Imperial College London

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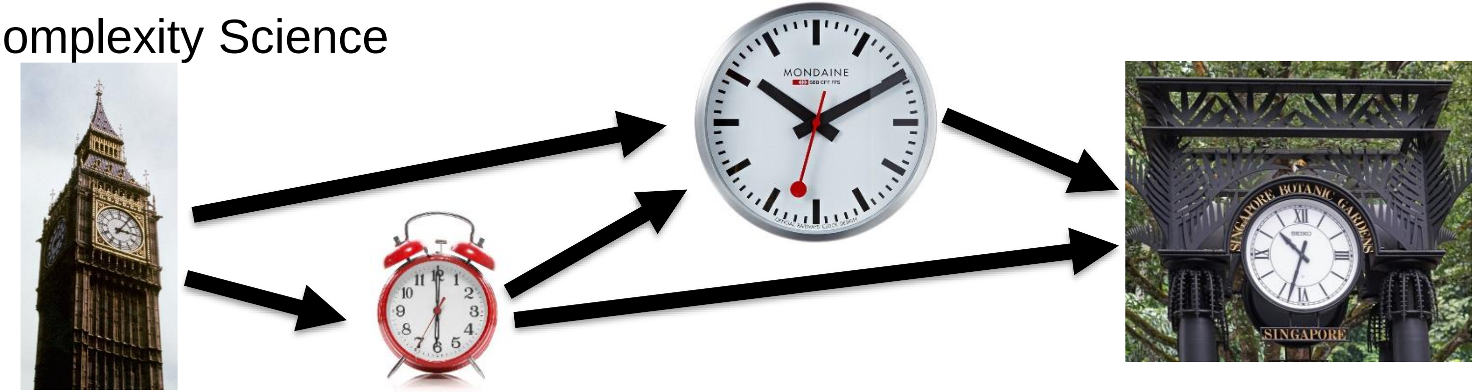
The Longest Path in the Price Model

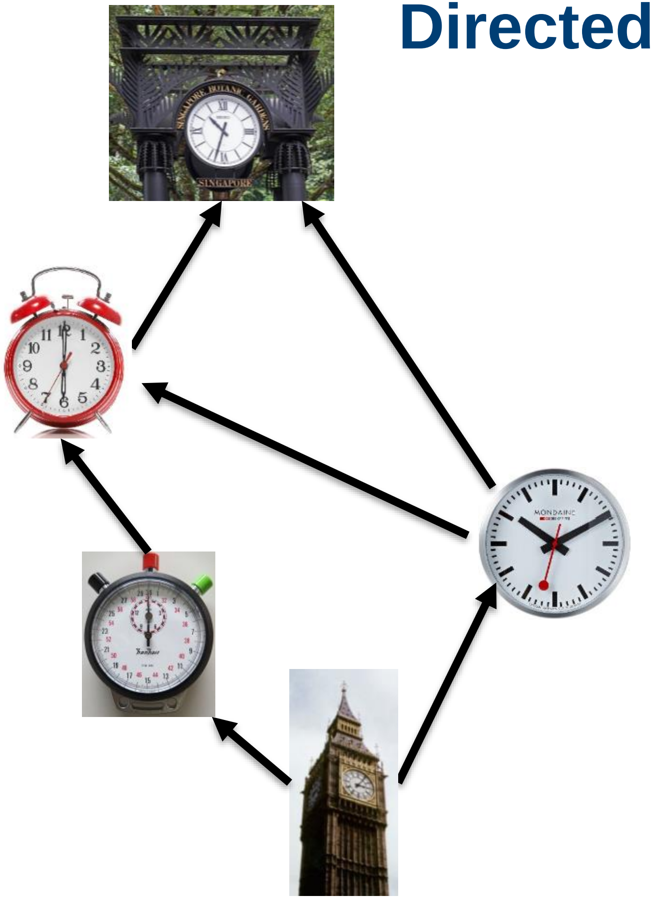
Complexity Science

TIM EVANS work with **Lucille Calmon, Vaiva Vasiliauskaite**

Centre for
Complexity Science



Directed Acyclic Graphs = DAG



- A directed network with no cycles
- Defines a ***Partial Order*** on set of nodes
  Order constrains direction of edges

e.g. **Temporal Vertex Networks**,
vertices assigned a time,
edges respect the ***arrow-of-time***

- Citation networks
  e.g. papers, patents, court judgements, blogs
- Task scheduling
- Food webs
- Cryptocurrency Transactions (e.g. IOTA)
- Causal set approach to quantum gravity

