Option Pricing with a Natural Equivalent Martingale Measure for Log-Symmetric Lévy Price Processes

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Presented by

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Notice 1

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Abstract

This thesis examines the pricing of options when the stock price follows a log-symmetric Lévy process. Models in continuous time and discrete time are considered. We identify situations when there is an equivalent change of measure that preserves the Lévy property, symmetry and the family of symmetric distributions of the returns (log-returns if discrete time), and makes the discounted price process into a martingale. We call such measures natural equivalent martingale measures.

In continuous time, when a natural equivalent martingale measure exists it is unique. It can be obtained by changing only the location or the scale parameter of the symmetric distribution if the Brownian component in present or absent, respectively, in the Lévy process. The analogous natural equivalent martingale measure in discrete time always exists but not unique. It can be obtained by changing the location and the scale parameters of the symmetric distribution.

Option pricing with natural equivalent martingale measure is arbitrage-free. We apply this approach to obtain new and elegant option pricing formulae for log-symmetric variance gamma and log-symmetric normal inverse Gaussian models.

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Contents

Chapter 1

Introduction

The discovery of the Black-Scholes option pricing formula [6] in the early 1970's signaled the beginning of a new era in the worlds of finance, economics and beyond. Option theory has since been used not only for stocks, bonds and other traded financial papers but also for valuation of various government guarantees and business decisions. It states that the price of a European call option at time t with maturity time T and strike price K is given by

$$\begin{aligned} C_t = S_t \Phi &\left(\frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right) \\ &\qquad - e^{-r(T-t)} K \Phi \left(\frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right), \end{aligned} \tag{1.1}$$

where S_t is the current stock price, r is the risk-free interest rate, σ is the volatility constant and Φ denotes the standard normal distribution function. It is well known that this formula admits no arbitrage (e.g. [34, p.304]). Although the original derivation of the formula is by solving partial differential equations, the preferred method nowadays is by the no-arbitrage and risk-neutral valuation approach (see Section 2.1.3). In particular, the Black-Scholes formula (1.1) can also be obtained by evaluation of the pay-off of the options under a unique equivalent martingale probability measure

(EMM)(risk neutral), i.e.,

$$C_t = e^{-r(T-t)}\mathbf{E}_Q\big[(S_T - K)^+ \big| \mathcal{F}_t\big], \tag{1.2}$$

where $\mathbf{E}_Q$ denotes the expectation under the EMM Q and $\mathcal{F}_t$ is the σ-field generated by the process $S_u, u \leq t$.

The success of the Black-Scholes formula lies in the assumption that the stock price follows a geometric Brownian motion and the return process is a Brownian motion with drift, i.e., the returns (or log-returns for discrete time) are normally distributed. However, it is widely believed that the returns (or log-returns) have distributions with more kurtosis, i.e., "fatter tails", than that of the normal distribution. Empirical evidence shows that the log daily returns of some assets (including stock) are well fitted by more general symmetric distributions, e.g. [43], [44], [19], [7], [29], [30]. Symmetric distributions belong to the more general elliptical family of distributions considered by Fang et al. [20]. The symmetric family encompasses not only the normal distributions, but also other classes of distributions such as Student-t, exponential power family, and mixtures. Furthermore, in 1996 Mc. Donald [42] suggested to use the distribution function of the underlying distribution for the returns in the Black-Scholes formula (1.1), instead of the standard normal, to reflect the specificity of the return distribution. However, according to Mc. Donald, this would violate the no arbitrage principle, and may result in prices that lead to arbitrage. Interestingly, Klebaner and Landsman [35] showed that option pricing with symmetric distributions for log-returns in discrete time led to modified Black-Scholes formula with an alternative distribution instead of standard normal, yet they are arbitrage-free. Central to this is the introduction of a change of measure that keeps the distribution of log-returns in the same symmetric family. They called it the *natural* EMM. A more detailed discussion of the works by Klebaner and Landsman [35] is given in Section 2.2.

In continuous time, replacing the assumption of normality for returns by symmetric while retaining all other assumptions in the Black-Scholes model, such as independence and stationarity of increments, leads to the case of symmetric Lévy process for returns, and the price process known as log-symmetric Lévy process (see (1.3)). It remains to see whether option pricing with such models can produce an arbitrage free option pricing formula with an alternative distribution that resembles Black-Scholes formula. This is the main objective of the thesis. We also consider the log-symmetric Lévy model for price processes in discrete time to complement [35].

