How does the period of oscillation of a cantilever relate with its mass?

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Background

The aim of this investigation is to determine the relationship between the period of oscillation of an oscillating cantilever and the weight of the said cantilever. Experimentally, this will be accomplished by measuring the period of oscillation of a chosen cantilever as added masses are incremented. The frequency of the cantilever will then be determined through its period of oscillation.

I first became interested in the prospect of the cantilever's properties during the coronavirus pandemic lockdowns. Due to the extra free time that I had, I spent some more time on my hobbies, one of which was at the pool with my cousins. Since I enjoy thrill rides, I thought that trying out the diving board would be an interesting experience. While we were at the pool, I noticed and interesting phenomena: after each person leaped off the springboard, the time it took for the springboard to return to its non-oscillating resting state would occasionally vary. Perhaps it was the summer heat, but this is what first got me interested in the effects of different forces on the movements of cantilevers. From common sense, I inferred that the weight of the diver probably played the largest effect, but then I wondered: could there be a difference between different cantilevers, rather than just the applied force of the divers? As I thought about this, I found more relations to this topic and my life. When me and my teammates construct our robot for robotics club, we often tend to alter the robot's arm (changing the type of metal from steel, a heavy metal, to aluminum, a lighter one) to make its movements under stress faster. I had never considered how the weight of the arm itself could affect how it oscillates. This then led me to think of industrial uses- what contributed to the usage of certain beam materials, which seem to not oscillate as much for load bearing walls? After these considerations, I have decided to investigate how the weight of the cantilever itself can affect the frequency of its oscillation.

As an essential part of mechanical engineering, and thus almost all construction, cantilevers can be found almost anywhere. A cantilever is defined as "any rigid construction extending horizontally well beyond its vertical support," usually formed as a beam[1]. Within that, cuboidal-shaped beams are by far the most commonly occurring, due to their practical uses, and as such, this investigation will be limited to them. A cantilever, nature, has two different sides: one where it is fixed (usually due to a clamp or other binding mechanism), and one where it is free, giving it range of movement[2].

As this investigation carries out the deflection, or a displacement of the cantilever's free end from the initial position, of the cantilever, it is important to recognize what causes the deflection to occur. While there are many different weight distributions, or loads, that can cause an increase in weight at the cantilever's end, thus causing it to deflect, for the purposes of this investigation, only the point load will be used to maintain consistency and stability. A point load is when the acting force, the load, is concentrated at a specific point on the beam- in this case, the cantilever's end[3].

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[1] "Cantilever Definition & Meaning." Dictionary.com, Dictionary.com, https://www.dictionary.com/browse/cantilever.
[2] Hool, George A.; Johnson, Nathan Clarke (1920). "Elements of Structural Theory - Definitions". *Handbook of Building Construction* (Google Books). vol. 1 (1st ed.). New York: McGraw-Hill. p. 2.
[3] "ASCE/SEI 7-05 Minimum Design Loads for Buildings and Other Structures". American Society of Civil Engineers. 2006. p. 1. ISBN 0-7844-0809-2.

A cantilever's deflection is caused when the beam has been depressed to a certain level below the original point, caused by the load weight of mg/Newtons [4]. In accordance with Newton's 3rd law, there is an equal and opposing reaction force present to act upwards, in the opposite direction of gravitational force. Once the load mass has been removed, the reaction force is released in order to stabilize the cantilever back to equilibrium, causing it to oscillate [5].

When the load force is affecting the cantilever's end, however, the cantilever itself remains in a state of precarious equilibrium. As the load force increases, so too does the reacting restoring force [6]. Using the equation F = PA (F represent force, P represents Pressure, and A represents Area), and with the area being affected, in this case the body of the cantilever itself, remaining constant, there exists a direct correlation between the applied force and the resulting pressure in the cantilever itself [7].

The Young's Modulus is a measure of the elasticity of any beam or cantilever, measured in pressure units (as the Young's Modulus increases, so does the pressure in any beam) [8]. This value is heavily dependent on the material of the cantilever. The equation below produces a rough approximation of the Young's Modulus value [9].

$$Y = \frac{\sigma}{\varepsilon} \approx \frac{16\pi^2 ML}{bd^3T^2}$$

σ = force per unit surface (pressure units), ε = proportional deformation (change in length divided by original length) (dimensionless), M = mass (kg), L = exposed length of cantilever (m), b = width of cantilever (m), d = thickness of cantilever (m), T = period of oscillation (s)

As the load force on the cantilever increases (seen in this investigation as an increase in the cantilever weight), so too should the pressure in the cantilever itself [10]. Thus, by artificially increasing the pressure in the beam through additional weights, we should be able to measure a change in the effective Young's Modulus value.

