Step Distributions

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1 Analytical step distribution of [1]

Hayashi's formulation of IC-XT as in [1] is:

$$A_n(N_{\mathrm{PM}}) = A_n(0) - j\sum_{l=1}^{N_{\mathrm{PM}}} \chi_{nm}(l)\exp\left[-j\phi_{\mathrm{rnd}(l)}\right] A_m(l-1) \tag{1}$$

$$\approx -j\sum_{l=1}^{N_{\mathrm{PM}}} \chi_{nm}\exp(-j\phi_{\mathrm{rnd},l}) \text{ Where: } \phi_{\mathrm{rnd},l} \backsim U(0,2\pi) \tag{2}$$

where A_n is the complex amplitude of the IC-XT of the target core n, A_m is the complex amplitude of the signal in the active core, χ_{nm} is the coupling coefficient between cores n and m, $\phi_{\mathrm{rnd},l}$ is the random phase shift (0 to 2π) at the l^{th} phase matching point (PMP) and N_{PM} is the total number of phase matching points between cores n and m in the MCF.

As proved in Convergence Proof, the IC-XT Intensity distribution will follow a 4 degrees of freedom χ^2 distribution, in the following form:

$$f_{\chi^2,4df}(x|\sigma) = \frac{x}{4\sigma^4}e^{\frac{-x}{2\sigma^2}} \tag{3}$$

where σ is the standard deviation of the gaussian random variable. This distribution has a mean of $4\sigma^2$ and a variance of $8\sigma^4$.

In this model subsequent samples are independent so they can be considered as two separate random variables.

X, Y are independent identical distributed (IID) random variables following the distribution described in Eq.3. Because they are considered independent they can be considered consecutive samples from the parent distribution Eq.3. Their corresponding step (w) distribution will be $p_W(w)$:

$$p_W(w) = p_W(x-y), \quad \text{where } W = X - Y \tag{4}$$

To find the derivative of a derived distribution it is first needed to find the cumulative density function (CDF) of the derived random variable and find its derivative.The CDF can be described as:

$$\begin{aligned}
P_W(w) &= \Pr[W \leq w] = \Pr[x - y \leq w] && (5)\\
&= \Pr[x \leq w + y] && (6)\\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{w+y} f_{X,Y}(x,y)dxdy. && (7)
\end{aligned}$$

Then finding the derivative in respect of w:

$$\begin{aligned}
p_W(w) &= \frac{d}{dw} P_W(w) = \frac{d}{dw} \int_{-\infty}^{\infty} \int_{-\infty}^{w+y} f_{X,Y}(x,y)dxdy && (8)\\
&= \int_{-\infty}^{\infty} \frac{d}{dw} \int_{-\infty}^{w+y} f_{X,Y}(x,y)dxdy && (9)\\
&= \int_{-\infty}^{\infty} \frac{d}{dw} \int_{-\infty}^{w} f_{X,Y}(t+y,y)dtdy, \quad \text{where } x = t + y \text{ and } dx = dt && (10)\\
&\implies \frac{d}{dz} \int_{-\infty}^{z} g(x,y)dx = g(z,y) \quad \text{Using the identity} && (11)\\
&= \int_{-\infty}^{\infty} f_{X,Y}(w+y,y)dy && (12)\\
&= \int_{-\infty}^{\infty} f_{X,Y}(x,x-w)dx, \quad \text{where } y = x - w \text{ and } dy = dx && (13)\\
&= \int_{-\infty}^{\infty} f_{X,Y}(x,x-w)dx && (14)
\end{aligned}$$

For model [1], X, Y are IID, so:

$$f_{X,Y} = f_X(x)f_Y(y) \quad \text{ for } X, Y \in [0, \infty) \tag{15}$$

$$f_X(x) = f_Y(y) \quad \text{ where } f_X(x) = \frac{x}{4\sigma^4} e^{-\frac{x}{2\sigma^2}} \tag{16}$$

$$p_w(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - w) dx \tag{17}$$

$$p_w(w|w < 0) = \int_0^{\infty} f_X(x) f_Y(x - w) dx \implies x - w > 0 \tag{18}$$

$$= \int_0^{\infty} \frac{x}{4\sigma^4} e^{-\frac{x}{2\sigma^2}} \frac{x - w}{4\sigma^4} e^{-\frac{x-w}{2\sigma^2}} dx \tag{19}$$

$$= \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^8} \int_0^{\infty} x^2 - wxe^{\frac{-x}{\sigma^2}} dx \tag{20}$$

$$= \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^8} \int_0^{\infty} x^2 e^{\frac{-x}{\sigma^2}} dx - w \int_0^{\infty} xe^{-\frac{x}{\sigma^2}} dx \tag{21}$$

$$= \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^8} g(x) - wh(x) \tag{22}$$

$$\implies g(x) = \int_0^{\infty} x^2 e^{-\frac{x}{\sigma^2}} dx, \quad h(x) = \int_0^{\infty} xe^{-\frac{x}{\sigma^2}} dx \tag{23}$$