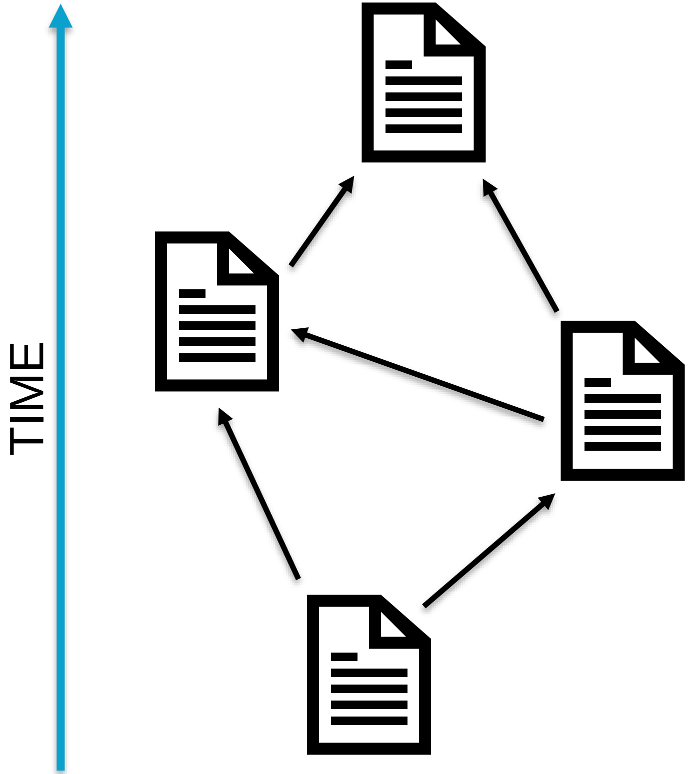
The Price model

Networks of Scientific Papers

The pattern of bibliographic references indicates the nature of the scientific research front.

(Science, 1965)

Derek J. de Solla Price

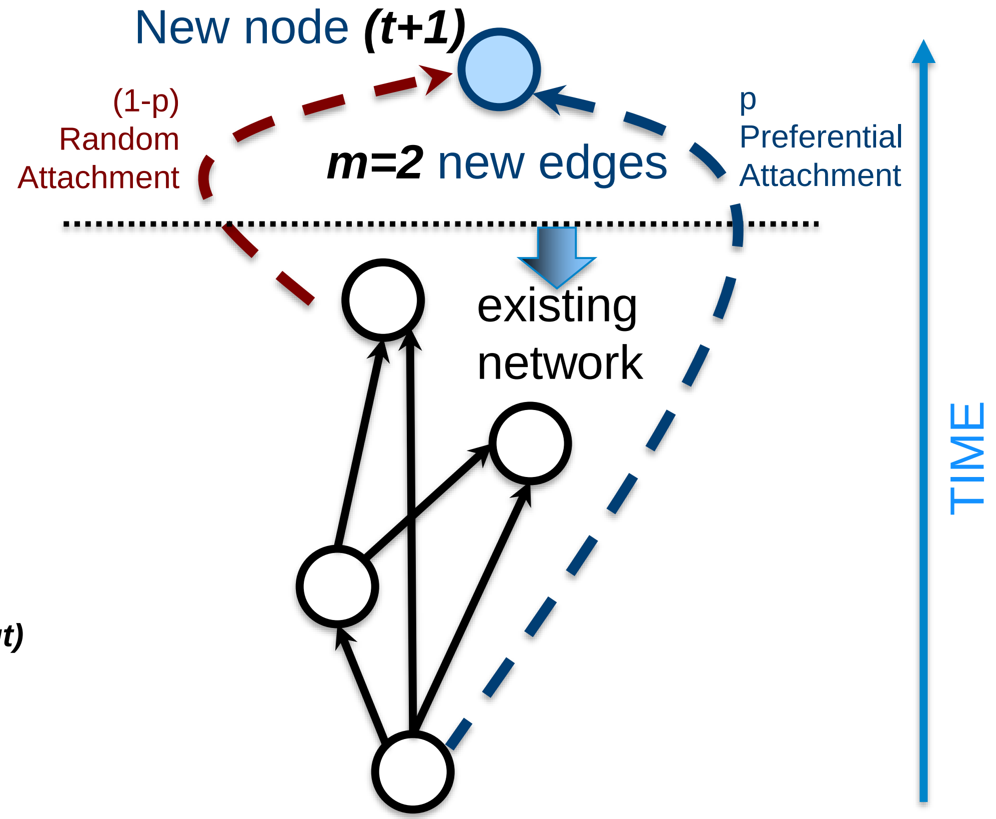


- Node = scientific publication
- Edge = from cited paper to citing paper
- Papers can only reference older papers, *arrow-of-time*
- Growing network model = Directed version of Barabási-Albert model

Price model

- Add new node ***(t+1)***
- Add ***m*** new edges to new node from existing node chosen using *either*
  - with probability ***p*** use Cumulative Advantage = Preferential Attachment Proportional to out-degree $k^{(out)}$