In order to determine the predicted relationship between the results and the manipulated variable, first the relationship of a cantilever's oscillation must be related with the oscillation behavior of another tool. In this case, the investigation of an oscillating cantilever bears significant similarity with one of an oscillating spring system. A spring system is one where the end of a spring, usually positioned towards the ground, is displaced a certain length and then released. Both contain a restoring force that aims to bring the end of the oscillator back to the initial positions. Hence, while they initially seem dissimilar, we can utilize the equation used for the period of a spring system in this investigation[11].

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[4] Cantilever Bridge Facts, Design and History. (n.d.). Retrieved August 26, 2017, from http://www.historyofbridges.com/facts-about-bridges/cantilever-bridge
[5] Newton, Isaac, 1642-1727. Newton's Principia: the Mathematical Principles of Natural Philosophy. New-York: Daniel Adee, 1846.
[6] Giordano, Nicholas (2009–2013). "Chapter 11, Harmonic Motion and Elasticity". *College Physics: Reasoning and Relationships*. Volumes 1 and 2 (1st, 2nd ed.). Independence, KY: Cengage Learning. p. 360.
[7] Beltrami, Edward J. (1998) [1988]. "Chapter 1, Simple Dynamic Models". *Mathematics for Dynamic Modeling* (2nd ed.). San Diego, CA: Academic Press. pp. 3–7.
[8] Jastrzebski, D. (1959). *Nature and Properties of Engineering Materials* (Wiley International). John Wiley & Sons, Inc.
[9] "Physics - Young's Modulus." *University of Birmingham*, University of Birmingham, https://www.birmingham.ac.uk/teachers/study-resources/stem/Physics/youngs-modulus.aspx.
[10] Boyd, Euan J.; Uttamchandani, Deepak (2012). "Measurement of the Anisotropy of Young's Modulus in Single-Crystal Silicon". *Journal of Microelectromechanical Systems*. Institute of Electrical and Electronics Engineers. **21** (1): 243–249.
[11] "13.1: The motion of a spring-mass system". Physics LibreTexts. 17 September 2019.

$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

The equation modeling the period of oscillation of a spring can be seen above. In this equation, T_s stands for the period of oscillation (s), m stands for the mass loaded to the spring (kg), and k represents the spring constant (N/m).

From this equation, we can extrapolate a similar, but not identical equation, used for the cantilever testing[12].

$$T_c = 2\pi\sqrt{\frac{m}{k}}$$

In this equation, T_c stands for the period of oscillation (s), m stands for the mass loaded to the cantilever (kg), and k represents the stiffness, or effective spring constant of the cantilever (N/m). As the investigation calls for a manipulation of the cantilever's mass, in this case m, by rearranging the equation to remove the radical we can get the equation:

$$T_c{}^2 = 4\pi^2 * \frac{m}{k}$$

$$T_c{}^2 = \frac{4\pi^2}{k} * m$$

$$\frac{T_c{}^2}{m} = \frac{4\pi^2}{k}$$

Hypothesis

Thus, we can determine that through testing the relationship between a cantilever's mass and period of oscillation, the slope of the linearized line of the period of oscillation vs cantilever mass will be equivalent to $4\pi^2$ divided by the effective spring constant of the cantilever, caused due to a decrease in the elasticity in the cantilever, modeled by an increase in the cantilever's effective Young's Modulus.

Independent Variable

During this investigation, I will be using the mass of the cantilever as the independent variable. Considering the cantilever's mass as negligible, I will be adding weights in increments of 100g, with a maximum of 700 grams. I will then repeat this process in 7 trials for the most accurate results.

Dependent Variable

The resulting variable that will be used in data analysis will be the period of oscillation of the cantilever, measured in seconds. In order to maintain experimental accuracy, I will be measuring the period of 10 oscillations through frame-by-frame video analysis, then dividing by 10 to calculate each oscillation's period to ensure that sources of error in any one oscillation do not become prevalent.

