A Supplemental Material for "Detecting Variance Change-Points for Blocked Time Series and Dependent Panel Data"

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This supplemental paper provides proofs for Theorem 5 in the paper. Some additional simulation results are also presented in Section 2.

1. PROOF OF THEOREM 5

To prove Theorem 5, we define some notations and prove some preliminary lemmas. For $l_1 < l < l_2$, where l_1, l_2 satisfy the following condition

$$k_{i_0} \leq l_1 < k_{i_0+1} < \cdots < k_{i_0+s} < l_2 \leq k_{i_0+s+1} \tag{1.1}$$

where $0 \leq i_0 \leq q - s$ and for some $0 < s \leq q$. Define

$$\tilde{U}^l_{l_1,l_2} = \frac{(l-l_1)^\eta (l_2-l)^\eta}{(l_2-l_1)^{2\eta-1/2}} \left(\frac{1}{l-l_1} \sum_{t=l_1+1}^{l} s_t^2 - \frac{1}{l_2-l} \sum_{t=l+1}^{l_2} s_t^2 \right) V^{-1}$$

$$\tilde{\Theta}^l_{l_1,l_2} = \frac{(l-l_1)^\eta (l_2-l)^\eta}{(l_2-l_1)^{2\eta-1/2}} \left(\frac{1}{l-l_1} \sum_{t=l_1+1}^{l} \sigma_t^2 - \frac{1}{l_2-l} \sum_{t=l+1}^{l_2} \sigma_t^2 \right) V^{-1}$$

$$\tilde{Z}^l_{l_1,l_2} = \frac{(l-l_1)^\eta (l_2-l)^\eta}{(l_2-l_1)^{2\eta-1/2}} \left(\frac{1}{l-l_1} \sum_{t=l_1+1}^{l} z_t^2 - \frac{1}{l_2-l} \sum_{t=l+1}^{l_2} z_t^2 \right) V^{-1}$$

where $z_t^2 = s_t^2 - \sigma_t^2$ and V is the population version of $\hat{V}$. Without loss of generality, assume that $\sum_{t=l_1+1}^{l_2} \sigma_t^2 = 0$. Otherwise, one can define $\tilde{\sigma}_t^2 = \sigma_t^2 - (l_2 - l_1)^{-1} \sum_{t=l_1+1}^{l_2} \sigma_t^2$ and change the value of σ_t^2 to be $\tilde{\sigma}_t^2$, which does not affect the value of $\tilde{\Theta}^l_{l_1,l_2}$ and $\tilde{Z}^l_{l_1,l_2}$ and the location of change-points. Under the assumption of $\sum_{t=l_1+1}^{l_2} \tilde{\sigma}_t^2 = 0$, we can write $\tilde{\Theta}^l_{l_1,l_2}$ as

$$\tilde{\Theta}^l_{l_1,l_2} = -V^{-1} \sum_{t=l_1+1}^{l} \tilde{\sigma}_t^2 / \frac{\{(l-l_1)(l_2-l)\}^{1-\eta}}{(l_2-l_1)^{3/2-2\eta}}.$$

The following Lemma 1-4 consider the behavior of function $\tilde{\Theta}^l_{l_1,l_2}$. Lemma 1 shows that the maximum of $\tilde{\Theta}^l_{l_1,l_2}$ can be only attained at change-points, whose proof can be found in Venkatraman (1992).

Lemma 1. *Let l_1, l_2 satisfy condition (1.1) for some $s > 0$ and $|\tilde{\Theta}^{l_0}_{l_1,l_2}| = \max_{l_1 \leq l \leq l_2} |\tilde{\Theta}^l_{l_1,l_2}|$. Then $l_0 = k_{i_0+i}$ for some $0 \leq i \leq s$.*

The following two conditions are needed for the following Lemmas regarding the function $\tilde{\Theta}^{l}_{l_1,l_2}$. The first condition requires the distance between l_1 and l_2 are not too close to each other and second condition requires that the l_1 and l_2 are close to some change-points.

$$l_1 < k_{i_0+i} - c\xi_{T,n} < k_{i_0+i} + c\xi_{T,n} < l_2 \text{ for } 1 \le i \le s; \tag{1.2}$$

$$\min(k_{i_0+1} - l_1, l_1 - k_{i_0}) \vee \min(k_{i_0+s+1} - l_2, l_2 - k_{i_0+s}) \le \epsilon_{T,n}. \tag{1.3}$$

where $\xi_{T,n} = T^{1-\beta} n^{-1/8}$ and $\epsilon_{T,n} = T^{\alpha} \log(T) n^{-1/4}$.