There is a large volume of literature on option pricing with Lévy processes, e.g. [4], [9], [11], [18], [39], [50]. It is well known that the Lévy market models, save the Brownian motion case, are incomplete [50, p.77], i.e., the EMM is not unique. In fact, there are infinitely many possible EMM's for option pricing with Lévy processes that produce arbitrage-free results, as it was shown in [13]. In such a case, the choice of EMM is fairly arbitrary and is motivated by various other considerations. Among the popular methods are the Esscher transform ([26], [33], [11]), minimum entropy martingale measure ([22], [23], [45]), minimal martingale measure ([21], [11]), minimax and minimal distance martingale measure [27], and variance-optimal martingale measure [51]. Although a Lévy process remains a Lévy process under these EMM's, some may fail to produce a probability law [17], or just a non-negative measure that satisfies a very mild no-arbitrage condition [51]. In some cases, the Esscher transform produces a continuum of EMM's that requires further refinement on the selection by optimizing the relative entropy or some other utility functions [37]. But most importantly, these approaches are technical and, in some cases, can be complicated, which make them unattractive to financial practitioners. For example, the optimization procedure may not have closed solution and numerical methods are required to estimate the parameter that leads to the martingale measure (e.g. [37], [45]). Therefore for practical reason, we propose a new approach for symmetric Lévy market models whereby the EMM is easily obtained by changing the location or the scale parameter of the underlying distribution of the returns (or log-returns).

In the case of symmetric Lévy market model, it turns out that there is a unique EMM that is occurring naturally within the same family of symmetric distributions as the real world distribution for the returns (or log-returns). The proposed EMM is a change of measure that preserves Lévy property, symmetry and the family of symmetric distributions of the returns (or log-returns), and make the discounted price process into a martingale. Following [35], we call this a *natural* change of measure, and the EMM a *natural* EMM. In continuous time, our main results (Theorems 3.2.3 and 3.2.4) show that it can be obtained by changing only the location (mean) parameter if a Brownian component is present, and the scale (variance) parameter if a Brownian component is absent, in the Lévy process. As such, they do not leave the family of the symmetric distributions. In discrete time, a natural EMM changes the location and the scale parameters (Theorems 3.3.3 and 3.3.4). Therefore, the natural EMM in discrete time is not unique in general. However, if the scale parameter is also fixed in addition to the symmetric family, a unique natural EMM that changes only the location parameter is obtained (Proposition 3.3.1, also see [35]).

Option pricing with a natural EMM is arbitrage-free (see Chapter 4.1). It leads to option pricing formulae akin to Black-Scholes with the distribution function of the underlying distribution for returns. In particular, new option pricing formulae are derived for log-symmetric variance gamma (VG) model (Chapter 5) and log-symmetric normal inverse Gaussian (NIG) model (Chapter 6). We also show that these new option pricing formulae contain the classical Black-Scholes formula as a special case. In the case of VG, an option pricing formula was given in [40, Eq. 6.7] and [39, Eq. 25] where a non-symmetric case was also included. They derived it by analytical methods and used the normal density function integrated with respect to a gamma density. While [40] presented the formula as a double integral of elementary functions and obtained the price by numerical integration, [39] provided a closed form formula in terms of the special functions involving the modified Bessel function of the second kind and the degenerate hypergeometric function. Our approach is purely probabilistic and identifies their integrated function in the symmetric case as the Bessel function distribution (Bessel distribution here after). This link Between VG process and the Bessel distribution is new and, to the best knowledge of the author, not known in the literature. All the results in this thesis have been submitted for publication [28].

The following is our model and assumptions in this thesis. We consider a model for the stock price process of the form

$$S_t = S_0 e^{Y_t}, \tag{1.3}$$

where S_0 is the initial price and $Y_t,\ t \geq 0$ is a time homogeneous symmetric Lévy process on $\mathbb{R}$ (in continuous time or discrete time) with finite first exponential moment (for martingale reasons which will become clear later) supported by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We assume that the filtered probability space satisfies the following usual conditions (e.g. [50, p.12]):

- $\mathcal{F}$ is P-complete,
- $\mathcal{F}_0$ contains all P-null sets of Ω,
- $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, is right continuous, i.e., $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

$\mathcal{F}_t$ is the σ-algebra generated by the process $S_u,\ u \leq t$, completed by the null sets. For finite planning horizon T, we can further assume $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ without loss of generality. In addition, the market is assumed to be frictionless, i.e., zero transaction costs, zero taxes and dividend, no restriction on borrowing and short selling, the same interest rate for both borrowing and lending, no transaction delays, and perfect liquid markets (e.g. [50, p.8]). As usual, when continuous time model is considered, the trading takes place in continuous time. When discrete time model is considered, the trading takes place in discrete time.