[12] "Simple harmonic motion | Formula, Examples, & Facts". Encyclopedia Britannica

Controlled Variables

Variable Controlled	Why it's important to control	How this experiment controls/monitors the variable
Maximum Amplitude of Oscillation (Initial point)	If the cantilever was released from varying initial points throughout the experiment, it would lead to major inconsistencies, as the total distance covered during the oscillation would be non-consistent, thereby affecting the total time of oscillation as well.	The cantilever will be released 35 cm ± 0.05 cm lower than the initial resting point; this is the maximum amplitude of the cantilever as well.
Length of Cantilever used	By keeping the length of the cantilever used consistent, any inconsistencies caused by the altered tension in the cantilever (due to a longer or shorter length) will be mitigated. Additionally, by keeping the length of the cantilever consistent using a G-clamp, errors caused by increased mass of the cantilever itself leading to a greater gravitational/applied force will not occur.	70% of full meter stick hanging off table, or 70 cm ± 0.05 cm
Force Applied	As the force applied contributes to the overall oscillation of the cantilever, keeping it consistent will allow for consistency in the overall distance traveled, the tension in the cantilever itself and also the reacting restoring force ultimately causing the cantilever to oscillate.	0.5 kg applied, or ≈ 4.91 N to 5 N of force will be utilized.
Cantilever	By keeping the cantilever that is used for the investigation consistent, the length, width, thickness, material and much more will be kept consistent; all physical characteristics of the cantilever will be the same. This means that external variables such as the force applied will stay consistent along with the initial amplitude and so on.	Meterstick, commonly found hard wood width: 2.5 cm or 0.025 m depth: 0.65 cm or 0.0065 m

Materials

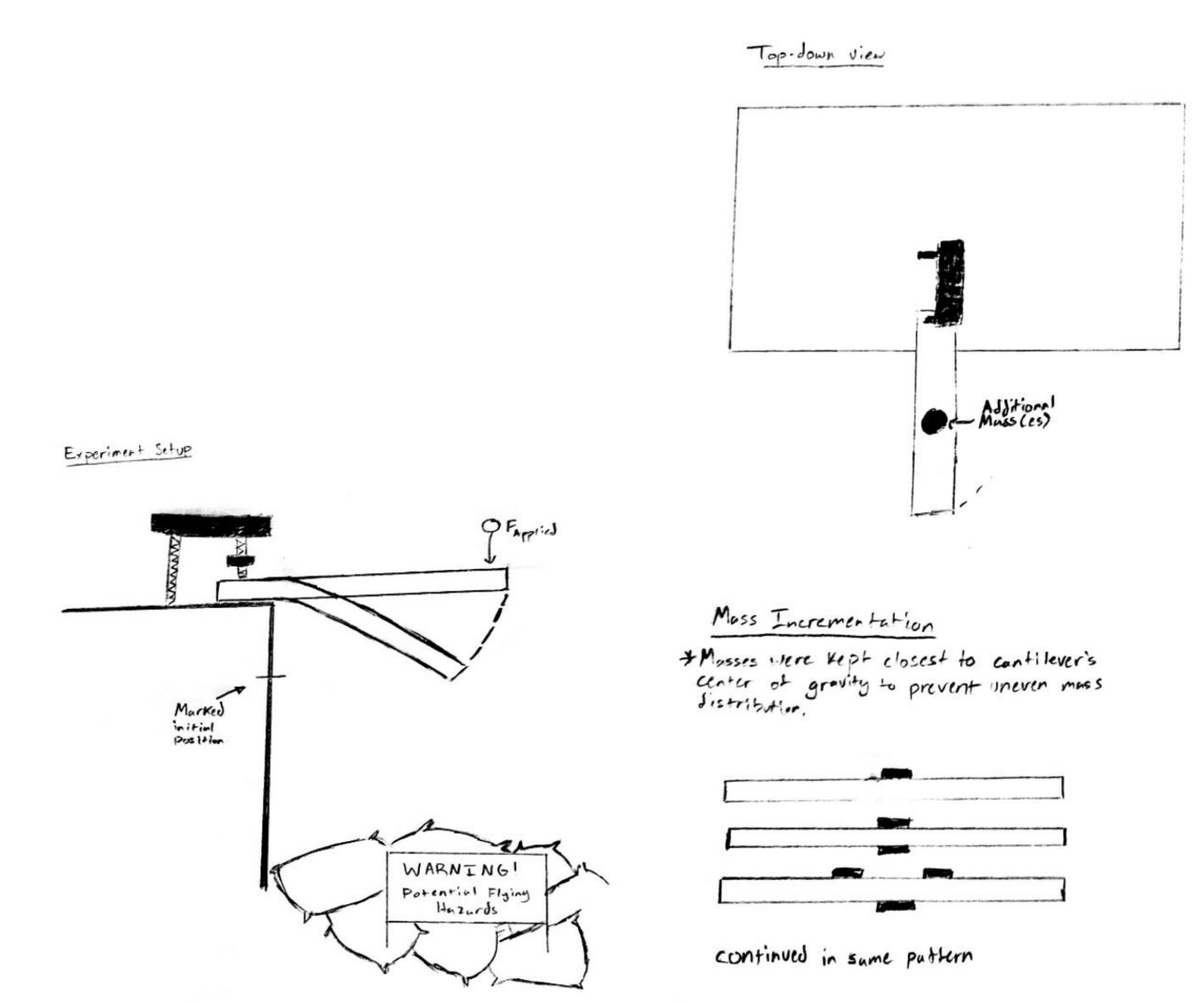
1. A cantilever (meter stick)
2. Tool for marking
3. 100-gram weights
4. Tape
5. Table (surface for cantilever to oscillate off of)
6. Clamp
7. Stopwatch
8. Ruler

