{R} + \frac{2R'}{rR} - \frac{l(l+1)}{r^2} - \frac{T''}{T} - \frac{2e^2 i T'}{\hbar r\,T} + \frac{e^4}{\hbar^2 r^2} - \frac{m^2}{\hbar^2} = 0 \quad,$$

or 

$$F(r) \equiv \frac{R''}{R} + \frac{2R'}{rR} - \frac{l(l+1)}{r^2} + \frac{e^4}{\hbar^2 r^2} - \frac{m^2}{\hbar^2}$$

$$= \frac{T''}{T} + \frac{2e^2 i T'}{\hbar r\,T} \quad . \qquad\qquad -- \;(6\text{-}52)$$

This equation is not immediately separable, but since we know it is true for all r and t we may assert it for two specific values of r:

$$\left.\begin{array}{l} F(r_1) = \dfrac{T''}{T} + \dfrac{2e^2 i T'}{\hbar r_1\,T} \\[3mm] F(r_2) = \dfrac{T''}{T} + \dfrac{2e^2 i T'}{\hbar r_2\,T} \end{array}\right\} \quad \text{for all } t;\ r_1 \neq r_2$$

$$=> \qquad F(r_1) - F(r_2) = \frac{2e^2 i}{\hbar}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]\frac{T'}{T}$$

$$=> \qquad \frac{T'}{T} = \frac{\hbar}{2e^2 i}\,\frac{F(r_1) - F(r_2)}{\{1/r_1 - 1/r_2\}} = \text{const} = \alpha$$

$$=> \qquad\begin{array}{l} T = T_0 \exp(\alpha t) \quad; \\[3mm] T' = \alpha T \quad;\quad T'' = \alpha^2 T \quad . \end{array} \left.\begin{array}{c}\\[3mm]\\ \end{array}\right\} \quad -- \;(6\text{-}53)$$

Note that from the second of (6-53) we immediately see that

$$\hat{p}_4 Z = \frac{\hbar}{i}\frac{\partial Z}{\partial t} = -\alpha i\hbar Z = -EZ \quad,$$

where E is the (constant) total energy of the electron. Thus we can make the identification

$$\left.\begin{array}{l}\alpha = -iE/\hbar \\[3mm] \text{and hence} \qquad T(t) = T_0 \exp\{-iEt/\hbar\} \quad .\end{array}\right\} \quad -- \;(6\text{-}54)$$

Note that this is a monochromatic wave stretching from $t=-\infty$ to $t=+\infty$, and so offers no information as to the coordinate time of the electron. However, this is a necessary consequence from (6-27) of the fact that the momentum $p_4$ takes the precise value -E. (6-54) is precisely the classical time-dependence, and is due here to the fact that $p_4$ is a conserved quantity of the

Kepler problem, so that $\overset{\cdot}{H}$ and $\hat{p}_4$ have simultaneous eigen-functions; this will __not__ be true of all systems. A similar situation obtains in the case of the conserved angular momentum, as we see if we set $\theta = \pi/2$ in (6-49).

Having obtained the equation (6-54) for $T(t)$ we can now substitute it into (6-52) to obtain an ordinary DE for $R(r)$:

$$R'' + \frac{2R'}{r} + \left\{ \frac{1}{\hbar^2} \left[ \frac{e^4}{r^2} - \frac{2Ee^2}{r} + (E^2 - m^2) \right] - \frac{l(l+1)}{r^2} \right\} R = 0 . \qquad \text{-- (6-55)}$$

For large values of $r$ (6-55) may be approximated by

$$R'' + \frac{(E^2 - m^2)R}{\hbar^2} = 0 ,$$

which has a solution $\qquad R = \exp\{-\beta r\}$ ,

where $\qquad\qquad\qquad \beta^2 = (m^2 - E^2)/\hbar^2$ . $\qquad\qquad$ -- (6-56)

We shall see (equation (6-64)) that $\beta^2 > 0$ and so $\beta$ is real; we take the positive root to ensure that the integral of $R$ con-verges. In view of this "solution at infinity" we try a solution for all $r$ of the form

$$R = \exp\{-\beta r\}\chi(x) \quad ; \quad x = 2\beta r , \qquad\qquad \text{-- (6-57)}$$

in which case (6-55) becomes

$$x\chi'' + (2-x)\chi' - \left[ 1 - \lambda + \frac{l(l+1) - e^4/\hbar^2}{x} \right] \chi = 0 ,$$

where $\qquad\qquad\qquad \lambda = -Ee^2/\hbar^2\beta$ . $\qquad\qquad\qquad$ -- (6-58)

