# Convergence of Models

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## 1 Formulation

Hayashi's formulation of IC-XT as in [1] is:

$$A_n(N_{\mathrm{PM}}) = A_n(0) - j\sum_{l=1}^{N_{\mathrm{PM}}} \chi_{nm}(l)\exp\left[-j\phi_{\mathrm{rnd}(l)}\right] A_m(l-1) \tag{1}$$

$$\approx -j\sum_{l=1}^{N_{\mathrm{PM}}} \chi_{nm}\exp(-j\phi_{\mathrm{rnd},l}) \text{ Where: } \phi_{\mathrm{rnd},l} \backsim U(0,2\pi) \tag{2}$$

where $A_n$ is the complex amplitude of the IC-XT of the target core n, $A_m$ is the complex amplitude of the signal in the active core, $\chi_{nm}$ is the coupling coefficient between cores n and m, $\phi_{\mathrm{rnd},l}$ is the random phase shift (0 to $2\pi$) at the $l^{th}$ phase matching point (PMP) and $N_{\mathrm{PM}}$ is the total number of phase matching points between cores n and m in the MCF.

The derived distribution of a sine and cosine function of a uniformly distributed random variable between 0 and $2\pi$ is the arcsine distribution:

$$f(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} \tag{3}$$

The function is defined between its support $[-1,1]$ and it is symmetric to its mean 0 and variance 0.5.
For this reason when summing all the random variable for the PMP, due to the central limit theorem (CLT) $A_n$ will follow Gaussian distribution in real and complex domain with $\mu = 0$ and $\sigma^2 = \frac{1}{2}N_{PM}\chi^2_{nm}$.

The intensity distribution of $A_n$ is found using $I_n = |A_n|^2 = ReA_n{}^2 + ImA_n{}^2$, which leads to a sum of two squared 0 mean normal distributions, leading to a second order ${}^2_{2df}$ distribution. Considering both polarisations, 4 squared IID Gaussian random variables need to be considered, leading to a fourth order ${}^2_{4df}$ distribution described as:

$$f_{\chi^2,4df}(x|\sigma) = \frac{x}{4\sigma^4}e^{\frac{-x}{2\sigma^2}} \tag{4}$$

where $\sigma$ is the standard deviation of the gaussian random variable. This distribution has a mean of $4\sigma^2$ and a variance of $8\sigma^4$.
Substituting $\sigma^2$ in our case:

$$f_{\chi^2,4df}(x) = \frac{x}{N_{PM}^2\chi_{nm}^4 2} e^{\frac{-x}{N_{PM}\chi_{nm}^2}} \tag{5}$$

with a mean of $2N_{PM}\chi_{nm}^2$ and variance of $2N_{PM}^2\chi_{nm}^4$.

# 2 Proposed model

The proposed model is described as follows:

$$\begin{aligned}
A_{n,t}(N_{\mathrm{PM}}) &= A_n(0,t) - j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}(l)\exp\left[-j\phi_{l,t}\right]A_m(l-1) && (6)\\
&\approx -j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}\exp(-j\phi_{l,t}) && (7)\\
&= -j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}\exp(-j\phi_{l,t-1}+\gamma) && (8)\\
&= -j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}\exp\left[-j\left(\phi_{l,0}+\sum_{k=1}^{t}\gamma\right)\right] \text{ Where: } \gamma \backsim N(\mu,\sigma^2) && (9)
\end{aligned}$$

Where $\phi_{l,0}$ are the theoretical phase shifts between the active and target core at the $l^{th}$ phase matching point derived from the equations described in [2], $\mu$ and $\sigma$ are the mean and standard deviation of the Gaussian distributed random variable $\gamma$ respectively.

Lets consider a single PMP and its dependence with time:

$$\chi_{nm}\exp\left[-j\left(\phi_{l,0}+\sum_{k=1}^{t}\gamma\right)\right] \tag{10}$$

Now consider the exponential part alone:

$$\exp\left[-j\left(\phi_{l,0}+\sum_{k=1}^{t}\gamma\right)\right] = cos\left[\phi_{l,0}+\sum_{k=1}^{t}\gamma\right] - j\cdot sin\left[\phi_{l,0}+\sum_{k=1}^{t}\gamma\right] \tag{11}$$