The following Lemma obtains the lower bound and upper bound for $\max_{l_1 \le l \le l_2} |\tilde{\Theta}^{l}_{l_1,l_2}|$ for l_1, l_2 satisfy different conditions.

Lemma 2. *Let l_1, l_2 satisfy condition (1.2) and the conditions (A3)-(A4) hold. Then*

$$\max_{l_1 \le l \le l_2} |\tilde{\Theta}^{l}_{l_1,l_2}| \ge 4^{1/2-\eta} \delta c \xi_{T,n} T^{-1/2} n^{1/2} = 4^{1/2-\eta} c \delta T^{1/2-\beta} n^{3/8}.$$

If l_1 and l_2 satisfy condition (1.1) such that $\min(k_{i_0+1} - l_1, l_2 - k_{i_0+1}) \le \epsilon_{T,n}$ for $s = 1$ or $\max(k_{i_0+1} - l_1, l_2 - k_{i_0+2}) \le \epsilon_{T,n}$ for $s = 2$, then

$$\max_{l_1 \le l \le l_2} |\tilde{\Theta}^{l}_{l_1,l_2}| < |n^{1/2} \sqrt{\epsilon_{T,n}} - 3c_0 \sqrt{\log(T)}| = |T^{\alpha/2} \sqrt{\log(T)} n^{3/8} - 3c_0 \sqrt{\log(T)}|.$$

Proof. Denote $Q_l = \sum_{t=1}^{l} \tilde{\sigma}_t^2$ and assume that $\tilde{\sigma}^2_{k_{i_0+i}} = \theta$ and $\tilde{\sigma}^2_{k_{i_0+i}+1} = \theta^*$. Because $\sum_{t=l_1}^{l_2} \tilde{\sigma}_t^2 = 0$, we have $Q_{l_2} - Q_{l_1} = 0$. This implies that

$$Q_{k_{i_0+i}+c\xi_{T,n}} - Q_{k_{i_0+i}-c\xi_{T,n}} = -(Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}}) - (Q_{k_{i_0+i}-c\xi_{T,n}} - Q_{l_1}) = c\xi_{T,n}(\theta^* + \theta).$$

This further implies that

$$\max\{|Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}}|, |Q_{k_{i_0+i}-c\xi_{T,n}} - Q_{l_1}|\} > \frac{\theta + \theta^*}{2} c\xi_{T,n}.$$

If $|Q_{k_{i_0+i}-c\xi_{T,n}} - Q_{l_1}| > \frac{\theta+\theta^*}{2} c\xi_{T,n}$, then

$$\begin{aligned}
|Q_{k_{i_0+i}} - Q_{l_1}| &= |(Q_{k_{i_0+i}} - Q_{k_{i_0+i}-c\xi_{T,n}}) + (Q_{k_{i_0+i}-c\xi_{T,n}} - Q_{l_1})| \\
&\ge ||(Q_{k_{i_0+i}-c\xi_{T,n}} - Q_{l_1})| - |(Q_{k_{i_0+i}} - Q_{k_{i_0+i}-c\xi_{T,n}})|| > \frac{|\theta^* - \theta|}{2} c\xi_{T,n}
\end{aligned}$$

If $|Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}}| > \frac{\theta+\theta^*}{2} c\xi_{T,n}$, then

$$\begin{aligned}
|Q_{k_{i_0+i}} - Q_{l_1}| &= |(Q_{k_{i_0+i}} - Q_{k_{i_0+i}-c\xi_{T,n}}) - c\xi_{T,n}(\theta^* + \theta) - (Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}})| \\
&= |(Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}}) - c\xi_{T,n}\theta^*| \\
&\ge ||(Q_{l_2} - Q_{k_{i_0+i}+c\xi_{T,n}})| - c\xi_{T,n}\theta^*| > \frac{|\theta^* - \theta|}{2} c\xi_{T,n}.
\end{aligned}$$

Therefore, $|Q_l - Q_{l_1}|$ is at least $c\delta\xi_{T,n}/2$. Since $\{(l-l_1)(l_2-l)\}^{1-\eta}/\{(l_2-l_1)\}^{3/2-2\eta} \le (l_2 - l_1)^{1/2}/4^{1-\eta} \le \sqrt{T}/4^{1-\eta}$, we have $\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| \ge 4^{1/2-\eta}c\delta\xi_{T,n}T^{-1/2}n^{1/2}$ if $V^{-1} = n^{1/2}$. This completes the first part of the proof.