$$h(x) = \int_0^{\infty} xe^{-\frac{x}{\sigma^2}} dx \tag{24}$$

$$= -\sigma^2 e^{-\frac{x}{\sigma^2}} x_0^{\infty} + \sigma^2 \int_0^{\infty} e^{-\frac{x}{\sigma^2}} dx \tag{25}$$

$$= -\sigma^2 e^{-\frac{x}{\sigma^2}} x_0^{\infty} - \sigma^4 e^{-\frac{x}{\sigma^2}}{}_0^{\infty} = \sigma^4 \tag{26}$$

$$g(x) = \int_0^{\infty} x^2 e^{-\frac{x}{\sigma^2}} dx \tag{27}$$

$$= -\sigma^2 e^{-\frac{x}{\sigma^2}} x^2{}_0^{\infty} + 2\sigma^2 \int_0^{\infty} e^{-\frac{x}{\sigma^2}} xdx \tag{28}$$

$$= -\sigma^2 e^{-\frac{x}{\sigma^2}} x^2{}_0^{\infty} + 2\sigma^2 h(x) \tag{29}$$

$$= -\sigma^2 e^{-\frac{x}{\sigma^2}} x^2{}_0^{\infty} + 2\sigma^2 \sigma^4 \tag{30}$$

$$= 2\sigma^6 \tag{31}$$

$$p_w(w|w < 0) = \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^8} g(x) - wh(x) \tag{32}$$

$$= \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^8} 2\sigma^6 - w\sigma^4 \tag{33}$$

$$= \frac{e^{\frac{w}{2\sigma^2}}}{16\sigma^4} 2\sigma^2 - w \tag{34}$$

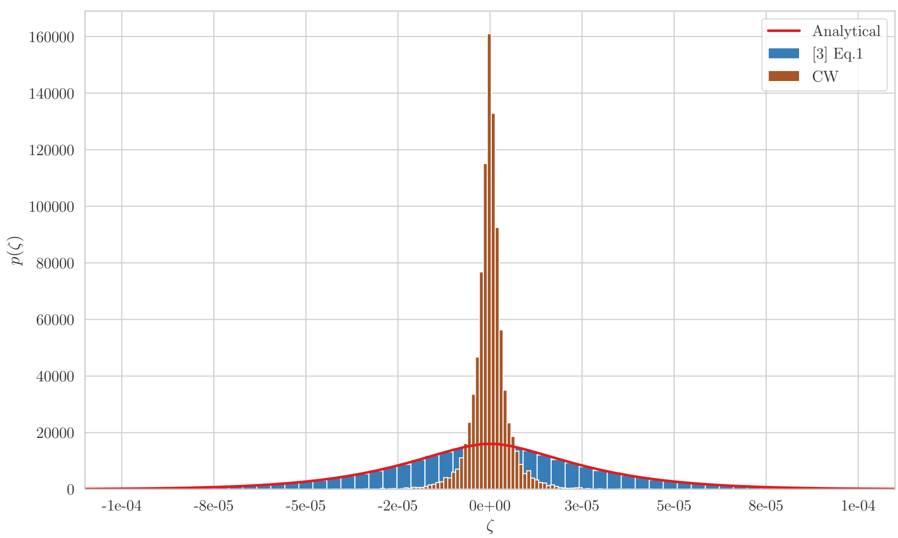

Figure 1: Experimental and theoretical step distribution for CW

X, Y are IID so, $X - Y \stackrel{d}{=} Y - X$ ($\stackrel{d}{=}$ means identically distributed). Therefore, $p_W(w = a) = p_W(w = -a)$. This shows that $p_W(w)$ is symmetrical at $w = 0$:

$$\therefore p_W(w) = \frac{e^{-\frac{|w|}{2\sigma^2}}}{16\sigma^4} 2\sigma^2 + |w| \tag{35}$$

Even though an analytical formulation for the step distribution has been found, the analysis has been done in dB because of the fact that the linear step distribution largely varies with σ, making it not easy to compare the experimental results for modulation that had not converged yet to the χ^2 distribution to the the distribution described in Eq.35. Even when the IC-XT converged to the χ^2 and the theoretical σ parameter has been found, the theoretical step distribution and the one derived from the experimentally measured one. This can be clearly seen in Fig.1, This is due to the fact that the model from which the theoretical distribution is derived considers subsequent samples independent.