As it is well known *sin* and *cos* are periodic function with a period of $2\pi$. This makes the random walk act on a circular plane represented by the finite Group in $\mathbb{R}$ with support $[0, 2\pi)$. This can be represented as a random walk in $\mathbb{R}_n$, the real modulo $n$, where in our case $n = 2\pi$. This random walk corresponds to a Markov chain which is irreducible, aperiodic and double stochastic (for more information read [3, 4]), thus its stationary probability will be uniformly

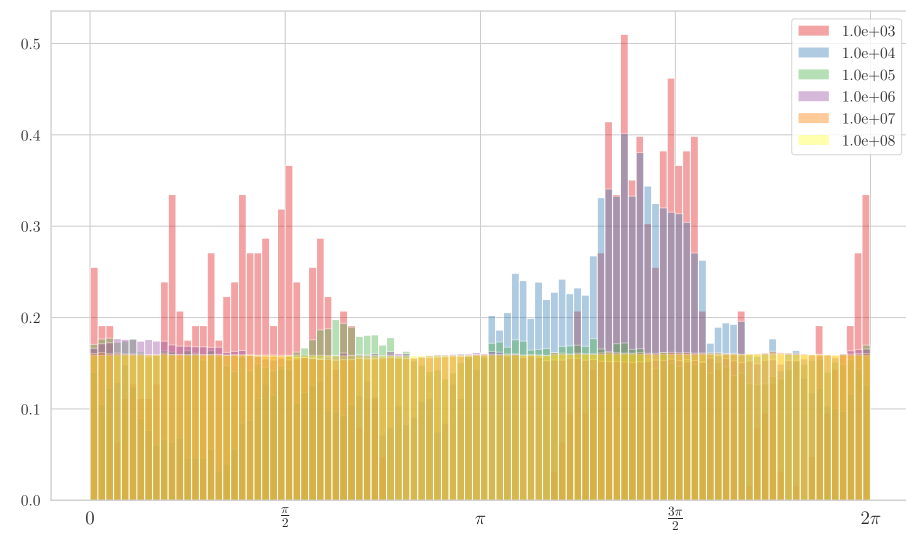


Figure 1: Random Walk distribution over the support with dfferent number of samples

distributed on $\mathbb{R}_n$, and therefore on the support $[0, 2\pi)$. Additional information and proof can be found in: [5, 6, 7, 8].

Analysis on the convergence can be performed but it is out of the scope of this paper. A simple conclusion is that the larger the *sigma* the faster it converges due to the fact that it is capable of better exploring the space. Similar conclusion can be taken into consideration whith $\mu$: the largest is the $||\mu||_1$ the faster it converges, because it will move more uniformly throughout the support.

This shows that when the number of samples and time windows tends to infinity the stationary distribution of the phase component correspond to the one described in [1], therefore the derived distribution for the IC-XT intensity derived above corresponds to the corresponding stationary distribution for the proposed model. This makes the IC-XT an ergodic process but not stationary.

$$p\left(\lim_{t\to+\infty} \exp\left[-j\left(\phi_{l,0} + \sum_{k=1}^{t}\gamma\right)\right]\right) = p(\exp[-j\phi_{\mathrm{rnd},l}]) \text{ Where: } \phi_{\mathrm{rnd},l} \backsim U(0, 2\pi) \tag{12}$$

So:

$$\lim_{t\to+\infty}[A_{n,t}(N_{\mathrm{PM}})] \approx \lim_{t\to+\infty}\left[-j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}\exp\left[-j\left(\phi_{l,0}+\sum_{k=1}^{t}\gamma\right)\right]\right] \tag{13}$$

$$= -j\sum_{l=1}^{N_{\mathrm{PM}}}\chi_{nm}\exp(-j\phi_{\mathrm{rnd},l}) \text{ Where: } \phi_{\mathrm{rnd},l} \backsim U(0,2\pi) \tag{14}$$

The stationary distribution will therefore be the same derived distribution:

$$f_{\chi^2,4df}(x|\sigma) = \frac{x}{4\sigma^4}e^{\frac{-x}{2\sigma^2}} \tag{15}$$

More specifically:

$$f_{\chi^2,4df}(x) = \frac{x}{N_{PM}^2\chi_{nm}^4 2}e^{\frac{-x}{N_{PM}\chi_{nm}^2}} \tag{16}$$

with a mean of $2N_{PM}\chi_{nm}^2$ and variance of $2N_{PM}^2\chi_{nm}^4$.

# References


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