Procedure

1. First, fasten the bare cantilever flat to the table using the clamp with of it 7/10ths length hanging off of the table.
2. Using the second ruler, measure a fixed distance down of 35 cm. This will be the distance that the ruler is pulled down to for each trial to ensure equal force is applied.
3. Pull the ruler down to the marked length.
4. Release the ruler and measure the period of oscillation using the stopwatch.
5. Repeat steps 3 and 4, 7 additional times to ensure the data's veracity.
6. Increment the ruler's mass by adding 100g weights evenly.

7. Fasten the weights to the ruler using a minimal amount of tape.
8. Repeat steps 3-6 until the ruler's mass reaches 600 grams.

<u>Diagram</u>



<u>Safety/Ethical/Environmental Issues</u>

After carefully considering all known variables and assessing the investigation's methodology, there were no major risks posed to anyone involved. The oscillating cantilever with the masses added could pose a threat due to its fast-moving speed combined with weight, however, written instructions along with physical safety barriers to ensure bystander safety have been implemented. In addition, weights have been safely secure to the oscillating cantilever using adhesives. Cushions (seen in diagram) have been placed around the experiment zone to ensure safety as well. This is an overall safe experiment with little to no safety, ethical and environmental concerns to be raised.

Data/Calculations/Graphs

Table 1; Raw data showing period of oscillation for 10 period of oscillations (s) vs mass loaded on the cantilever (g)

	Average Time Taken for 10 periods of oscillation (s) ± 0.01s								
Mass Loaded on Cantilever (g)	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Average Trial Time (s)	Average Uncertainty (±)
0	1.71	1.75	1.68	1.72	1.72	1.69	1.73	1.714	0.070
100	5.21	5.26	5.22	5.19	5.26	5.25	5.23	5.231	0.062
200	7.38	7.45	7.40	7.42	7.44	7.39	7.43	7.416	0.072
300	8.99	9.03	9.01	8.97	9.05	8.99	9.00	9.005	0.060
400	10.51	10.45	10.49	10.46	10.44	10.50	10.47	10.474	0.066
500	11.77	11.71	11.75	11.74	11.72	11.79	11.76	11.749	0.078
600	12.82	12.84	12.87	12.86	12.88	12.79	12.83	12.841	0.076

Sample Data Calculation (0 g):

The average period of oscillation in column 9 was calculated using the following formula:

$$a_{average} = \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7}{7}$$

$$= \frac{1.71 + 1.75 + 1.68 + 1.72 + 1.72 + 1.69 + 1.73}{7}$$

$$= 1.714286$$

$$= 1.71\ (2\ significant\ figures)$$

The average uncertainty in column 10 was calculated using the following formula:

$$u_{average} = \left(\frac{0.01}{a_1} + \frac{0.01}{a_2} + \frac{0.01}{a_3} + \frac{0.01}{a_4} + \frac{0.01}{a_5} + \frac{0.01}{a_6} + \frac{0.01}{a_7}\right) * a_{average}$$

$$= \left(\frac{0.01}{1.71} + \frac{0.01}{1.75} + \frac{0.01}{1.68} + \frac{0.01}{1.72} + \frac{0.01}{1.72} + \frac{0.01}{1.69} + \frac{0.01}{1.73}\right) * 1.71$$