If l_1 and l_2 satisfy condition (1.1) such that $\min(k_{i_0+1}-l_1, l_2-k_{i_0+1}) \le \epsilon_{T,n}$ for $s=1$. Then by Lemma 1, if $k_{i_0+1}-l_1 < l_2-k_{i_0+1}$, then

$$\begin{aligned}
&\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| = |\tilde{\Theta}^{k_{i_0+1}}_{l_1,l_2}| \le V^{-1}(k_{i_0+1}-l_1)^{\eta}(l_2-l_1)^{3/2-2\eta}B/(l_2-k_{i_0+1})^{1-\eta}\\
&\le V^{-1}(k_{i_0+1}-l_1)^{\eta}2^{3/2-2\eta}(l_2-k_{i_0+1})^{3/2-2\eta}B/(l_2-k_{i_0+1})^{1-\eta} \le V^{-1}(k_{i_0+1}-l_1)^{1/2}2^{3/2-2\eta}B\\
&\le 2^{3/2-2\eta}Bn^{1/2}\sqrt{\epsilon_{T,n}}.
\end{aligned}$$

If $k_{i_0+1}-l_1 > l_2-k_{i_0+1}$, then using the relation $\sum_{t=l_1+1}^{l}\tilde{\sigma}_t^2 = -\sum_{t=l+1}^{l_2}\tilde{\sigma}_t^2$, we have

$$\begin{aligned}
&\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| = |\tilde{\Theta}^{k_{i_0+1}}_{l_1,l_2}| \le V^{-1}(l_2-k_{i_0+1})^{\eta}(l_2-l_1)^{3/2-2\eta}B/(k_{i_0+1}-l_1)^{1-\eta}\\
&\le V^{-1}2^{3/2-2\eta}(l_2-k_{i_0+1})^{\eta}(k_{i_0+1}-l_1)^{3/2-2\eta}B/(k_{i_0+1}-l_1)^{1-\eta} \le 2^{3/2-2\eta}Bn^{1/2}\sqrt{\epsilon_{T,n}}.
\end{aligned}$$

Similarly, one can obtain the same bound for $\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}|$ for l_1, l_2 satisfies $\max(k_{i_0+1}-l_1, l_2 - k_{i_0+2}) \le \epsilon_{T,n}$ when $s=2$. □

Lemma 3. *Let l_1, l_2 satisfy condition (1.2) and (1.3) for $\alpha < 1-2\beta$. Let ν be a change-point such that $\tilde{\Theta}^{\nu}_{l_1,l_2} > \max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| - 6c_0\sqrt{\log(T)}$. Then*

$$\tilde{\Theta}^{\nu}_{l_1,l_2} = A(l_2-l_1)^{3/2-2\eta}/\{(\nu-l_1)(l_2-\nu)\}^{1-\eta} \textit{ with } A > \delta c n^{1/2}\xi_{T,n}/5 = \delta c n^{3/8}T^{1-\beta}/5.$$

Proof. Similar to the second part of the proof in Lemma 2, we note that $\tilde{\Theta}^{\nu}_{l_1,l_2} \le 2^{3/2-2\eta}B\sqrt{\min(\nu-l_1, l_2-\nu)}$. By condition (A3) and (1.3), we know that $\min(\nu-l_1, l_2-\nu)$ is either less than $\epsilon_{T,n}$ or larger than $2c\xi_{T,n}-\epsilon_{T,n}$. However, if $\min(\nu-l_1,l_2-\nu) \le \epsilon_{T,n}$, then $\tilde{\Theta}^{\nu}_{l_1,l_2} \le \sqrt{2}B\sqrt{\epsilon_{T,n}}$, which implies that $\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| \le 2^{3/2-2\eta}B\sqrt{\epsilon_{T,n}} + 6\sqrt{\log(T)} < c\delta T^{1/2-\beta}n^{3/8}$ for large T, n by condition $\alpha < 1-2\beta$. Therefore, $\min(\nu-l_1, l_2-\nu)$ must be greater than $2c\xi_{T,n}-\epsilon_{T,n}$.