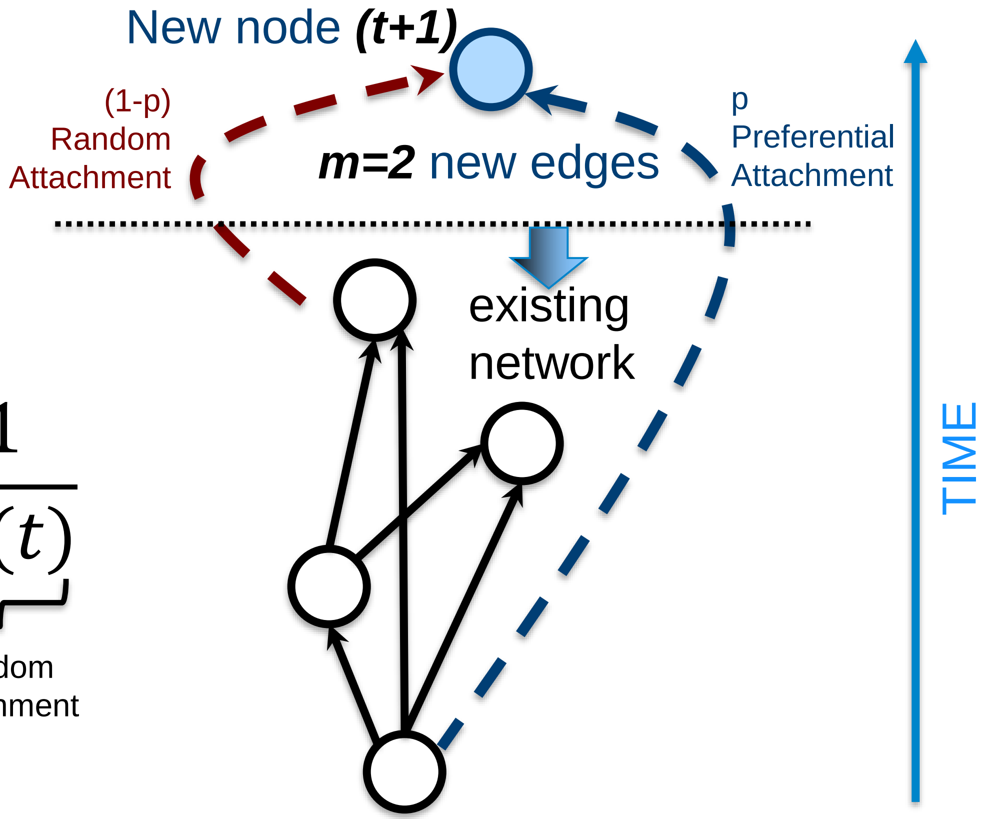
  or
  - with probability ***(1-p)*** use Random Attachment



Price model

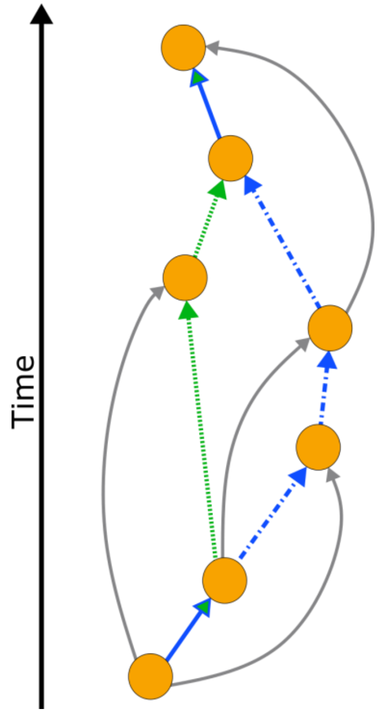
Choosing an existing node with out-degree $\boldsymbol{k}^{(out)}$ with probability Π

$$\Pi = p \underbrace{\frac{k^{(\text{out})}}{E(t)}}_{\substack{\text{cumulative advantage =}\\ \text{preferential attachment}}} + (1-p) \underbrace{\frac{1}{N(t)}}_{\substack{\text{random}\\ \text{attachment}}}$$