$$= 0.070\ (2\ significant\ figures)$$

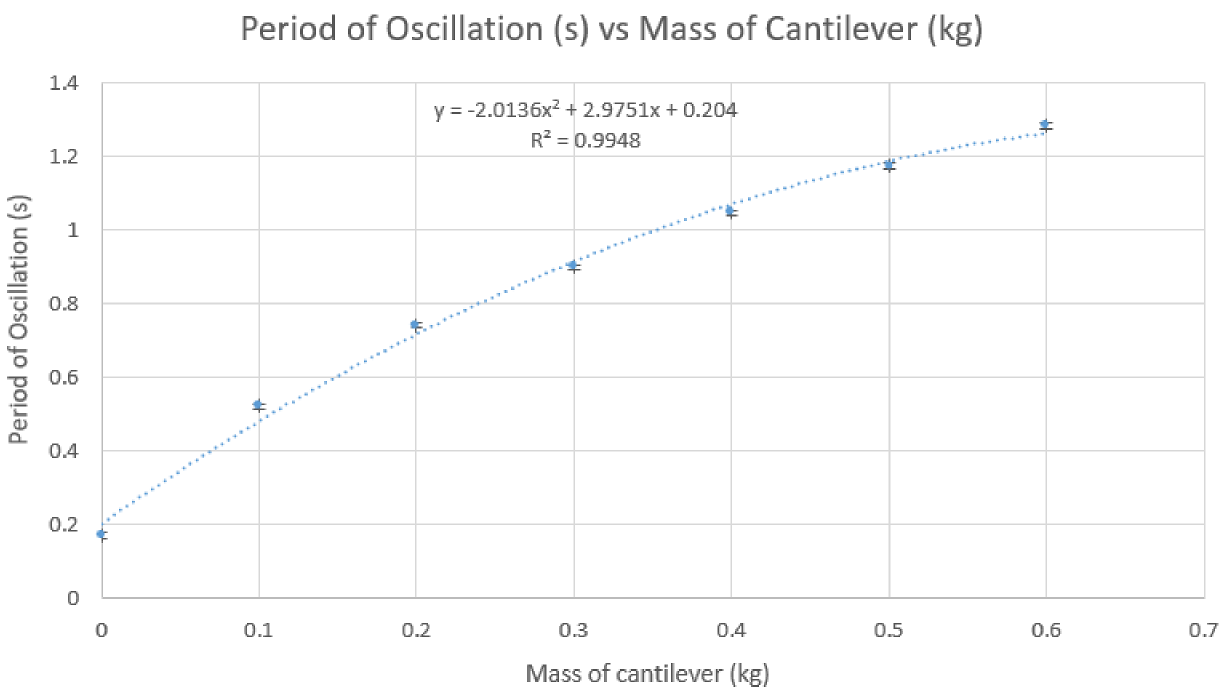
In order to properly qualify the data, two main changes must be made to process it. First, each of the data values must be divided by 10, as the current values reflect the period of oscillation for 10 whole oscillations (in order to reduce possibilities of error). Second, the masses must be converted to kilograms from grams in order to be properly used in the final equation for data analysis.

Table 2; Processed data for period of oscillation for 1 period of oscillation (s) vs mass loaded on the cantilever (kg)

	Average Time Taken for 1 period of oscillation (s) ± 0.0001s								
Mass Loaded on Cantilever (kg)	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Average Trial Time (s)	Average Uncertainty (± s)
0.00	0.17	0.18	0.17	0.17	0.17	0.17	0.17	0.171	0.070
0.10	0.52	0.53	0.52	0.52	0.53	0.53	0.52	0.524	0.062
0.20	0.74	0.75	0.74	0.74	0.74	0.74	0.74	0.741	0.072
0.30	0.90	0.90	0.90	0.90	0.91	0.90	0.90	0.901	0.060
0.40	1.05	1.05	1.05	1.05	1.04	1.05	1.05	1.049	0.066
0.50	1.18	1.17	1.18	1.17	1.17	1.18	1.18	1.176	0.078
0.60	1.28	1.28	1.29	1.29	1.29	1.28	1.28	1.284	0.076

In order to properly display the data spread, the average period of oscillation, the dependent variable, will be graphed along with the independent variable, the mass.

Graph 1; Shows a quadratic relationship between average period of oscillation (s) and mass of cantilever (kg). The R^2 value is very high, implying that the data is veritable.