2 IC-XT dB intensity distribution

The PDF of a function (y) of a single continuous random variable (x) in the form:

$$y = g(x) \tag{36}$$

where g(x) is a monotonically increasing function with unique inverse ($x = g^{-1}(y)$) can be found as:

$$f_Y(y) = f_X\left[g^{-1}(y)\right]\left|\frac{dg^{-1}(y)}{dy}\right| \tag{37}$$

So considering as $f_X(x)$ Eq.3 and $g(x)$ the linear to dB transorfamtion, which is a monotonically increasing with unique inverse.

$$f_X(x) = \frac{x}{4\sigma^4}e^{\frac{-x}{2\sigma^2}} \tag{38}$$

$$g(x) = 10log_{10}(x) \tag{39}$$

$$y = g(x) \tag{40}$$

$$g^{-1}(y) = 10^{(}\frac{x}{10}^{)} \tag{41}$$

$$\frac{dg^{-1}(y)}{dy} = 10^{\frac{x}{10}-1}log(10) \tag{42}$$

Substituting into Eq.37:

$$f_Y(y) = \frac{10^{\frac{y}{10}}}{4\sigma^4}exp\left[-\frac{10^{(\frac{y}{10})}}{2\sigma^2}\right]10^{\frac{y}{10}-1}log(10) \tag{43}$$

$$= \frac{log(10)}{4\sigma^4}10^{\left(\frac{y}{10}+\frac{y}{10}-1\right)}exp\left[-\frac{10^{(\frac{y}{10})}}{2\sigma^2}\right] \tag{44}$$

$$= \frac{log(10)}{4\sigma^4}10^{\left(\frac{y}{5}-1\right)}exp\left[-\frac{10^{(\frac{y}{10})}}{2\sigma^2}\right] \tag{45}$$

The IC-XT distribution in dB will follow Eq.45.

Unfortunately, an analytical expression for the moments of the distribution cannot be found, due to the divergence of the integral. Still Monte Carlo analysis will be performed in section **??**.

2.1 Step distribution

Using part of the analysis performed above, we can find the step distribution for the IC-XT intensity in dB. Eq.17 shows that the derived distribution of a difference of two independent random variables is:

$$p_w(w) = \int_{-\infty}^{\infty} f_X(x)f_Y(x-w)dx \tag{46}$$

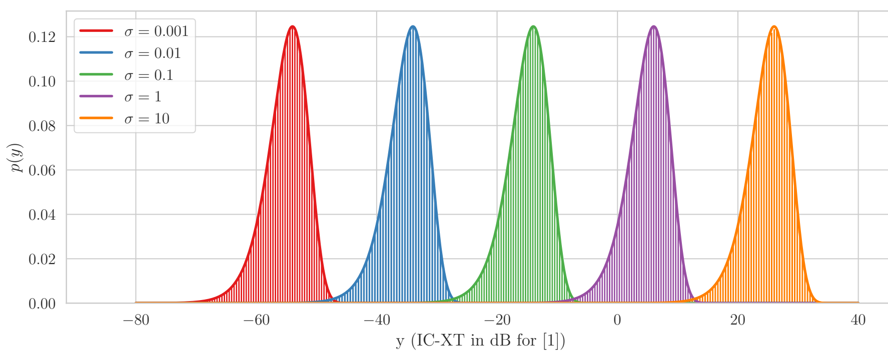

Figure 2: PDF of samples generated with different σ and theoretical fitting

Substituting $f_X(x)$ and $f_Y(y)$ with Eq.45, we get:

$$p_w(w) = \int_{-\infty}^{\infty} \frac{log(10)}{4\sigma^4} 10^{\left(\frac{x}{5}-1\right)} exp\left[-\frac{10^{\left(\frac{x}{10}\right)}}{2\sigma^2}\right] \cdot \frac{log(10)}{4\sigma^4} 10^{\left(\frac{x-w}{5}-1\right)} exp\left[-\frac{10^{\left(\frac{x-w}{10}\right)}}{2\sigma^2}\right] \tag{47}$$

$$= \frac{log(10)^2}{16\sigma^8} \int_{-\infty}^{\infty} 10^{\left(\frac{2x-w}{5}-2\right)} exp\left[-\frac{10^{\left(\frac{x}{10}\right)}\left(1+10^{\left(-\frac{w}{10}\right)}\right)}{2\sigma^2}\right] dx \tag{48}$$

$$= \frac{log(10)^2}{16\sigma^8} 10^{\left(\frac{-w}{5}-2\right)} \int_{-\infty}^{\infty} 10^{\left(\frac{2x}{5}\right)} exp\left[-\frac{10^{\left(\frac{x}{10}\right)}\left(1+10^{\left(-\frac{w}{10}\right)}\right)}{2\sigma^2}\right] dx \tag{49}$$

Unfortunately the integral does not converge, and therefore a closed form representation of the dB step distribution is not feasible.

2.2 Monte Carlo Analysis

Even though the analytical analysis of the distribution moments was not possible, a Monte Carlo Analysis has been performed. Multiple observations of synthetic IC-XT following the model presented in [1] have been generated and their PDF plotted Fig.2. This shows that the synthetic data are perfectly described by the analytical representation in Eq.45.