The thesis is organized as follows: In chapter 2, we study the fundamental concepts of option pricing. A review of the paper by Klebaner and Landsman [35], which motivated the works in this thesis, is given. Some properties of Lévy processes and symmetric distributions necessary for later chapters are also given. In Chapter 3, we give the construction of a natural EMM for log-symmetric Lévy processes. The differences between discrete time and continuous time cases are discussed. In Chapter 4, we consider option pricing with a natural EMM for log-symmetric Lévy processes in continuous time and in discrete time. Chapters 5 and 6 contain applications of this approach to log-symmetric VG and log-symmetric NIG models respectively. Some numerical comparisons between these formulae and the Black-Scholes formula are given at the end of their respective chapters. Finally, we conclude in Chapter 7 with discussions and suggestions for future research.

Chapter 2

Preliminaries

2.1 Option Pricing

In this section, we study the fundamentals of option pricing and the important concepts related to this topic. We first give a brief introduction of what is an option. Then we highlight the mathematical theory for pricing of options.

2.1.1 Financial Derivatives and Options

Generally speaking, a financial derivative or a contingent claim on an asset is a contract that allows transaction (purchase or sale) of this asset in the future on terms that are specified in the contract. The underlying risky asset can be a security (stock or bond); a currency; an index portfolio; a future price; or some measurable state variable, such as the volatility of a market index. A basic example of a financial derivative is the option on stock.

Definition 2.1.1. An option is a contract that gives the holder the right, but not the obligation, to buy or sell the stock at an agreed exercise price K, called the *strike price*.

A *call* option is a right to buy, whereas a *put* option is a right to sell. The options can be European style or American style. A European option only allows the holder to exercise his right at a particular date T, called the maturity date of the contract. In contrast, an American option allows the holder to exercise his right at any time within the lifespan of the contract. The European options and American options are *plain vanilla options*, i.e., has expiration date and straightforward strike price. Other *exotic options* (see e.g. [50], Chapter 9) include Asian options, Barrier options, lookbacks and swaps to name a few.

In this thesis, we generally work with (but not limited to) European call options on stock. Because exercise is a right and not an obligation, the exercise payoff for a call option is

$$(S_T - K)^+ \equiv \max\{0, S_T - K, \}, \tag{2.1}$$

where S_T is the price of the stock at exercise. Due to the volatility of the stock price, the future price S_T is not known at the start of the contract. Remarkably, the price of an option that depends on the unknown price S_T can be computed deterministically at the start of the contract under an EMM Q as in (1.2).

2.1.2 Change of Measure

The Change of measure is a technique for obtaining equivalent measures. If the equivalent measure also makes the discounted price $e^{-rt}S_t$ into a martingale, then it is an EMM. The technique is based on the next theorem that provides a way to construct measures that are absolutely continuous with respect to the base measure P (e.g. [34, p. 274]).

Definition 2.1.2. A probability measure Q is absolutely continuous with respect to probability measure P, denoted $Q \ll P$, if $Q(A) = 0$ whenever $P(A) = 0$. Moreover, if $Q \ll P$ and $P \ll Q$, i.e., they have the same null sets, then Q and P is said to be equivalent (denoted $Q \sim P$).

Theorem 2.1.1. (Radon-Nikodym)
Let $Q \ll P$, then there exists a nonnegative random variable Λ such that

$E_p(\Lambda) = 1$, and

$$Q(A) = \mathbf{E}_P\big(\Lambda I(A)\big) = \int_A \Lambda dP \tag{2.2}$$

for any measurable set A. The random variable Λ is unique P-almost surely. Conversely, if there exists Λ with the above properties and Q is defined by (2.2), then it is a probability measure and $Q \ll P$.

Therefore, if P and Q are equivalent, i.e., they have the same null sets, then there exists a nonnegative random variable, denoted $\Lambda = \frac{dQ}{dP}$, called the Radon-Nikodym derivative (which is the likelihood ratio for the density of Q with respect to P), such that the probabilities under Q are given by (2.2). Girsanov's theorem gives the form of the likelihood ratio. We leave the model specific likelihood ratio to a later stage. More specifically, we will deal with the likelihood ratio for the change of measure for Lévy processes in Chapter 3. Here, we give a general result for calculation of expectations and conditional expectations under a change of measure (e.g. [34, p.275]).