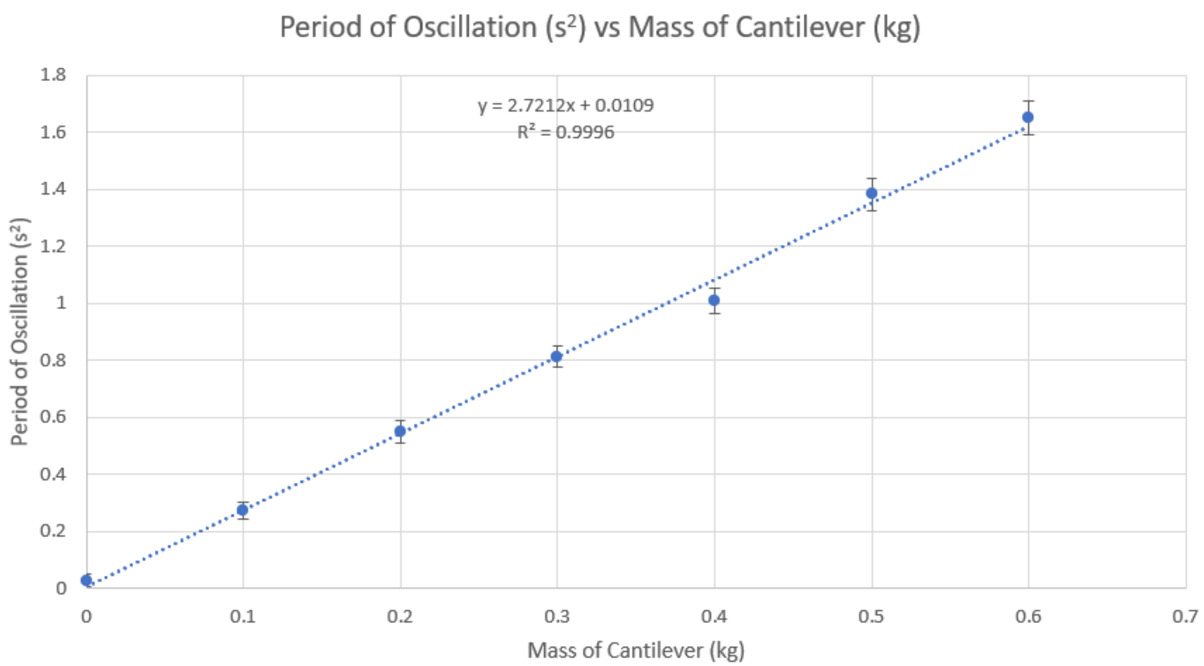


In order to determine whether the data increases by the predicted amount, we must recall the hypothesis. T^2 / m must be equal to $4\pi^2$ divided by the effective spring constant of the cantilever. In order to linearize the equation according to the equation then, the average period of oscillation must be squared by the mass.

Table 3; Shows a linearized values of average period of oscillation squared (s^2) and mass of cantilever (kg).

Mass Loaded on the Cantilever (kg)	Average Trial Time squared (s^2)	Average Uncertainty (± s)
0.00	0.029	0.024
0.10	0.274	0.028
0.20	0.550	0.038
0.30	0.813	0.036
0.40	1.010	0.044
0.50	1.382	0.056
0.60	1.649	0.059

Graph 2; Shows a linear relationship between average period of oscillation squared (s^2) and mass of cantilever (kg).



Based on the prediction, the slope above of 2.7212, representing T^2 / m, must be equal to $4\pi^2$ divided by k, the stiffness or effective spring constant of the cantilever. In order to calculate k, we must look at the units that would typically be used, N/m (as this equation is extrapolated from the period of an oscillating spring, k would originally be the spring constant and thus keeps the same units.

Thus, we can determine the stiffness of the cantilever by using an application of Hooke's Law, F = k*x, As Newtons are units of force and meters are units of length, k can thus be determined by dividing the total force applied over the maximum amplitude of oscillation. As the applied force was 5 Newtons, and the maximum amplitude of oscillation was 0.35 m, the effective spring constant of the cantilever equals 5N/0.35m, or 14.28 N/m.

Using this in the previously derived equation, plugging in 14.28 N/m for k, we get the following result of 2.7635, which is the same as the slope of the equation, 2.7212, with a percent error of approximately ≈ 1.5%. Thus, this result is statistically significant, shows that the period of oscillation of a cantilever increases with the mass just as the original equation predicts, and establishes the relationship between the two variables and qualifying the original hypothesis [13].

The main factor causing this relationship is caused due to the pressure in the beam, namely measure by the Young's Modulus of the beam. By examining trends in this value, we can further explain why mass and period of oscillation are related in such a way.

Using the equation Y = (16π^2 ML)/ (bd^3 T^2) and remembering L = 0.7m, b = 0.025m and, d = 0.0065m, discounting the inherent mass of the cantilever and using an arbitrarily small number (such as 0.01 kg) to prevent errors with zero, we can determine the relative Young's Modulus during each mass interval used:

Table 4; The relationship between avg period of oscillation (s), mass of cantilever (kg), and Young's Modulus (Pa)

Mass Loaded on the Cantilever (kg)	Average Period of Oscillation (s)	Young's Modulus (Pa)
0.00 (used 0.01)	0.17	$5.57 * 10^9$
0.10	0.52	$5.95 * 10^9$
0.20	0.74	$5.88 * 10^9$
0.30	0.90	$5.96 * 10^9$
0.40	1.05	$5.84 * 10^9$
0.50	1.17	$5.88 * 10^9$
0.60	1.28	$5.90 * 10^9$