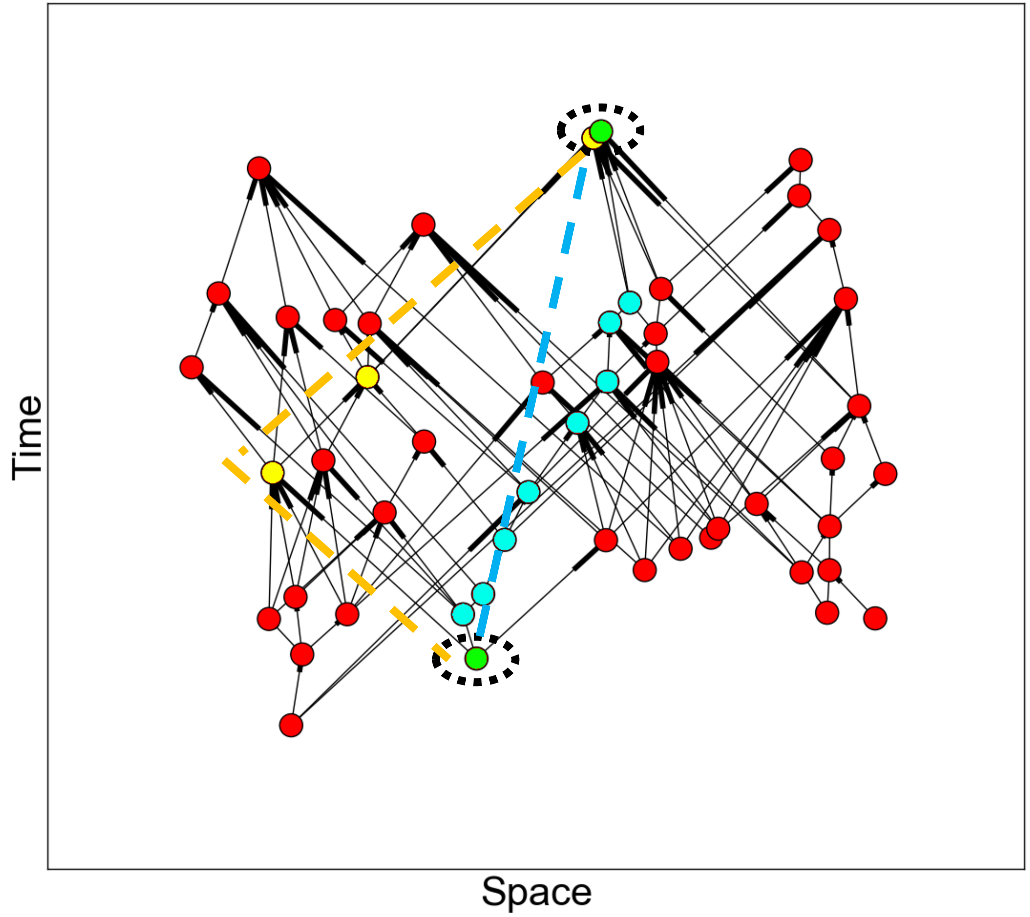


The Longest Path in Price Model

- The longest path is well-defined in a DAG
- Approximates geodesic in Minkowski random DAGs [Brightwell & Gregory 91]
- Similar to “Main Path” of bibliometrics?
- Longest path length ***L*** ~ ℓ greedy path length



Grey – SHORTEST =3
Green – GREEDY ℓ =4
Blue – LONGEST L= 5



Longest Path

Longest Path (9)

≈ Geodesic

Shortest Path (4)

≈ Edges of Light Cones

Minkowski PPP model

e.g. see Brightwell & Gregory 1991; Reid 2003; Evans and Clough 2016

The Greedy Path in Price Model

DEFINITION

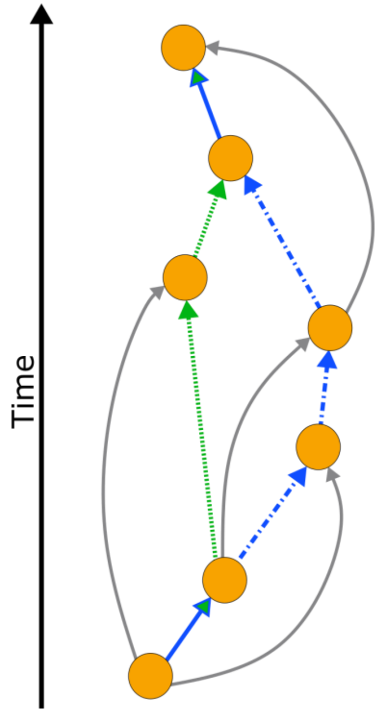
The next node on a Greedy Path
is the closest in time

Conjecture:

Longest path length ***L*** ~ ℓ greedy path length

$$\lim_{N \to \infty} \frac{\ell}{L} = c < 1$$

* Known for Minkowski Space PPP [Brightwell & Gregory 91]



Grey – SHORTEST =3
Green – GREEDY ℓ =4
Blue – LONGEST L= 5

The Master Equation for the length of the greedy path

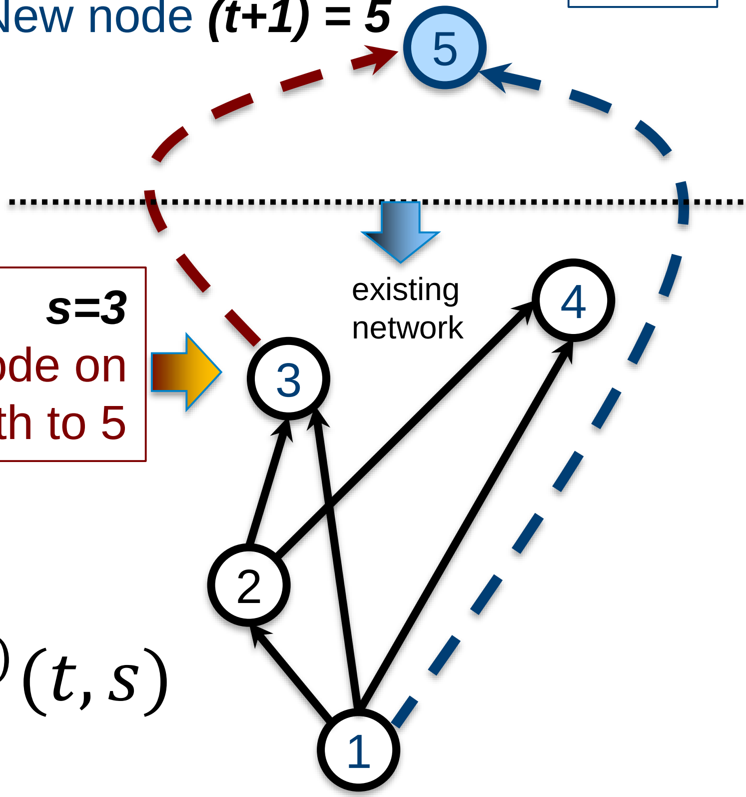
m=2 new edges

$\ell(\boldsymbol{t})$ = Length of greedy path
from source node ***t=1***
to node ***t***