As it can be seen from Fig.2, the distribution shape does not change with the σ parameters but just shifts, thus only changing the mean. To prove a Monte Carlo Simulation has been performed, where multiple observations (of 1e8 samples) for different values of σ have been generated and the correspondent moments measured. The results have been plotted in Fig. 3. Fig. 3 empirically prove that the σ factor only varies the mean of the distribution. Therefore only shits the PDF. A regression process has been performed to find an analytical

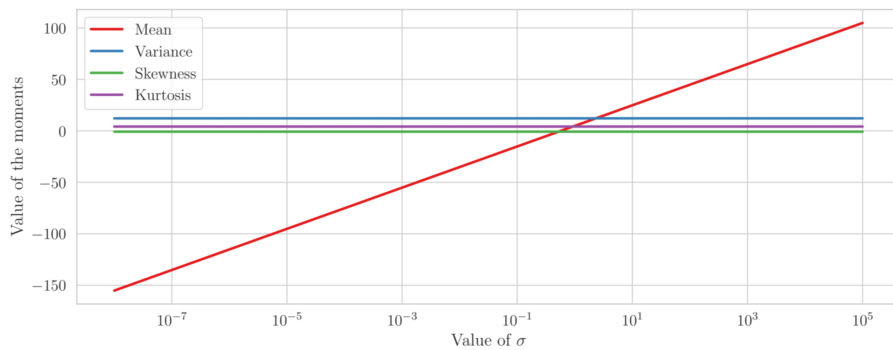

Figure 3: Moments of IC-XT Intensity with σ

representation of the trend of the mean. The mean μ can be approximated as:

$$\mu = 20log_{10}(\sigma) + 4.846. \tag{50}$$

The other moments stayed constants (variance= 12.16, skewness= -0.78, kurtosis= 4.19), therefore all the central moments are constant.
The distribution of the difference between two (independent) observations (step distribution) from distribution with constant central moments will always be the same because the difference operation will remove the non-central information. The step distribution in dB should therefore be independent on σ, as it can be seen from Fig.4, the distribution is always identical independent on the σ. To further prove that the step distribution would be constant, no matter σ (and therefore the characteristics of IC-XT signal),we find the moments for the step distribution with 1e8 observation for 100 different σ logspaced, and the value were always constant, Fig.5.

The fact that the theoretical step distribution never changes allows us to

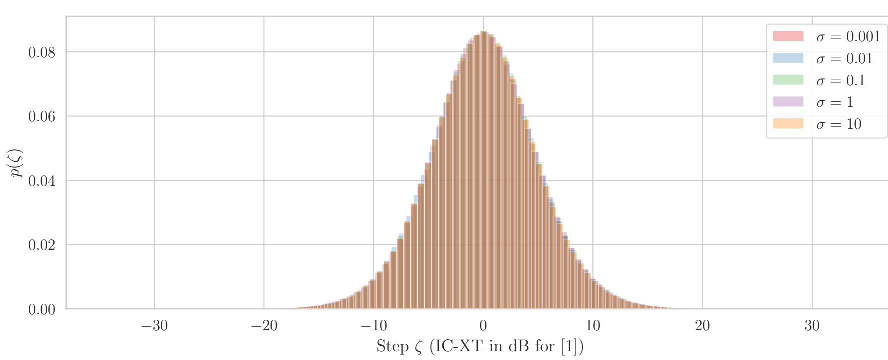

Figure 4: Step distribution of IC-XT in dB with different σ

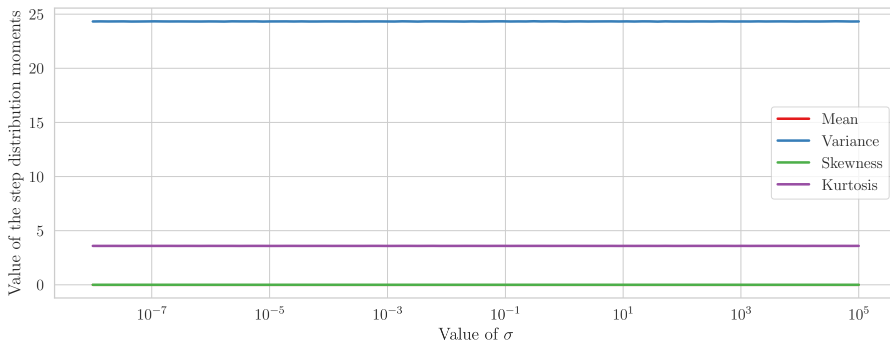

Figure 5: Moments of step distribution with σ

clearly show the difference between experimental and theoretical step distribution independently on the source signal parameters and measurement conditions.

3 Random Walk Step Distribution

We introduce a new model, a generalized version of the [1], in which the random phase components are described as random walks in time. The model is described as:

$$A_{n,t}(N_{\text{PM}}) = A_n(0,t) - j\sum_{l=1}^{N_{\text{PM}}} \chi_{nm}(l)\exp\left[-j\phi_{l,t}\right] A_m(l-1) \tag{51}$$

$$\approx -j\sum_{l=1}^{N_{\text{PM}}} \chi_{nm}\exp(-j\phi_{l,t}) \tag{52}$$

$$= -j\sum_{l=1}^{N_{\text{PM}}} \chi_{nm}\exp(-j\phi_{l,t-1}+\gamma) \tag{53}$$

$$= -j\sum_{l=1}^{N_{\text{PM}}} \chi_{nm}\exp\left[-j\left(\phi_{l,0}+\sum_{k=1}^{t}\gamma\right)\right] \quad \text{Where: } \gamma \backsim N(\mu,\sigma^2) \tag{54}$$

Where $\phi_{l,0}$ are the theoretical phase shifts between the active and target core at the l^{th} phase matching point derived from the equations described in [2], μ and σ are the mean and standard deviation of the Gaussian distributed random variable γ respectively.