Sample Calculation of Young's Modulus (0.4 kg):

$$Y = \frac{\sigma}{\varepsilon} = \frac{16\pi^2 ML}{bd^3T^2}$$

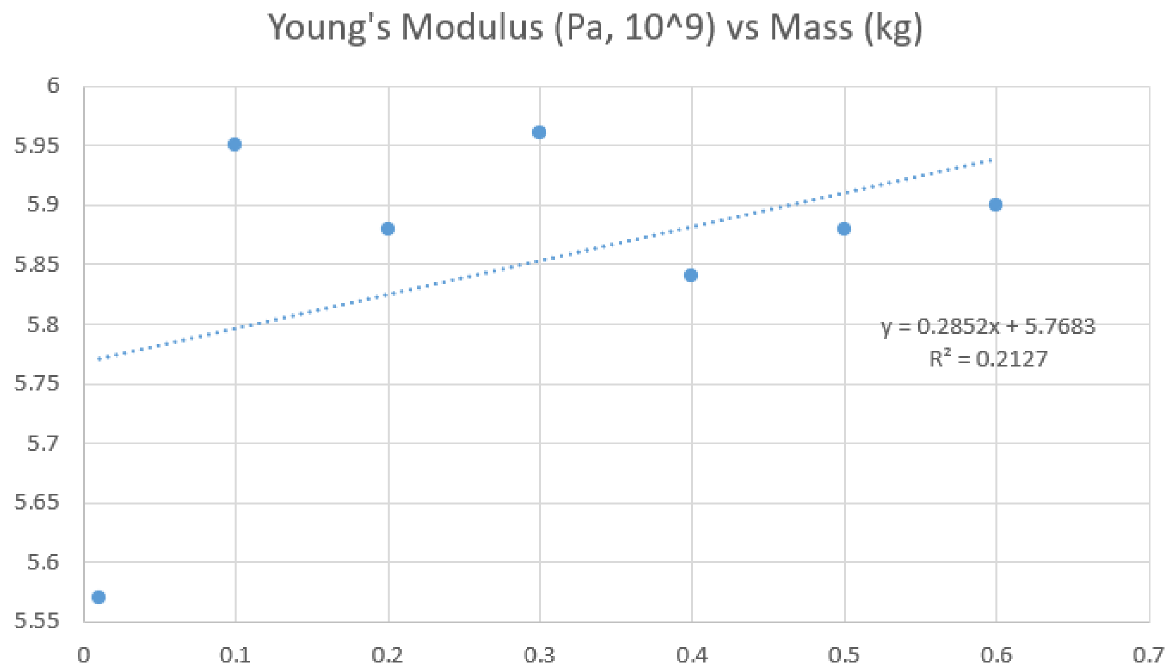
$$Y = \frac{16\pi^2\,(0.4\,kg)(0.7\,m)}{(0.025\,m)(0.0065\,m)^3(1.05\,m)^2}$$

$$Y = 5.84 * 10^9\,Pa$$

Graph 3; Shows Young's Modulus (Pa, 10^9) vs Mass (kg). The graph appears to display a zero-correlation relationship between the two variables.

From Graph 3 (see below), we can see that discounting the first test value, there seems to be no positive or negative correlation between the mass of the cantilever and the Young's Modulus of the cantilever during that mass interval (likely zero correlation). Thus, we can conclude that the effective Young's Modulus of a cantilever has little to no bearing on its period of oscillation.

[13] Error Propagation in Arithmetic Calculations. https://terpconnect.umd.edu/toh/models/ErrorPropagation.pdf



<u>Evaluation and Concluding Statements</u>

This was an experiment to apply the cantilever theory on an actual cantilever and determine the relationship between mass (kg) and period of oscillation (s), and to determine the effects of Young's Modulus (Pa) on the results. After collecting and processing the data from the experiment, **it can be concluded that the results only partially support the hypothesis**. When observing the effects of the mass of the cantilever (kg) on the period of oscillation (s), there is a positive relationship between the two variables that follows the theoretically derived equation from the period of an oscillation $T_c = 2\pi * \sqrt{(m/k)}$, which is

$$\frac{{T_c}^2}{m} = \frac{4\pi^2}{k}$$

However, the Young's Modulus (Pa) of the cantilever during any mass interval seems to have no effect on the period of oscillation, due to zero correlation between the variables being present.