Without loss of generality, assume $2(\nu-l_1) \le l_2-l_1$ and assume $\nu = k_{i_0+i}$ for some $1 \le i \le s$. Let $\theta = \tilde{\sigma}^2_{\nu}$ and $\theta^* = \tilde{\sigma}^2_{\nu+1}$. Then by assumption (A4), $\min(|\theta|, |\theta^*|) > \delta/2$.

If $|\theta| > \delta/2$, we consider two cases. (a) if $i=1$, then $A = V^{-1}(\nu-l_1)\theta$, which is greater than $c\xi_{T,n}V^{-1} = c\delta T^{1-\beta}n^{3/8}/2$; (b) if $i>1$ and $A < \delta c n^{3/8}T^{1-\beta}/3$. Let $\nu' = k_{i_0+i-1}$ be another change-point before ν. Since $(x-l_1)(l_2-x)$ is increasing function for $x < (l_2-l_1)/2$ and $(\nu-l_1) \le (l_2-l_1)/2$, we have $(\nu' - l_1)(l_2 - \nu') \leq (\nu - l_1)(l_2 - \nu)$ and

$$|\tilde{\Theta}_{l_1,l_2}^{\nu'}| = \frac{|A - \theta(\nu' - \nu)V^{-1}|}{\{(\nu' - l_1)(l_2 - \nu')\}^{1-\eta}/(l_2 - l_1)^{3/2-2\eta}} \geq \frac{2\delta c n^{3/8} T^{1-\beta}/3}{\{(\nu - l_1)(l_2 - \nu)\}^{1-\eta}/(l_2 - l_1)^{3/2-2\eta}} > 2|\Theta_{l_1,l_2}^{\nu}|$$

which implies that $|\tilde{\Theta}_{l_1,l_2}^{\nu}| \leq |\tilde{\Theta}_{l_1,l_2}^{\nu'}|/2 \leq \max_{l_1 \leq l \leq l_2} |\tilde{\Theta}_{l_1,l_2}^{l}|/2 < \max_{l_1 \leq l \leq l_2} |\tilde{\Theta}_{l_1,l_2}^{l}| - 6c_0\sqrt{\log(T)}$ for large T, n. This is a contradiction to the condition $\tilde{\Theta}_{l_1,l_2}^{\nu} > \max_{l_1 \leq l \leq l_2} |\tilde{\Theta}_{l_1,l_2}^{l}| - 6c_0\sqrt{\log(T)}$. Therefore, $A > \delta c n^{3/8} T^{1-\beta}/3$. Using similar ideas, it can be shown that the results of this Lemma are still true under the case of $|\theta^*| > \delta/2$. This finishes the proof of this Lemma. □

Lemma 4. *Let l_1, l_2 satisfy conditions (1.2) and (1.3) for some $\alpha < 1 - 2\beta$ with $\beta < 1/8$. Let ν be a change-point such that $\tilde{\Theta}_{l_1,l_2}^{\nu} > \max_{l_1 \leq l \leq l_2} |\tilde{\Theta}_{l_1,l_2}^{l}| - 6c_0\sqrt{\log(T)}$. Then under the conditions (A3)-(A4), for some $0 < r < \epsilon_{T,n}$, Then $\tilde{\Theta}_{l_1,l_2}^{\nu} > \tilde{\Theta}_{l_1,l_2}^{\nu+r} + 6c_0\sqrt{\log(T)}$.*

Proof. From the proof of Lemma 3, we had $\min(\nu - l_1, l_2 - \nu) > 2cT^{1-\beta}n^{-1/8} - T^{\alpha}\log(T)n^{-1/4}$. Let $\nu' > \nu$ be another change-point next to ν. There are two possible cases (a): $\nu' = l_2$ and (b) $\nu' < l_2$.