$\Pi^{(max)}(t,s)$ = Probability ***s*** is largest time connected to ***t***

s=3
next node on greedy path to 5

New node ***(t+1) = 5***



Probability node at time ***(t+1)*** has length ℓ

$$P(\ell, t+1) = \sum_{s=1}^{t} P(\ell-1, s)\ \Pi^{(max)}(t,s)$$

The Master Equation for the length of the greedy path

$\ell(\boldsymbol{t})$ = Length of greedy path from source node ***t=1*** to node ***t***

Probability node at time ***(t+1)*** has length ℓ

$$P(\ell, t+1) = \sum_{s=1}^{t} P(\ell - 1, s)\ \Pi^{(max)}(t, s)$$

Probability that closest node chosen at time ***(t+1)*** is ***s***

$$\Pi^{(max)}(t, s) = \left(\Pi_{\leq}(t, s)\right)^m - \left(\Pi_{\leq}(t, s-1)\right)^m$$

cdf of attachment probability $\Pi_{\leq}(t, s) = \sum_{r=1}^{s} \Pi(t, r)$

The Longest Path in the Price Model

The Generating Function solution

$$G(z,t) = \sum_{z=0}^{\infty} z^{\ell} P(\ell,t)$$

$$G(z,t) = \prod_{s=1}^{t-1} \left[z + (1-z) \left(\frac{s-1+p}{s} \right)^{m} \right]$$

Simple linear form for attachment probability Π allows for exact solution at finite time within mean field approximation.

Generating function is a product of ***m*** Gamma function ratios

The Average Greedy Path Length

***m*=**
number of new edges at each t

$\bar{p} = (1-p)$ probability to connect to existing nodes using random attachment

$$\lim_{t\to\infty} \ell(t) = m\bar{p}\ln(t) - m\bar{p}\,\psi(m\bar{p}+1) + \sum_{n=2}^{m}\binom{m}{n}(-1)^{n-1}(\bar{p})^{n}\xi(n) + O(t^{-1})$$

digamma function

Riemann's zeta function

The Greedy Path Length Distribution

***m*=**
number of new edges at each t

$\overline{p} = (1-p)$ probability to connect to existing nodes using random attachment

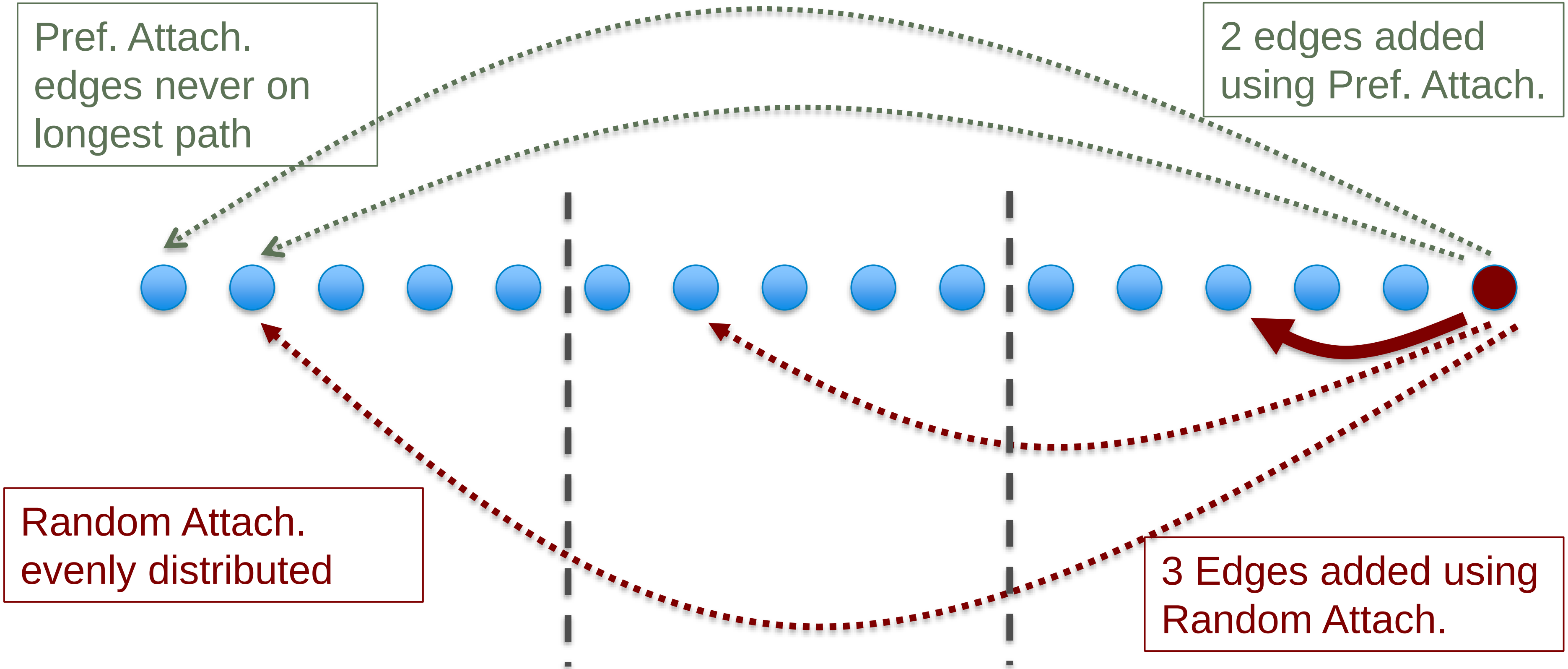
Average:

$$\lim_{t\to\infty} \ell(t) = m\,\bar{p}\,\ln(t) + O(1)$$

Distribution: **BINOMIAL** in long time limit

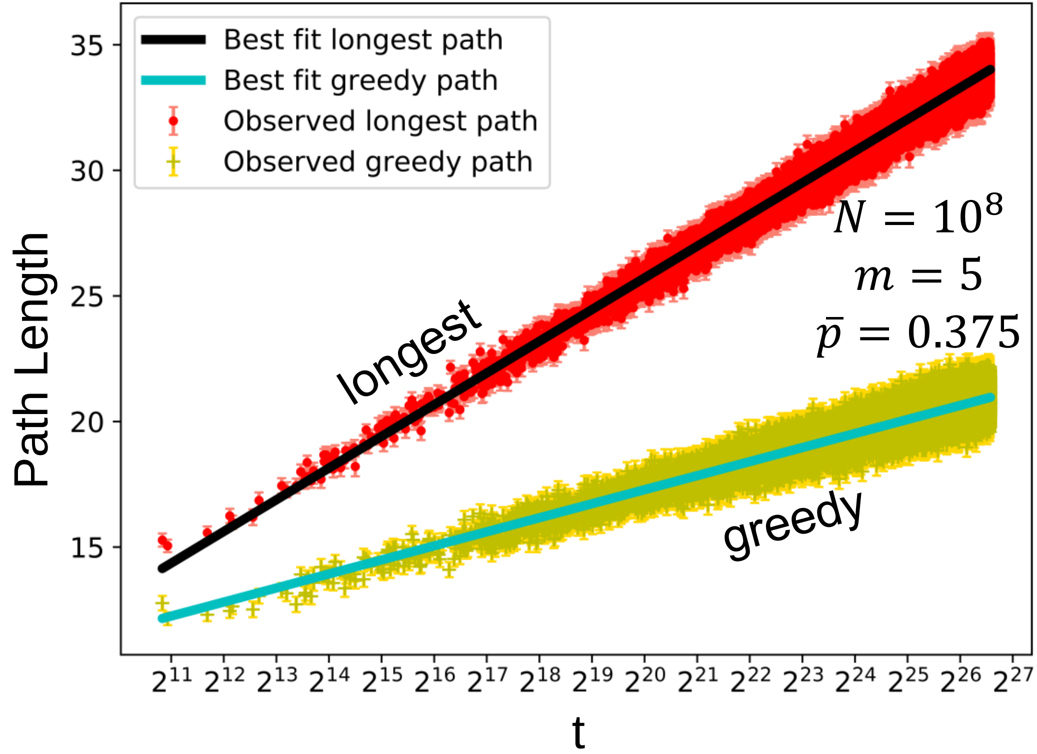
Random Attachment gives Longest Path

m=5, p=2/5



Numerical Example

$$\boxed{\ell(t) = a \ln(t) - b}$$



Average over 100 runs

Greedy path length scales very close to prediction:

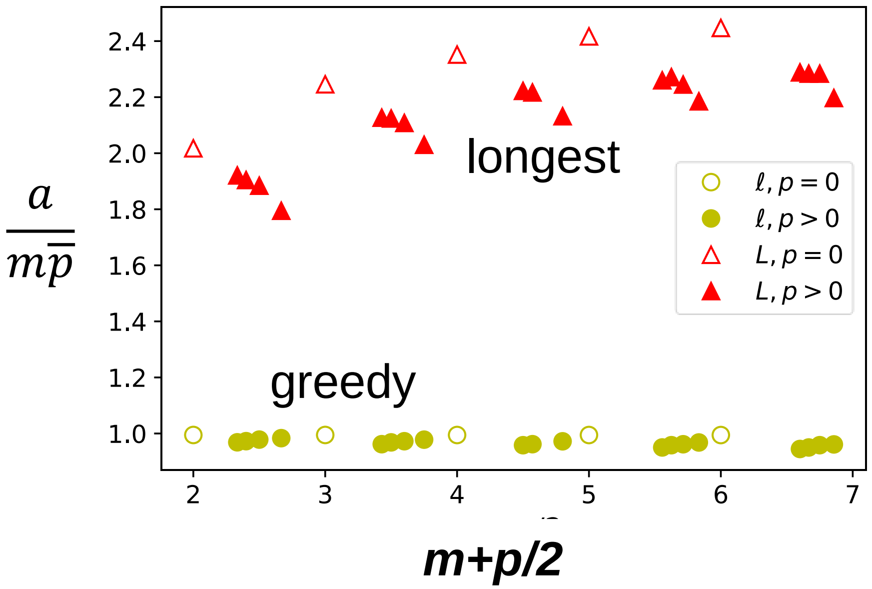
$$\frac{a_{\text{obs}}}{a_{\text{theo}}} = 0.96$$

For the longest path the slope is more than twice as large as for the greedy path:

$$\frac{a(L)}{a(\ell)} = 2.37$$

Numerical Results

Fit to $a \ln t + \mathrm{b}$

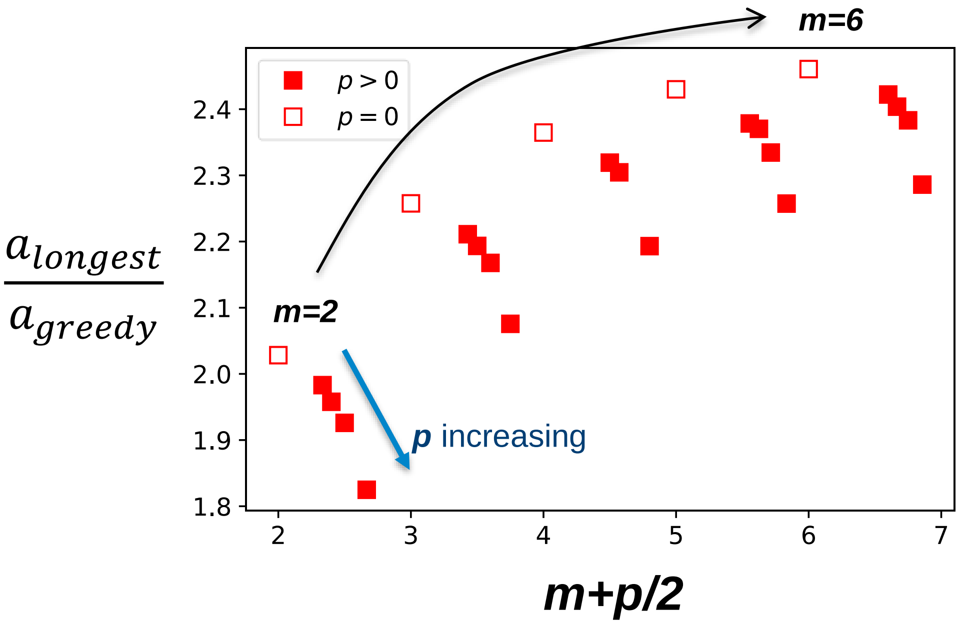


Numerical errors are smaller than symbols

Greedy path ℓ scales as $m\bar{p}$ ln(t)

Numerical Results

Fit to $a \ln t + b$



Numerical errors are smaller than symbols

Longest path $L \approx$ twice ℓ

 Tim EVANS, Imperial College

Summary

- An analytical solution for greedy path length in the Price model
- The length of longest and greedy paths in the Price model scale as ***log(N)*** in a network of ***N*** nodes
- The analytical and numerical results in excellent agreement.