First, the average period of oscillation (s) was measured, stated in Table 1, which was then processed in Table 2. Then, by linearizing the data (shown in Table 3 and Graph 2), calculating the effective spring constant of the cantilever, and analyzing the trendline the relationship between the mass (kg) and period of oscillation squared (s^2) was established. The data was then used for calculations in Young's Modulus (Pa), seen in Table 4 and Graph 3. After statistical analysis, it was found that Young's Modulus had no bearing on the relationship between the other two variables. While this could be caused due to a genuine non-relationship between the two variables, it is possible that this could be caused due to issues in procedure (see *Sources of Error and Improvements*).

This research is significant for a multitude of reasons. Analysis of cantilever theory such as this investigation could not only benefit diving board manufacturers as they determine to what extent that they must make their products more resilient to continued deterioration caused by oscillation, it could also benefit industrial manufacturers in the construction of buildings, and how the applied mass of the people in the building affects the oscillation, and therefore deterioration, of building internals and foundations, all interesting ways of applying Physics to real life that capture my interest. The findings of how oscillation relates with the mass of the cantilever is consistent with the building strategies of many industrial companies, which is why many buildings have beams of very heavy materials, (especially near the foundations)- they oscillate much less and provide a much more stable foundation in carrying the building's total weight. As a result, this experiment is not only consistent with fundamental Newtonian physics and Hooke’s principles of elastic restoring force but is also reflected in real world structures and designs due to its extremely practical nature. Real world applications are endless, and the principles examined here can ensure that oscillation of parts of any complex tool is kept to a minimum, prioritizing safety and efficiency. Some avenues of further research could be measuring to exactly what extent cantilever oscillation affects the deterioration of the cantilever itself, in the form of variables such as the overall deformation of the cantilever.

Sources of Error and Improvements

While the experiment was carried out with the utmost care and consideration, there are still large areas of improvement in methodology an investigation in hindsight. Overall, the experiment seems to have been conducted to an acceptable standard of accuracy and precision for an investigation of this scale.

Weakness/Source of Error	Significance	Suggested Procedural Improvement
Cantilever Strain – After being repeatedly deflected by a consistently powerful force with masses straining the body of the cantilever itself, it is possible that the strain may have caused damage to the cantilever which could have carried onto future measurements and unavoidable alteration to the final data.	Low significance, as the cantilever is the core to the investigation, any significant damage to it could have an effect on the future measurements of the period of oscillation, thereby altering the final conclusion. However, as there were no visual alterations to the cantilever itself, no sounds or alternate signs of physical damage, it is safe to say that the cantilever was not heavily affected by the experiment’s procedure.	Use a cantilever that is made from a sturdier material that is also equally as long and thick, such as a layered aluminum meterstick. Carefully observe the cantilever for deformation after every trial and restore it back to its original state if damage is indeed found.
Parallax Error – Although video analysis software was employed when measuring the period of oscillation, it is possible that mistakes could have occurred during the actual recording of the video itself, leading to inconsistent measurements for the period of oscillation being taken.	High significance, as it cannot be quantified in any other form than the base uncertainty accompanying any measurements and is a large portion of error in many experiments. However, it is an unlikely cause of error in this experiment because the camera was aligned directly in front of the cantilever, giving a near-perfect perpendicular viewing angle. Regardless, the factor of human error involved (ex: accidentally moving the camera a bit, causing misalignment) is present, causing its significance to rise.	The best way to reduce this error’s significance would be to take multiple recordings simultaneously from different angles and average the data collected from each, and also having a helper to assist during data processing to ensure that all of the correct data is noted down.
Mass Placement – Due to issues relating to funding and availability, the masses that were used in the experiment were all in 100gram increments. Regardless of how the masses were spaced, those points would become the most heavily weighed and might’ve increased the period of oscillation.	Low significance, as while mass placement could have a significant error in other investigations that focus on placement and position, the current investigation is primarily focused on the overall mass, which then affects the gravitational restoring force. The equation of $F_g = mg$ does not consider mass placement, so in this case, its effects are likely minimal.	Acquire masses of much smaller increments and when each mass interval is progressed, add as many masses needed to reach the interval spaced on the cantilever evenly.
Mass Size – As the masses themselves occupied physical space that was then added onto the material, there was a change in the total amount of oscillating material as the number of masses was incremented, which could have affected the calculation for the effective Young’s Modulus.	Medium significance – While the additional material could have added additional resistance to slow down the period of oscillation through aerodynamic resistance and similar factors the masses were all small and did not protrude from the cantilever, meaning that the additional material most likely did not affect the calculations for the effective spring constant, and thus Young’s Modulus very much.	Use masses of much smaller size and scale and place in indentations within the cantilever itself so that that the average material of the oscillating system is the same and external factors are thus unable to affect the cantilever’s oscillation.

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