This equation comes very close to the Laguerre-type equation of classical quantum theory, the only difference being the final term in $e^4/\hbar^2$. Since in our units $e^4/\hbar^2 \approx 5 \times 10^{-6}$, the approxim-ation is good. We may recast this in the form of the associated Laguerre equation if we let

$$\chi = x^j y(x) \quad ; \quad j(j+1) = l(l+1) - e^4/\hbar^2 , \qquad \text{-- (6-59)}$$

whence $\qquad\qquad xy'' + (2j+2-x)y' + (\lambda-j-1)y = 0$ . $\qquad\qquad$ -- (6-60)

Since $x=0$ is a regular singular point of (6-60) we can seek

a series solution for y of the form

$$y = \sum_{0}^{\infty} a_s x^{\varrho+s} \quad ; \quad a_o \neq 0 \quad , \qquad \text{-- (6-61)}$$

from which we obtain the following indicial equation:

$$\varrho(\varrho + 2j + 1) = 0 \quad .$$

Both roots of this equation lead to the same solution, as in the classical theory, so we take $\varrho=0$. The recurrence relation for the coefficients of the series is

$$a_{s+1} = \frac{(s+j+1-\lambda)}{(s+1)(s+2j+2)} a_s \quad , \qquad \text{-- (6-62)}$$

valid for s=0,1,2,... The situation is now exactly the same as for the classical hydrogen atom, with the exception that j (as defined in (6-59)) is no longer an integer. The solution (6-61) is meaningful only if the numerator in (6-62) is some integer ⩽s, in which case the series terminates as a polynomial of degree $(\lambda-j-1)$. Using the definition (6-59) of j we find that $\lambda$ can therefore be written to the first approximation as

$$\left. \begin{array}{l} \lambda \approx n - \dfrac{e^4}{\hbar^2 (2l+1)^2} \quad , \\[3mm] \text{where} \qquad n \geqslant l + 1 \quad . \end{array} \right\} \qquad \text{-- (6-63)}$$

n is the quantum number of (6-55), and leads to quantisation of the energy levels exactly as in the classical hydrogen atom (see for example Lawden,1967). To calculate these levels we substitute (6-63) and (6-56) into (6-58) to obtain

$$\frac{-Ee^2}{\hbar\sqrt{(m^2-E^2)}} = n - \frac{e^4}{\hbar^2 (2l+1)^2}$$

$$\Rightarrow \quad E^2 = \frac{1}{n^2\hbar^2 + e^4} \left[ \hbar^2 m^2 n^2 - \frac{2ne^4}{\hbar^2 (2l+1)^2} \right]$$

$$\Rightarrow \quad E = E_n \approx m - \frac{me^4}{2n^2\hbar^2} \quad . \qquad \text{-- (6-64)}$$

This approximation is the same as the classical energy levels,

and shows that β is real.

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   After   the   initial  successes  of  our  study  of  the  hydrogen

atom  we  would  do  well  now  to  look  at  the  more  problematical  feat-

ures   of  the  quantum  theory  developed  in  section  6.2.   First  we

should   note   that   we  have  made  no  reference  to  the  electron/

positron   duality   which  arose  necessarily  from  Dirac's  (1928)

theory.   Dirac's  stated  intention  in  this  paper  was  "to  find  some

incompleteness   in   the   previous  methods  of  applying  quantum

mechanics  to  the  point  charge  electron,  such  that,  when  removed,

the   whole   of  the  duplexity  phenomena  follow  without  arbitrary

assumptions."   The  above  theory  makes  no  reference  at  all  to  spin

or   the   splitting  of  energy  levels.   In  fact  it  _is_  possible  to

obtain  a  result  for  the  fine  structure  as  the  next  approximation

in   the  series  truncated  in  (6-64)  (see  Schroedinger,1926),  but

this   result  indicates  a  much  larger  spread  than  the  experiment-

ally   observed  value.   This  might  be  rectified  by  taking  account

of   spin,  but  even  then  the  fact  that  spin  must  be  independently

introduced   must  be  considered  a  failing  of  the  theory.   On  the

other   hand  the  assumptions  which  Dirac  makes  in  order  to  derive

spin   are  hardly  the  most  natural  ones,  and  on  the  whole  we  feel

that   our   theory  is  fundamentally  sound  in  a  way  which  Dirac's

theory   is  not  and  that  this  is  sufficient  to  counterbalance  the

above  disadvantages.