(a) $\nu' = l_2$. Let $i = \nu - l_1$ and $h = l_2 - \nu$. Similar to Venkatraman (1992), we can show that

$$\tilde{\Theta}_{l_1,l_2}^{\nu} - \tilde{\Theta}_{l_1,l_2}^{\nu+r} \geq \frac{rA(i+h)^{3/2-2\eta}}{(ih)^{1-\eta}(i+r)} \geq \frac{crA}{\min(i,h)^{3/2}}.$$

for a constant c. Taking $r = T^{\alpha}\log(T)n^{-1/4}$, we have, if $\alpha \geq 1/2 + \beta$,

$$\tilde{\Theta}_{l_1,l_2}^{\nu} - \tilde{\Theta}_{l_1,l_2}^{\nu+r} \geq cT^{\alpha}\log(T)n^{-1/4}T^{1-\beta}n^{3/8}T^{-3/2} \geq c\log(T)n^{1/8}.$$

(b) $\nu' < l_2$. Let $i = \nu - l_1$, $h = cT^{1-\beta}n^{-1/8}$ and $j = l_2 - \nu - h$. Note that $j \geq h$ and $h < i$. Similar to Venkatraman (1992), we can show that

$$\tilde{\Theta}_{l_1,l_2}^{\nu} - \tilde{\Theta}_{l_1,l_2}^{\nu+r} \geq \Delta_{1r} + \Delta_{2r}.$$

where

$$\Delta_{1r} = \frac{Arh(i+j+h)^{3/2-2\eta}}{\{i(j+h)\}\{(i+r)(j+h-r)\}^{1-\eta}}; \text{ and } \Delta_{2r} = -\frac{br}{h}\frac{\{(i+h)j\}^{1-\eta}}{(i+r)^{1-\eta}(j+h-r)^{1-\eta}}.$$

Here $b = \tilde{\Theta}_{l_1,l_2}^{\nu+h} - \tilde{\Theta}_{l_1,l_2}^{\nu}$. Similar to Venkatraman (1992), we have, if $\alpha \geq 1/2 + 2\beta$,

$$\Delta_{1r} \geq \frac{cAr(h-r)}{[\max(i,j+h)]^{3/2}[\min(i,j+h)]} \geq cT^{1-\beta}n^{3/8}T^{\alpha}\log(T)n^{-1/4}cT^{1-\beta}n^{-1/8}T^{-5/2} \geq c\log(T)$$

for any constant c. Following the proof in Venkatraman (1992), we can also show that $\Delta_{1r} \geq -1/2$ and $\Delta_{3r} \geq 0$. This completes the proof of this Lemma. □

The next Lemma 5 establishes the upper bound of $\max_{0\le l_1<l<l_2\le T}|\tilde{Z}^l_{l_1,l_2}|$ and provides the convergence rate of estimated change-point. The proof is omitted because it is very similar to the proof in Venkatraman (1992).

Lemma 5. *Let $B_n = \{\max_{1\le l_1<l<l_2\le T}|\tilde{Z}^l_{l_1,l_2}| \le 3c_0\sqrt{\log(T)}\}$ for any $l_1 < l_2$ and some constant c_0 as $n, T \to \infty$. Then $\lim_{n\to\infty} P(B_n) = 1$. Let l_1, l_2 satisfy conditions (1.2) and (1.3). Let l_0 be the estimated change-points such that $\tilde{U}^{l_0}_{l_1,l_2} = \max_{0\le l_1<l<l_2\le T}|\tilde{U}^l_{l_1,l_2}|$. Then in the event of B_n, for some $1 \le i \le s$, $|k_{i_0+i} - l_0| \le \epsilon_{T,n}$.*

Proof of Theorem 5. At the beginning of the ISWDA algorithm, $l_1 = k_0 = 1$ and $l_2 = k_{q+1} = T$ satisfy condition (1.3) for any $\alpha \ge 0$ and $i_0 = 0, s = q$, and satisfy condition (1.2) for any $1 \le i \le s$ by condition (A3). Then by Lemma 1, $\max_{l_1\le l\le l_2}|\tilde{\Theta}^l_{l_1,l_2}| \ge \delta c\delta T^{3/8}n^{3/8}$, which implies that $\max_{l_1\le l\le l_2}|\tilde{U}^l_{l_1,l_2}| \ge \delta c\delta T^{3/8}n^{3/8} - 3c_0\sqrt{\log(T)} > U^*_{T,\alpha_n}$ in the event B_n defined in Lemma 5. Therefore, one change-point l_0 will be detected and by Lemma 5, l_0 will close to one of the change-points satisfy the condition (1.3). Thus each subsequence satisfy conditions (1.2) and (1.3) and hence the detection continues.