Tim Evans,
Lucille Calmon,
Vaiva Vasiliauskaite

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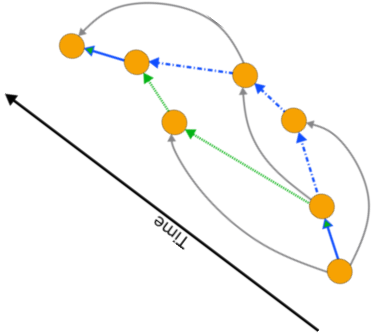
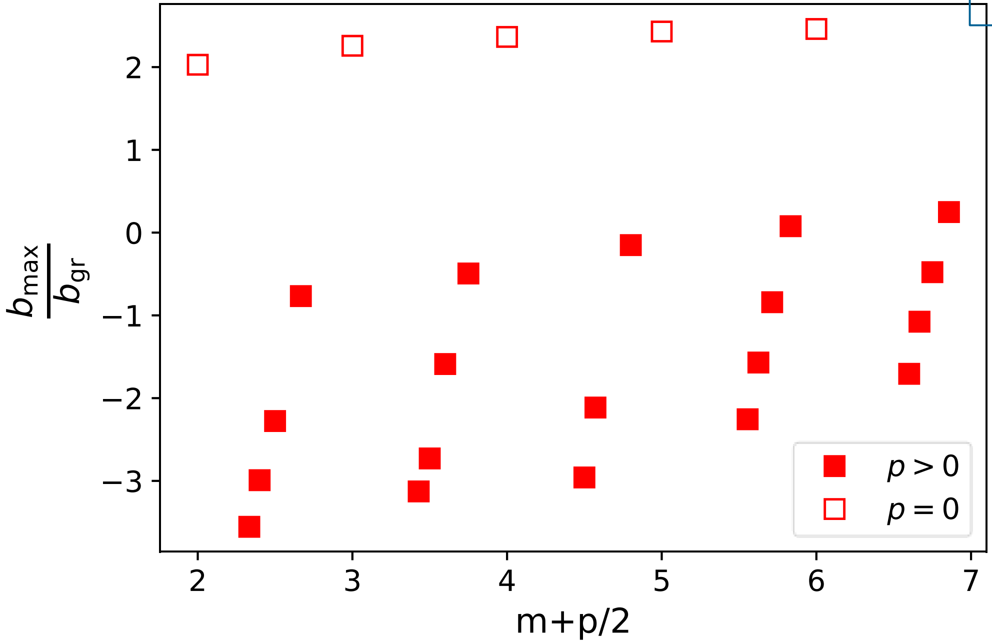
The arrow-of-time inherent in growing network models produces new distinctive features.



Extra Slides

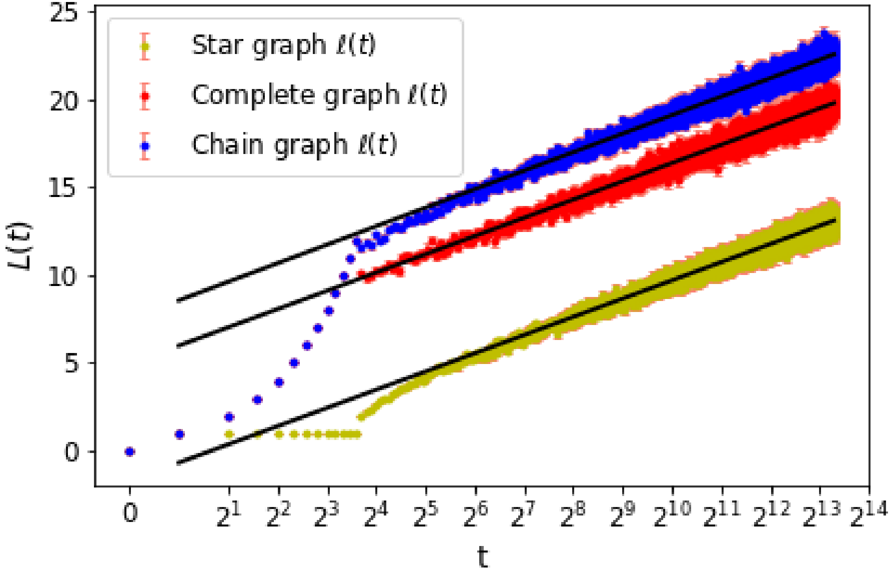
The second-to-leading order term

$$\ell(t) = a\ln(t) - b$$

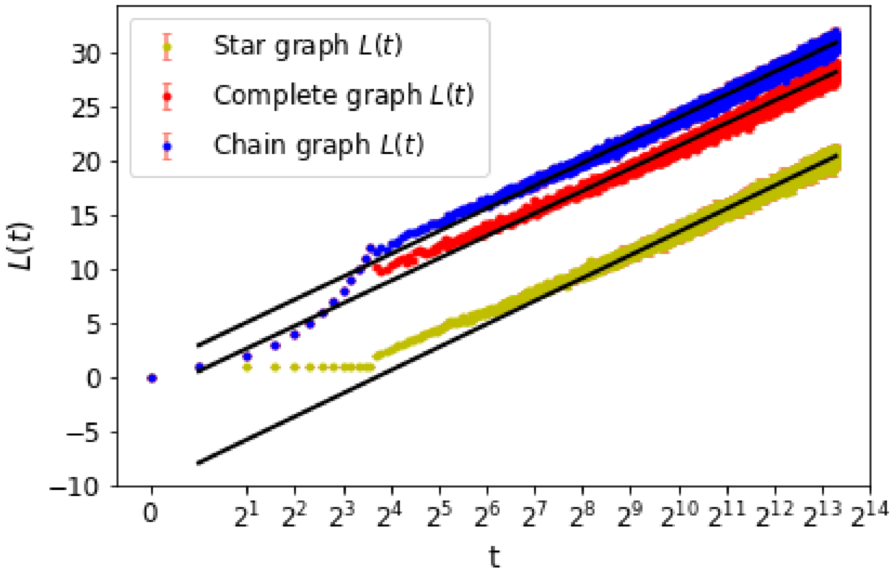


No clear scaling pattern

Initial graph effect



Greedy path ℓ scales as $m\bar{p}$



Longest path $L \approx$ twice ℓ

Just a shift in the constant

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