   A   further   failing  of  our  theory  is  associated  with  the

problems  of  time  and  evolution.   Looking  back  at  (6-45)  we  see

the   fundamental   parameter-independence   condition   $\dot{H}=0$  as  a

necessary   consequence  of  a  quantum  theory  based  on  homogeneous

mechanics.   Because  of  this  all  reference  to  τ  vanishes  from  the

Schroedinger  equation.   The  same  holds  true  of  the  Heisenberg

picture, in which the evolution equation is given by (6-40):

$$\delta \hat{a}_\tau = \delta \tau [\hat{a}_\tau, \hat{H}] = 0 \quad . \qquad \qquad -- \text{(6-65)}$$

Thus we find that this quantum theory is completely independent of $\tau$ and indeed contains no reference to it at all. This seems at first a little strange, until we note from (6-65) that $\hat{H}$ and $\tau$ are conjugate variables and so satisfy the criteria for the uncertainty principle (6-25). Since we specify that the Hamiltonian has a sharp value, $\hat{H}=0$, we can have no knowledge of the parameter $\tau$. This unsatisfactory situation is also true in standard quantum theory, where we look for steady states of energy and so consign all time dependence to the monochromatic term $\exp\{-iEt/\hbar\}$.

Another major failing of our formulation of quantum theory lies in the rôle of coordinate time. The dynamical equation (6-46) is simply the Klein-Gordon equation (see Schroedinger,1926), which was discarded very early in the history of quantum theory on the grounds that it involves second time derivatives of the state vector $\psi$. This means that in order to solve (6-46) in the general case we require the value of $\partial\psi/\partial t$ at an initial instant: the subsequent motion of a particle is not fully determined by the state $\psi(x,0)$ at the initial instant. However, we see from the solution procedure leading to (6-53) that the second time derivatives played no part in the hydrogen atom problem and this will continue to be true for the same reason in a number of useful cases. So we do not always need $\partial\psi/\partial t$. The prototype case where $\partial^2\psi/\partial t^2$ does enter in to the problem is the case of free rectilinear motion.

Consider a particle moving in 2-dimensional spacetime $(x,t)$ whose momentum has been measured precisely at an initial instant $t=0$. Then we know

$$-i\hbar\partial\psi/\partial x = p\psi \quad ,$$

so $$\psi = \exp\{ipx/\hbar\} \; T(t) \quad . \qquad \text{-- (6-66)}$$

Suppose there is no external field, so we are concerned with the following Schroedinger equation:

$$-\hbar^2 \frac{\partial^2\psi}{\partial x^2} + \hbar^2 \frac{\partial^2\psi}{\partial t} + m^2\psi = 0 \quad . \qquad \text{-- (6-67)}$$

Substituting now from (6-67) we obtain

$$\frac{T''}{T} = -\frac{1}{\hbar^2}(p^2 + m^2) = -E^2/\hbar^2$$

$$\Rightarrow \qquad T = A \exp\{iEt/\hbar\} + B \exp\{-iEt/\hbar\} \quad .$$

Here $\psi$ is undetermined unless $\partial\psi/\partial t|_0$ is given, and yet we see that the ambiguity is in a sense irrelevant. In both (6-53) and here the time dependence of $\psi$ is a monochromatic wave - the only difference here is that the wave describes an ellipse in the Argand diagram, rather than a circle.

We have seen that there are serious problems in our formulation of quantum theory, yet the above remarks lead us to believe that these problems are not insurmountable. To investigate this more thoroughly would be to digress to an unwarranted extent from the main theme of this thesis, but we feel that the above formulation of quantum theory has much to recommend it and is structurally more sound than existing relativistic quantum theories of particle mechanics. It should be remembered that the Lorentz-Dirac equation for the motion of a relativistic charged particle also involves initial specification of the acceleration (Teitelboim, 1970); also, initial specification only of $\partial\psi/\partial x^\alpha$ implies that we know exactly <u>when</u> the specification takes place, which is far from being a valid assumption. Indeed such knowledge is expressly prohibited by our interpretation of the time-energy relation (6-27). The whole problem of initial conditions is a

far-reaching one, and could possibly be resolved in the above by the incorporation of τ into the theory and by a clearer demarcation between dynamics and epistemology in measurements on physical particles.

## REFERENCES

Adler,R.J., Bazin,M.J. & Schiffer,M. (1965): "Introduction to general relativity", McGraw-Hill.

Bergmann,P.G. (1949): Phys. Rev., 75, 680.

Bluman,G.W. & Cole,J.D. (1974): "Similarity methods for differential equations", Springer.

Caratheodory,C. (1935): "Calculus of variations", Chelsea, NY, 1982 (English translation of German original).

Chang,T.S. (1945): Proc. Roy. Soc., A183, 316.

Chang,T.S. (1946): Proc. Camb. Phil. Soc., 42, 132.