Suppose we have detected less than q change-points, then there exists a segment $\{l_1+1, \cdots, l_2\}$ such that (1.2) holds. Therefore, by Lemma 2, $\max_{l_1\le l\le l_2}|\tilde{U}^l_{l_1,l_2}| > U^*_{T,\alpha_n}$. Hence, a change-point will be detected in the segment. Thus $\hat{q} \ge q$. Once $\hat{q} = q$, all the subsequent segments have end points satisfy condition $\min(k_{i_0+1}-l_1, l_2-k_{i_0+1}) \le \epsilon_{T,n}$ for $s = 1$ or $\max(k_{i_0+1}-l_1, l_2-k_{i_0+2}) \le \epsilon_{T,n}$ for $s = 2$. Then, by Lemma 2, $\max_{l_1\le l\le l_2}|\tilde{U}^l_{l_1,l_2}| < U^*_{T,\alpha_n}$, which implies that no change-points will be detected further and all the detected change-points satisfy $|\hat{k}_i - k_i| \le \epsilon_{T,n}$. In addition, by Lemma 5, the event B_n happens with probability 1. This implies that $\lim_{n\to\infty} P(\hat{q} = q; |\hat{k}_j - k_j| \le T^{3/4}n^{-1/4},\ 1 \le j \le q) = 1$. The proof of this theorem is complete. □

2. SIMULATION RESULTS

2.1 Empirical Sizes

To illustrate that the asymptotic null distribution of the SWDA based test and compare it with Inclán and Tiao's and Chen and Gupta's methods in finite sample case, we simulated dependent samples from the following model

$$Y_{it} = Z_{it} \text{ where } Z_{it} = \rho Z_{i(t-1)} + \sqrt{\sigma_t^2 - \rho^2 \sigma_{t-1}^2}\varepsilon_{it}, \tag{2.1}$$

where $\rho = 0.3$, ε_{it} and Z_{i1} were independent standard normal, t_5 or χ_3^2 distributed, and ε_{it} were independent of $Z_{i(t-1)}$, for $i = 1, \cdots n$, and $t = 2, \cdots, T$. Here all the σ_t^2 were equal, that is, $\sigma_t^2 = \sigma_1^2 = \mathrm{Var}(Z_{i1})$, for $t = 1, 2, ..., T$. We ran simulations with $n = 100$, $T = 100$, 300, 500, $\varepsilon_T = \log\log(T)$ and 10,000 replications to calculate the empirical size. The sizes of the proposed test were calculated as the proportions of test statistics $\max_{\varepsilon_T \le k \le T-\varepsilon_T} |\sqrt{T}U_k^0|$ that are larger than $U_{T,\alpha}^*$ among 10,000 replicates under different significance levels α=0.01, 0.05, and 0.1. The values of empirical sizes are listed in Table 1. From the table, we can see that no matter for Gaussian or non-Gaussian distributions, the sizes from the SWDA based test are close to the specified nominal levels α's, while the sizes from Inclán and Tiao's and Chen and Gupta's methods are not well controlled for non-Gaussian distributions.

Table 1. Empirical Sizes of the SWDA test, Inclán-Tiao's test and Chen-Gupta's test in model (2.1).

		SWDA			Inclán-Tiao			Chen-Gupta		
		α			α			α		
	Length	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
	T=100	0.002	0.017	0.038	0.035	0.079	0.186	0.000	0.018	0.051
Normal	T=300	0.002	0.026	0.075	0.027	0.105	0.163	0.001	0.021	0.086
	T=500	0.002	0.023	0.079	0.020	0.089	0.159	0.001	0.026	0.091
	T=100	0.012	0.046	0.079	0.395	0.625	0.708	0.230	0.497	0.612
t_5	T=300	0.021	0.053	0.099	0.417	0.649	0.746	0.313	0.600	0.756
	T=500	0.020	0.063	0.106	0.420	0.658	0.739	0.311	0.646	0.810
	T=100	0.002	0.027	0.065	0.349	0.577	0.696	0.140	0.459	0.612
χ_3^2	T=300	0.004	0.034	0.071	0.362	0.568	0.694	0.216	0.559	0.691
	T=500	0.004	0.042	0.096	0.384	0.583	0.682	0.239	0.601	0.758

We also simulated dependent samples from the following regression model with predictor X_{it},

$$Y_{it} = 1 + 2X_{it} + Z_{it}, \tag{2.2}$$

where Z_{it} were generated by (2.1) and X_{it} were generated from standard normal distribution independent with Z_{it} for $i = 1, \cdots n$, and $t = 2, \cdots, T$. The values of empirical sizes are listed in Table 2. We can observe similar phenomena as those in Table 1.