To find the step distribution in respect of the random variable γ, the derived distribution for every possible parameter of γ and for every possible state $\phi_{l,t-1}$, need to be found.
Lets start finding the derived distribution for:

$$y = \exp(-j\phi_{l,t-1}+\gamma) \quad \text{Where: } \gamma \backsim N(\mu,\sigma^2) \tag{55}$$

$$= \exp(-j\gamma_1) \quad \text{Where: } \gamma_1 \backsim N(\mu_1 = \phi_{l,t-1}+\mu,\sigma^2) \tag{56}$$

$$= cos\left[\gamma_1\right] - j\cdot sin\left[\gamma_1\right] \tag{57}$$

Unfortunately a close analytical form for the real and imaginary part of the random variable y does not seem to exist. But the analysis can continue due to the fact that the formulation of A_n includes the sum of the components for all the PMPs. Following the central limit theorem (CLT), the sum of independent random variable with different means and variances will converge to a Gaussian distribution with mean the sum of the means of the composing random variables and as variance the sum of the variances. Therefore to find the total overall distribution it is needed to find the mean and the variance of the random variable y in respect of $\mu 1, \sigma$ of the random variable γ_1. Again an analytical solution for the problem was not possible to be found, on the other hand a formulation of the moments can be found using a Monte Carlo approach.
The mean and the variance of the sine and cosine of a Gaussian distributed random variable has been found generating 40000 combinations (200 values of

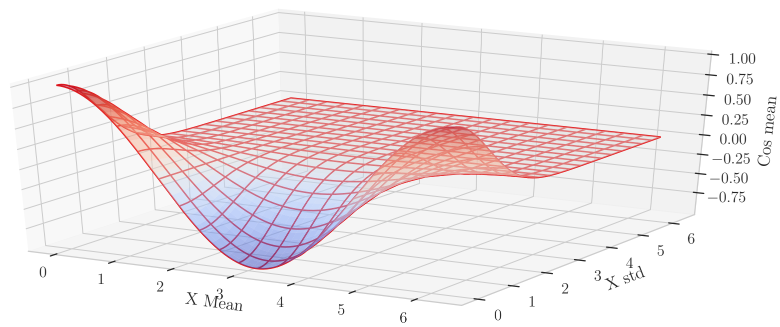

Figure 6: Mean of $cos(\gamma_1)$ for different μ_1, σ. Surface plot with colour-map corresponds to measured data, the red wire-frame plot correspond to the fitting

μ, σ between the support $0, 2\pi)$ of 1e6 samples each. The moments have been plotted and a perfect fitting for each has been found.

We started analysing the mean of the cosine function on γ_1. From Fig.6, it can be clearly seen that the behaviour follows the proposed fitting:

$$\mathrm{E}\left[cos(x)\right] = cos(\mu) \cdot e^{-\frac{1}{2}\sigma^2} \text{ where } x \sim N(\mu, \sigma^2). \tag{58}$$

This behaviour is justified by the fact that when the standard deviation is small the function will almost be deterministic, and when the variance increases (as explained in Supplementary Material Convergence) the derived distribution will tend to the one of the cosine of a uniform distribution over the overall support (arcsine distibution between -1 and 1) due to property of distribution on function with modular support ($\mathbb{R}_n$, periodic). Because the derived distribution of the cosine of a uniform distribution is symmetric around 0, the mean of $cos(\gamma_1)$ will tend to zero when σ increases.

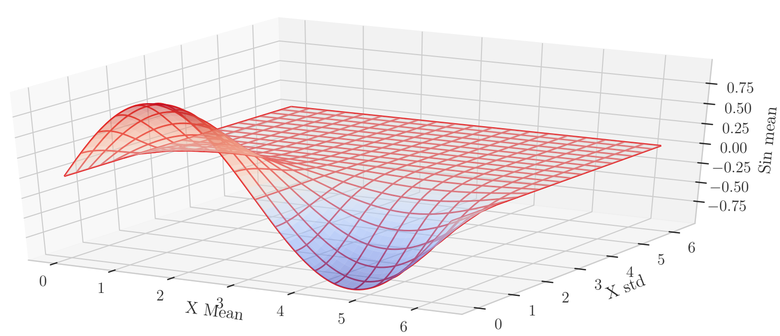

Figure 7: Mean of $sin(\gamma_1)$ for different μ_1, σ.