Dieudonne,J. (1960): "Foundations of modern analysis", Academic Press.

Dirac,P.A.M. (1927): Proc. Roy. Soc., A114, 243.

Dirac,P.A.M. (1928): Proc. Roy. Soc., A117, 610.

Dirac,P.A.M. (1958): "The principles of quantum mechanics", 4th ed., Oxford.

Feshbach,H. & Villars,F. (1958): Rev. Mod. Phys., 30, 24.

Fuchs,K. (1939): Proc. Roy. Soc. Edin., A59, 109.

Goldstein,H. (1981): "Classical mechanics", 2nd ed., Addison-Wesley.

Hawking,S.W. & Ellis,G.F.R. (1973): "The large-scale structure of space-time", Cambridge.

Heisenberg,W. & Pauli,W. (1929): Z. Phys., 56, 1.

Heisenberg,W. & Pauli,W. (1930): Z. Phys., 59, 168.

Kilmister,C.W. & Reeve, J.E. (1966): "Rational mechanics", Longmans.

Lawden,D.F. (1967): "The mathematical principles of quantum mechanics", Methuen.

Leach,P.G.L. (1981): J. Australian Math. Soc., B23, 173.

Liedloff,J. (1977): "The continuum concept", Alfred A. Knopf, Inc., NY.

Menzel,D.H. (1961): "Mathematical physics", Dover, NY.

Minkowski,H. (1908): in "The principle of relativity", Dover, 1923.

Pearson,C.E. & Carrier,G.F. (1976): "Partial differential equations: theory and technique", Academic Press, London.

Prince,G.E. & Leach,P.G.L. (1980): Hadronic J., $\underline{3}$, 941.

Riesz,F. & Sz.-Nagy,B. (1955): "Functional analysis", Ungar, NY.

Rohrlich,F. (1965): "Classical charged particles", Addison-Wesley.

Rund,H. (1959): "The differential geometry of Finsler spaces", Springer.

Rund,H. (1966): "The Hamilton-Jacobi theory in the calculus of variations", Van Nostrand.

Schroedinger,E. (1926): Ann. d. Phys., $\underline{81}$, 109.

Sommerfeld,A. (1928): "Atomic structure and spectral lines", Methuen, London.

Stone,M.H. (1932): Ann. of Math. (2), $\underline{33}$, 643.

Sudarshan,E.C.G. & Mukunda,N. (1974): "Classical dynamics: a modern perspective", Wiley.

Synge,J.L. (1954): "Geometrical mechanics and De Broglie waves", Cambridge.

Synge,J.L. (1960): "Classical dynamics", in "Principles of classical mechanics and field theory", Encyclopedia of Physics, Vol.III/1, Springer. Ed: S.Fluegge.

Teitelboim,C. (1970): Phys. Rev. D, $\underline{1}$, 1572.

Volterra,V. (1931): "Theory of functionals of integral and integro-differential equations", Blackie, London.

Von Neumann,J. (1955): "The mathematical foundations of quantum mechanics", Springer.

Weiss,P. (1936): Proc. Roy. Soc., $\underline{A156}$, 192.

Weiss,P. (1938a): Proc. Roy. Soc., $\underline{A169}$, 102.

Weiss,P. (1938b): Proc. Roy. Soc., $\underline{A169}$, 119.

## SYNOPSIS

Like any major breakthrough in thinking, the theory of relativity caused a great upheaval in our attitude to science. Seventy years after the advent of relativity we are still coming to terms with the changes it has brought in our outlook. Part of this process is simply the valid translation of pre-relativistic laws and concepts into the 4-dimensional language of relativity - a problem by no means as easy as would at first seem; the aim of this thesis is to survey the ways in which the methods of analytical mechanics may be translated into a relativistic setting.

Chapter 1 provides an introduction to the work in the form of a non-rigorous discussion of the historical and mathematical development of electromagnetism, analytical mechanics and relativity, and ends with a presentation of the basics of the functional calculus. This is needed in the presentation of field theory given in chapter 2. We see two possibilities for the relativistic formulation of analytical mechanics, and field theory represents the first of these possibilities. In the absence of any real grounds for continuing on this tack we then move on to the other possibility in chapter 3, where we review the attempts of a number of authors to formulate relativistic particle mechanics as a Hamiltonian system. This then leads in chapter 4 to our own such attempt, based mainly on the work of Synge, which we have named homogeneous mechanics. After the main exposition of the theory the work of the remaining chapters 5 and 6 is then to apply the above theory (not always successfully) to a number of cases where analytical mechanics has in the past proven itself an invaluable tool: namely, the areas of symmetries and quantum theory.