Table 2. Empirical Sizes of the SWDA test, Inclán-Tiao's test and Chen-Gupta's test in model (2.2).

		SWDA			Inclán-Tiao			Chen-Gupta		
		α			α			α		
	Length	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
	T=100	0.002	0.017	0.042	0.031	0.094	0.197	0.001	0.016	0.048
Normal	T=300	0.002	0.028	0.074	0.034	0.099	0.178	0.001	0.029	0.086
	T=500	0.005	0.030	0.068	0.020	0.107	0.165	0.001	0.028	0.092
	T=100	0.010	0.039	0.066	0.429	0.633	0.709	0.192	0.478	0.625
t_5	T=300	0.014	0.063	0.095	0.437	0.690	0.726	0.294	0.613	0.741
	T=500	0.024	0.057	0.104	0.439	0.641	0.759	0.316	0.633	0.791
	T=100	0.002	0.022	0.060	0.373	0.569	0.718	0.144	0.462	0.628
χ_3^2	T=300	0.006	0.032	0.092	0.356	0.577	0.699	0.232	0.517	0.732
	T=500	0.003	0.040	0.085	0.368	0.554	0.706	0.247	0.582	0.737

2.2 Empirical Power

The simulation studies are designed to evaluate the performance of the proposed ISWDA (Iterated Standardized Weighted Differences of Averages) method and compare it with Inclán and Tiao's CUSUM method and Chen and Gupta's SIC method. We conducted simulations for two variance change patterns under normal, t_5 and χ_3^2 distributions.

First, we generated data Z_{it} from an AR(1) model specified in (2.1) without predictors. We generated sequences of data with length $T = 120$ and sample size $n = 100$. The average number of wrong rejections and the empirical power for each change-point are reported based on nominal level $\alpha = .05$ and 10,000 simulation iterations.

In Table 3, $\rho = 0.30$, ε_{it} and Z_{i1} were independently standard normal, t_5 or χ_3^2 distributed, and ε_{it} were independent of $Z_{i(t-1)}$, for $i = 1, \cdots n$, and $t = 2, \cdots, T$. This table studied the cases of multiple abrupt variance changes. Specifically, each row corresponds to the true variances with pattern (σ^2,

..., σ^2, $2\sigma^2$, ..., $2\sigma^2$, $4\sigma^2$, ..., $4\sigma^2$, $2\sigma^2$, ..., $2\sigma^2$), where the variance σ^2 is the variance of Z_{i1}, and the variance σ^2 jumps to $2\sigma^2$ and $4\sigma^2$ at time positions 30 and 60, then drops to $2\sigma^2$ at time position 100. From the table, we can see that ISWDA procedure has good detection power for the normal cases. For the non-normal cases, Inclán-Tiao's and Chen-Gupta's methods make many wrong rejections, and even in such cases, our method performs no worse in power, suggesting that the method is valid to apply to both Gaussian and non-Gaussian sequences.

In Table 4, we explored the performances of the three methods for the cases of gradual variance changes using the same model as in Table 3. Each row in this table shows the simulation result for the case that the variance σ^2 changes to $2\sigma^2$, $3\sigma^2$, $4\sigma^2$, $5\sigma^2$ and $6\sigma^2$ at time positions 100, 101, 102, 103 and 104, then stays at $6\sigma^2$. From the table, we can conclude that, first, when the distribution shape departs away from normal distribution, Inclán-Tiao's and Chen-Gupta's methods make many wrong rejections, again indicating that these procedures are not suitable for non-Gaussian sequences. Second, our procedure not only makes less number of type I errors, but also makes less number of type II errors than the other two methods.

Table 3. Number of wrong detections and empirical powers for abrupt change-points in model (2.1).