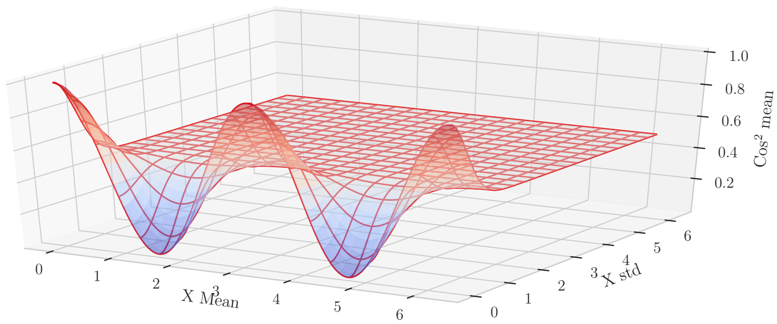

Figure 8: Mean of $cos(\gamma_1)$ for different μ_1, σ.

Similar analysis has been performed for the sine, Fig. 7, where the mean was described as:

$$\mathrm{E}\left[sin(x)\right] = sin(\mu) \cdot e^{-\frac{1}{2}\sigma^2} \text{ where } x \sim N(\mu, \sigma^2). \tag{59}$$

We now perform the analysis on the variance of the functions. Because an analytical representation is not trivial from the data itself, to find the formulation we use the identity:

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \tag{60}$$

$$= \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2. \tag{61}$$

We already have the mean of the derived random variable, now it is needed to find the mean of the random variable squared.

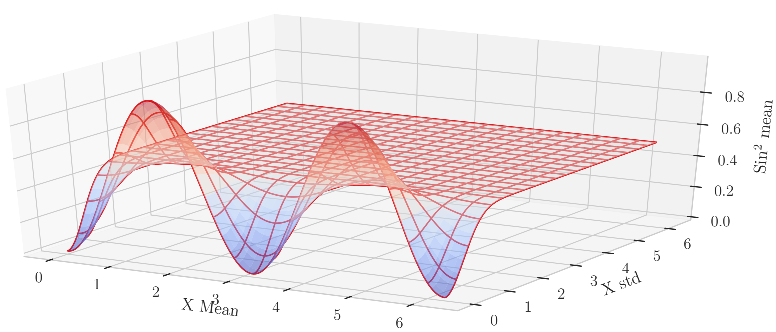

Figure 9: Mean of $sin(\gamma_1)^2$ for different μ_1, σ.

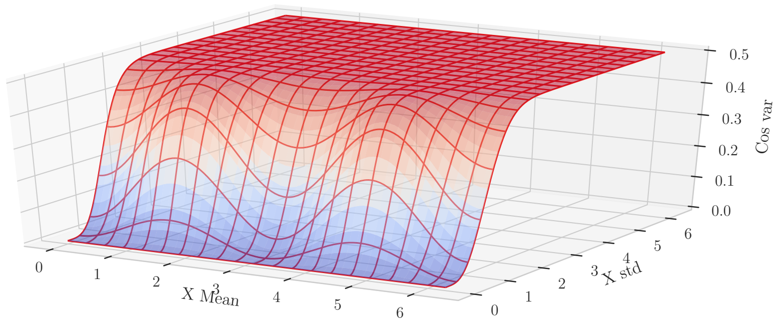

Figure 10: Variance of $cos(\gamma_1)$ for different μ_1, σ.

The analytical result are:

$$\mathrm{E}\left[cos^2(x)\right] = \frac{1}{2}\left[1 + cos(2\mu) \cdot e^{-2\sigma^2}\right] \text{ where } x \sim N(\mu, \sigma^2) \text{ Fig.8} \tag{62}$$

$$\mathrm{E}\left[sin^2(x)\right] = \frac{1}{2}\left[1 - cos(2\mu) \cdot e^{-2\sigma^2}\right] \text{ where } x \sim N(\mu, \sigma^2) \text{ Fig.9} \tag{63}$$

Now that the mean of the square function have been found we substitute the expectations in Eq.61 to find the variances.

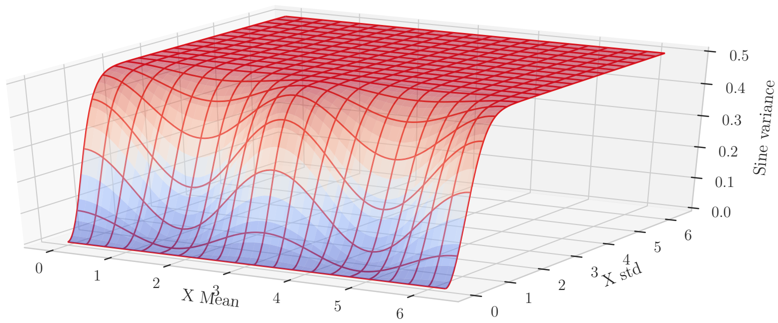

Figure 11: Variance of $sin(\gamma_1)^2$ for different μ_1, σ.

