# Derivation of original phase shift for every core pair

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From [1] it is possible to derive the theoretical phase shift between coupled core in a multi-core fibre due to fibre bend.

The following derivation considers the difference between central core (n=1) and any adjacent core (n=n).

## 1 Find PMP location

For [1] the Phase matching Point (PMP) will take place when the equivalent refractive index between core pair is 0. We use this to find the location in space where the phase matching appears.

Find period of contribution from core 1 to n

$$\begin{aligned}
\delta n_{eq} &= n_{eff1} - n_{effn}\left(1 + \frac{D}{R}cos\theta_n\right) && (1)\\
&= n_{eff1} - n_{eff1}(1+a)\left(1 + \frac{D}{R}cos\theta_n\right) && (2)\\
&= n_{eff1} - n_{eff1}\left(1 + a + (1+a)\cdot\frac{D}{R}cos\theta_n\right) && (3)\\
&= n_{eff1}\left(-a - (1+a)\cdot\frac{D}{R}cos\theta_n\right) && (4)\\
&= 0 \implies \text{XT condition} && (5)\\
0 &= a + (1+a)\cdot\frac{D}{R}cos\theta_n && (6)\\
-a &= (1+a)\cdot\frac{D}{R}cos\theta_n && (7)\\
cos\theta_n &= \frac{-a}{1+a}\frac{R}{D} && (8)\\
\theta_n &= 2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) \quad k \in \mathbb{Z} && (9)\\
\text{Where} \implies \theta_n(z) &= \gamma z + (n-2)\frac{\pi}{3} + \theta_c \quad (n = 2, ..., 7) && (10)\\
2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) &= \gamma z + (n-2)\frac{\pi}{3} + \theta_c && (11)\\
z &= \frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} \quad k \in \mathbb{Z}, k \geq 0 && (12)
\end{aligned}$$

As it can be seen from the derivation the phase matching point appear periodically with the fibre length (two phase matching point for integer $k$).

## 1.1 PMP between any core pair

As can be seen from the derivation below a simple analytical formulation between any core pair PMP is not easy to find. To use it in simulations it is recommended to put the last equation in a root finding system varying the output periodic parameter till the length of the fibre.
Find period of contribution from core m to n

$$\delta n_{eq} = n_{effm}\left(1+\frac{D}{R}cos\theta_m\right) - n_{effn}\left(1+\frac{D}{R}cos\theta_n\right) \tag{13}$$

$$= n_{effm}\left(1+\frac{D}{R}cos\theta_m\right) - n_{effm}(1+a)\left(1+\frac{D}{R}cos\theta_n\right) \tag{14}$$

$$= n_{effm}\left(1+\frac{D}{R}cos\theta_m - (1+a)\left(1+\frac{D}{R}cos\theta_n\right)\right) \tag{15}$$

$$= n_{effm}\left(\frac{D}{R}cos\theta_m - a + (1+a)\left(\frac{D}{R}cos\theta_n\right)\right) \tag{16}$$

$$= 0 \implies \text{ XT condition} \tag{17}$$

$$0 = \frac{D}{R}cos\theta_m - a - (1+a)\frac{D}{R}cos\theta_n \tag{18}$$

$$0 = cos\theta_m - \frac{R}{D}a - (1+a)cos\theta_n \tag{19}$$

$$0 = cos\left(\gamma z + (m-2)\frac{\pi}{3} + \theta_c\right) - \frac{R}{D}a - (1+a)cos\left(\gamma z + (n-2)\frac{\pi}{3} + \theta_c\right) \tag{20}$$

$$0 = cos\left(\gamma z + (m-2)\frac{\pi}{3} + \theta_c\right) - \frac{R}{D}a - (1+a)cos\left(\gamma z + (m-2)\frac{\pi}{3} + \theta_c + l\frac{\pi}{3}\right) \quad l = n - m \tag{21}$$

$$0 = cos\,(u) - \frac{R}{D}a - (1+a)cos\left(u + l\frac{\pi}{3}\right) \quad u = \gamma z + (m-2)\frac{\pi}{3} + \theta_c \tag{22}$$

## 2 Phase at core $n$ in respect of distance

From [1] it can be easily derived the phase of the signal at every in any core at distance $z$.

$$\phi_n(z) = \int_0^z \beta_n\left\{1+\frac{D}{R}cos\theta_n(z')\right\}dz' \quad (n = 2, ..., 7) \tag{23}$$

$$= \beta_n z + \beta_n\frac{D}{R}\int_0^z cos\theta_n(z')dz' \tag{24}$$

$$= \beta_n\left(z + \frac{D}{R}\frac{sin(\gamma z + (n-2)\frac{\pi}{3} + \theta_c) - sin((n-2)\frac{\pi}{3} + \theta_c)}{\gamma}\right) \tag{25}$$

## 3 Phase difference at every PMP

Substitute $z$ for PMP with the phase equation.

$$z = \frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} \tag{26}$$

$$\delta\phi(k) = \phi_1(z(k)) - \phi_n(z(k)) \tag{27}$$

$$= \beta_1 z(k) - \beta_n\left(z(k) - \frac{D}{R}\frac{sin(\gamma z(k) + (n-2)\frac{\pi}{3} + \theta_c) - sin((n-2)\frac{\pi}{3} + \theta_c)}{\gamma}\right) \tag{28}$$

$$= \beta_1 z(k) - \beta_n z(k) - \beta_n \frac{D}{R}\frac{sin(\gamma z(k) + (n-2)\frac{\pi}{3} + \theta_c)}{\gamma} + \beta_n \frac{D}{R}\frac{sin((n-2)\frac{\pi}{3} + \theta_c)}{\gamma} \tag{29}$$

$$= (\beta_1 - \beta_n) z(k) - \beta_n \frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} + C \tag{30}$$

$$= (\beta_1 - \beta_n)\frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} - \beta_n \frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} + C \tag{31}$$

$$= \frac{2\pi(n_{eff1} - n_{effn})}{\lambda}\frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} + \tag{32}$$

$$- \frac{2\pi(n_{eff1} + a)}{\lambda}\frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} + C \tag{33}$$

$$= \frac{2\pi(a)}{\lambda}\frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} + \tag{34}$$

$$- \frac{2\pi(n_{eff1} + a)}{\lambda}\frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} + C \tag{35}$$

$$= \frac{a\omega}{c}\frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} + \tag{36}$$

$$- \frac{\omega(n_{eff1} + a)}{c}\frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} + C \tag{37}$$

$$= \frac{a\omega}{c}\frac{2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right) - (n-2)\frac{\pi}{3} - \theta_c}{\gamma} + \tag{38}$$

$$- \frac{\omega(n_{eff1} + a)}{c}\left[\frac{D}{R}\frac{\sin\left(2k\pi \pm arcos\left(\frac{-a}{1+a}\frac{R}{D}\right)\right)}{\gamma} - \frac{D}{R}\frac{sin((n-2)\frac{\pi}{3} + \theta_c)}{\gamma}\right] \tag{39}$$

The maximum number of phase matching points periods ($k$) for a fiber can be found as follows:

$$k = int \left( \frac{L\gamma \left| arcos(\frac{-a}{1+a}\frac{R}{D}) \right| + \frac{(n-2)\pi}{3} + \theta_c}{2\pi} \right) \tag{40}$$

## References


[1] T. Hayashi et al. "Crosstalk variation of multi-core fibre due to fibre bend". In: *36th European Conference and Exhibition on Optical Communication.* Sept. 2010, pp. 1–3. DOI: `10.1109/ECOC.2010.5621143`.