Distribution	Test	No. of wrong detections	Empirical power change1	change2	change3
	ISWDA	0.20	0.97	0.91	0.97
$Normal$	Inclán-Tiao	0.54	0.96	0.93	0.87
	Chen-Gupta	0.22	0.93	0.96	0.98
	ISWDA	0.67	0.82	0.72	0.81
t_5	Inclán-Tiao	**4.11**	0.78	0.76	0.74
	Chen-Gupta	**2.39**	0.81	0.85	0.88
	ISWDA	0.67	0.82	0.71	0.81
χ_3^2	Inclán-Tiao	**3.89**	0.77	0.75	0.73
	Chen-Gupta	**2.12**	0.81	0.83	0.86

Second, we generated data Y_{it} from the regression model specified in (2.2) with predictor X_{it}, which were generated from standard normal distribution independent with Z_{it} for $i = 1, \cdots n$, and

Table 4. Number of wrong detections and empirical powers for gradual change-points in model (2.1)

Distribution	Test	No. of wrong detections	Empirical power				
			change1	change2	change3	change4	change5
	ISWDA	0.03	0.81	1.00	0.84	0.17	0.01
$Normal$	Inclán-Tiao	0.92	0.37	0.15	0.16	0.23	0.28
	Chen-Gupta	0.09	0.90	0.65	0.26	0.47	0.20
	ISWDA	0.11	0.57	0.95	0.80	0.30	0.04
t_5	Inclán-Tiao	**3.34**	0.50	0.24	0.14	0.15	0.20
	Chen-Gupta	**1.65**	0.72	0.65	0.27	0.37	0.23
	ISWDA	0.06	0.57	0.96	0.78	0.30	0.04
χ_3^2	Inclán-Tiao	**3.04**	0.49	0.23	0.14	0.15	0.19
	Chen-Gupta	**1.33**	0.72	0.61	0.28	0.35	0.22

$t = 2, \cdots, T$. Z_{it} were generated from the AR(1) model specified in (2.1), in which $\rho = 0.30$, ε_{it} and Z_{i1} were independently standard normal, t_5 or χ_3^2 distributed, and ε_{it} were independent of $Z_{i(t-1)}$, for $i = 1, \cdots 100$, and $t = 2, \cdots, 120$.

Table 5 studied the cases of multiple abrupt variance changes with the same change pattern of Table 3, while Table 6 explored the performances of the three methods for the cases of gradual variance changes using the same change pattern of Table 4. Results in these two tables show similar phenomena as those in Table 3 and Table 4.

Table 5. Number of wrong detections and empirical powers for abrupt change-points in model (2.2).

Distribution	Test	No. of wrong detections	Empirical power		
			change1	change2	change3
$Normal$	ISWDA	0.20	0.97	0.91	0.96
	Inclán-Tiao	0.57	0.96	0.93	0.86
	Chen-Gupta	0.23	0.93	0.96	0.98
t_5	ISWDA	0.66	0.83	0.73	0.80
	Inclán-Tiao	**4.18**	0.78	0.75	0.72
	Chen-Gupta	**2.39**	0.82	0.84	0.88
χ^2_3	ISWDA	0.68	0.82	0.71	0.81
	Inclán-Tiao	**3.95**	0.77	0.75	0.73
	Chen-Gupta	**2.14**	0.81	0.83	0.86

Table 6. Number of wrong detections and empirical powers for gradual change-points in model (2.2).

Distribution	Test	No. of wrong detections	Empirical power				
			change1	change2	change3	change4	change5
$Normal$	ISWDA	0.87	0.94	1.00	0.98	0.57	0.31
	Inclán-Tiao	0.72	0.60	0.31	0.02	0.04	0.15
	Chen-Gupta	0.60	0.96	0.77	0.38	0.50	0.26
t_5	ISWDA	0.11	0.57	0.95	0.80	0.30	0.04
	Inclán-Tiao	**3.36**	0.50	0.24	0.14	0.15	0.20
	Chen-Gupta	**1.66**	0.72	0.64	0.27	0.37	0.22
χ^2_3	ISWDA	0.17	0.56	0.96	0.77	0.29	0.04
	Inclán-Tiao	**3.39**	0.49	0.23	0.14	0.16	0.19
	Chen-Gupta	**2.33**	0.72	0.61	0.28	0.35	0.22

REFERENCES

Venkatraman, E. S. (1992), "Consistency results in multiple change-point situations," Technical report , Department of Statistics. Stanford University.