The variance of the cosine will follow the form:

$$
\begin{aligned}
\mathrm{Var}\left[cos(x)\right] &= \mathrm{E}\left[cos^2(x)\right] - \mathrm{E}\left[cos(x)\right]^2 && (64)\\
&= \frac{1}{2}\left[1 + cos(2\mu)\cdot e^{-2\sigma^2}\right] - \left(cos(\mu)\cdot e^{-\frac{1}{2}\sigma^2}\right)^2 \text{ where } x \sim N(\mu,\sigma) && (65)\\
&= \frac{1}{2}\left[1 + cos(2\mu)\cdot e^{-2\sigma^2}\right] - \frac{1}{2}\left[(1 + cos(2\mu))\cdot e^{-\sigma^2}\right] && (66)\\
&= \frac{1}{2}\left[1 + cos(2\mu)\cdot e^{-2\sigma^2} - (1 + cos(2\mu))\cdot e^{-\sigma^2}\right] && (67)\\
&= \frac{1}{2}\left[1 + e^{-\sigma^2}(cos(2\mu)\cdot e^{-\sigma^2} - 1 - cos(2\mu))\right] && (68)\\
&= \frac{1}{2}\left[1 + e^{-\sigma^2}(cos(2\mu)\cdot(e^{-\sigma^2} - 1) - 1)\right]. && (69)
\end{aligned}
$$

As it can be seen from Fig.10 the fitting is perfect. The same is done for the sine, Fig.11.

$$
\begin{aligned}
\mathrm{Var}\left[sin(x)\right] &= \mathrm{E}\left[sin^2(x)\right] - \mathrm{E}\left[sin(x)\right]^2 && (70)\\
&= \frac{1}{2}\left[1 - cos(2\mu)\cdot e^{-2\sigma^2}\right] - \left(sin(\mu)\cdot e^{-\frac{1}{2}\sigma^2}\right)^2 \text{ where } x \sim N(\mu,\sigma) && (71)\\
&= \frac{1}{2}\left[1 - cos(2\mu)\cdot e^{-2\sigma^2}\right] - \frac{1}{2}\left[(1 - cos(2\mu))\cdot e^{-\sigma^2}\right] && (72)\\
&= \frac{1}{2}\left[1 - cos(2\mu)\cdot e^{-2\sigma^2} - (1 - cos(2\mu))\cdot e^{-\sigma^2}\right] && (73)\\
&= \frac{1}{2}\left[1 + e^{-\sigma^2}(cos(2\mu)(1 - e^{-\sigma^2}) - 1)\right]. && (74)
\end{aligned}
$$

The distribution of the Complex amplitude of IC-XT will follow the following distribution:

$$
\begin{aligned}
&A_{n,t}(N_{\mathrm{PM}}) = X - jY && (75)\\
&\text{Where: } X \backsim N\left(e^{-\frac{1}{2}\sigma^2}\sum_{l=1}^{N_{PM}} cos(\mu_{l,t-1}), \frac{1}{2}\left[N_{PM} - e^{-\sigma^2}\left(N_{PM} - (e^{-\sigma^2} - 1)\sum_{l=1}^{N_{PM}} cos(2\mu_{l,t-1})\right)\right]\right) && (76)\\
&\text{Where: } Y \backsim N\left(e^{-\frac{1}{2}\sigma^2}\sum_{l=1}^{N_{PM}} sin(\mu_{l,t-1}), \frac{1}{2}\left[N_{PM} - e^{-\sigma^2}\left(N_{PM} - (1 - e^{-\sigma^2})\sum_{l=1}^{N_{PM}} cos(2\mu_{l,t-1})\right)\right]\right) && (77)
\end{aligned}
$$

Where $\mu_{l,t-1}$ corresponds to $\phi_{l,t-1}$, therefore the previous phase value at the l^{th} PMP.

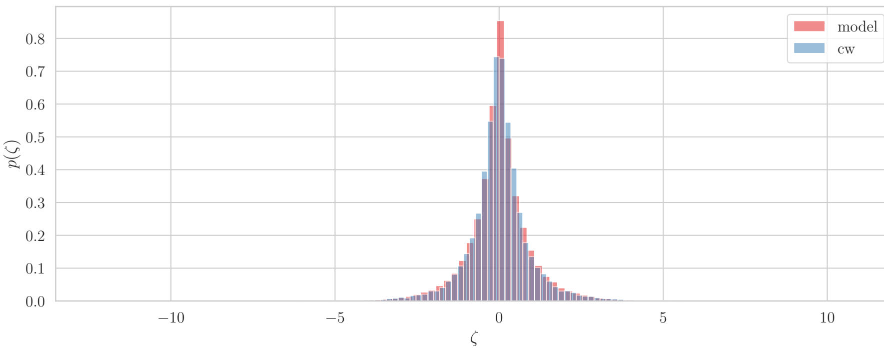

Figure 12: Experimental and Model step distribution normalized

The relationship between intensity and complex amplitude is:

$$I_{n,t} = \left|A_{n,t}(N_{\mathrm{PM})}\right|^2 \tag{78}$$
$$= \mathbf{Re}\left\{A_{n,t}(N_{\mathrm{PM}})\right\}^2 + \mathbf{Im}\left\{A_{n,t}(N_{\mathrm{PM}})\right\}^2 \tag{79}$$
$$= X^2 + Y^2 \tag{80}$$

So it follows the sum of the square a two independent, differently distributed Gaussian Random Variables, where $X \backsim N(\mu_x, \sigma_x^2)$ and $Y \backsim N(\mu_y, \sigma_x^y)$ (Note real values of the mean and variances described before).
The PDF of the square of a Gaussian random variable is found as follow:

$$Z = X^2 \tag{81}$$
$$|X| = \sqrt{Z} \tag{82}$$
$$F_Z(z) = P(|X| \leq \sqrt{z}) \tag{83}$$
$$= P(-\sqrt{z} \leq X \leq \sqrt{z}) \tag{84}$$
$$= F_X(\sqrt{z}) - F_X(-\sqrt{z}) \tag{85}$$
$$f_Z(z) = \frac{dF_Z(z)}{dz} \tag{86}$$
$$= f_X(\sqrt{z}) - f_X(-\sqrt{z}) \tag{87}$$
$$= \frac{1}{2}\frac{exp\left[-\frac{1}{2}\frac{\left(\sqrt{z}-\mu_x\right)^2}{\sigma_x^2}\right]}{\sqrt{2\pi z\sigma_x^2}} + \frac{1}{2}\frac{exp\left[-\frac{1}{2}\frac{\left(-\sqrt{z}-\mu_x\right)^2}{\sigma_x^2}\right]}{\sqrt{2\pi z\sigma_x^2}} \tag{88}$$
$$\therefore \frac{1}{2}\frac{1}{\sqrt{2\pi z\sigma_x^2}}\left(exp\left[-\frac{1}{2}\frac{\left(\sqrt{z}-\mu_x\right)^2}{\sigma_x^2}\right] + exp\left[-\frac{1}{2}\frac{\left(-\sqrt{z}-\mu_x\right)^2}{\sigma_x^2}\right]\right) \tag{89}$$

So the Intensity can be rewritten as:

$$I_{n,t} = Z_x + Z_y \tag{90}$$

$$\text{Where: } Z_x \backsim f_Z(z|\mu_x, \sigma_x^2), Z_y \backsim f_Z(z|\mu_y, \sigma_y^2) \tag{91}$$

The derived distribution of the sum of two independent random variables is the convolution of the 2 distribution. Unfortunately the convolution does not converge, and therefore no analytical representation can be found.

Even though the analytical form was not able to be found an heuristic approximation of the distribution in dB as been found using Pseudo-Voigt Profile (PVP).
The similarity between the simulated step distribution and the measured one is noticeable, Fig.12. To prove the validity of the model analysis of the information carried by the step distribution in dB has been performed as shown in the journal.

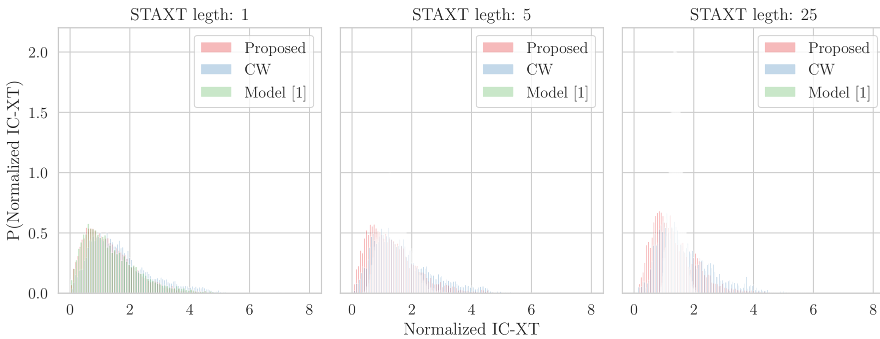

Figure 13: PDF of normalized IC-XT with different STAXT windows

4 STAXT analysis

Short Time Average Cross-Talk (STAXT) is a widely used analytical method to analyse properties of IC-XT. It consists on averaging subsequent samples in a moving average manner. This type of analysis is widely used in stochastic processes to better understand the drift property of stochastic processes.

The moving average acts as a low pass filter, therefore keeping the information about the low frequency components. When analysing a non stationary random process, using short time average (smaller than the auto-correlation window) keeps information about the infinitesimal drift, and keeps information about the auto-correlation, analysis similar to the one performed in [3]. When the moving average window is large and the process is ergodic, due to central limit theorem the process distribution will converge to a Gaussian distribution. For stationary processes, the window is extremely small and the CLT is applicable immediately.

Our proposed model, as shown in convergence, is ergodic and non-stationary, the model in [1] is both ergodic and stationary. We analyse the effect of STAXT with different averaging windows on both processes in comparison with the experimental data, Fig. 13. As expected the PDF of the CW and the proposed model is not largely effected by the averaging time window, while the Model of [1] quickly converges to a gaussian distribution following CLT. This analysis further prove that the proposed model is valid and in agreement with analysis such as the one performed in [3].

References

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