Theorem 2.1.2. *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space on $\mathbb{R}$. Let Λ_t, $0 \le t \le T$, be a positive P-martingale such that $\mathbf{E}_P(\Lambda_T) = 1$. Define the new probability measure Q on the same space by the relation $\frac{dQ}{dP} = \Lambda_T$, then Q is absolutely continuous with respect to P. And for any random variable X,*

$$\mathbf{E}_Q(X) = \mathbf{E}_P(\Lambda_T X).$$

Furthermore,

$$\mathbf{E}_Q(X|\mathcal{F}_t) = \mathbf{E}_P\left(\frac{\Lambda_T}{\Lambda_t} X \middle| \mathcal{F}_t\right),$$

and if X is $\mathcal{F}_t$ measurable, then for $s \le t$,

$$\mathbf{E}_Q(X|\mathcal{F}_s) = \mathbf{E}_P\left(\frac{\Lambda_t}{\Lambda_s} X \middle| \mathcal{F}_s\right).$$

Following the theorem above, by taking the indicator function $I(X \in A)$, we obtain the distribution of X under Q

$$Q(X \in A) = E_P(\Lambda_T I(X \in A)). \tag{2.3}$$

2.1.3 No-arbitrage and Risk-Neutral Pricing

Here, we highlight the concepts of pricing options by no-arbitrage approach for continuous time model. The concepts for discrete time model are analogous (e.g. [52], Chapter V, or [34], Chapter 11.2).

Arbitrage in Continuous Time Model

Arbitrage is defined in finance as a trading strategy that allows making a profit out of nothing without taking any risk. Therefore, to make economic sense, the pricing of options should be done in a way that is consistent with no-arbitrage

However, there are different versions of the no-arbitrage concept in continuous time model: No Arbitrage (NA), No Free Lunch (NFL), No Free Lunch with Bounded Risk (NFLBR), No Free Lunch with Vanishing Risk (NFLVR), No Feasible Free Lunch with Vanishing Risk (NFFLVR), see e.g. [12], [36], [52, p.651], [5, p.140]. But the main premise remains the same, that is, the market is fair, rational, and does not allow one to make riskless profit there. Underlying, this means there is an admissible self-financing strategy (defined later) that replicates the value of the claim. What differs is the kind or class of admissible self-financing strategies considered for the replication (e.g. [52, p.651]).

Moreover, in continuous time settings with general semimartingale price processes, existence of EMM implies the absence of arbitrage in a market, but the converse is not true. It is only when the semimartingales are bounded, then the absence of arbitrage is equivalent to the existence of a martingale measure (e.g. [12] or [52, p.657]).

Therefore, for our purpose, we consider a two-asset market model on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with finite time horizon T. The market model consists of a semimartingale S_t representing the stock price process, and a money (savings) account β_t with $\beta_0 = 1$, and both are adapted to $\mathbb{F}$. We further assume $\beta_t > 0$ P-almost surely and of finite variation. Since price processes are bounded, we have the following result that guarantees no-arbitrage for the model [10, p. 303].

Theorem 2.1.3. (First Fundamental Theorem of Asset Pricing) *A market model does not have arbitrage opportunities if and only if there exists at least one equivalent martingale measure Q such that the discounted price process $\frac{S_t}{\beta_t}$, $t \in [0,T]$ is a* Q*-martingale.*

In what follows, we look at some of the key ideas behind the concept of no-arbitrage following ([34], Chapter 11.3) using a simpler formulation.

An investment or trading strategy in the market is described by a *portfolio*. Let a_t and b_t be, respectively, the number of shares of S_t and the amount of cash (in units of β_t) held at time t. Assume a_t and b_t are $\mathbb{F}$-predictable, i.e., $a_t, b_t \in \mathcal{F}_{t_-} = \bigcup_{s<t} \mathcal{F}_s$.

Definition 2.1.3. A strategy or portfolio is a $\mathbb{F}$-predictable 2-dimensional process $\pi_t = (a_t, b_t)$.

Note that both a_t and b_t are allowed to assume any positive or negative values. A negative value of a_t means *short sale* of stock, i.e., sell the stock at time t; and a negative value of b_t means borrowing money (from banks) at some riskless interest rate r. The self-financing property of a strategy can be defined as the following.

Definition 2.1.4. A strategy π_t is said to be self-financing if the changes in its value V_t^π, $t \in [0,T]$ comes only from the changes in the prices of the assets, i.e.,

$$dV_t^\pi = a_t dS_t + b_t d\beta_t,$$

or equivalently,

$$V_t^\pi = V_0^\pi + \int_0^t a_u dS_u + \int_0^t b_u d\beta_u. \tag{2.4}$$

The integrals on the right-hand-side of (2.4) are Itô stochastic integrals (in general, both S_t and β_t can be stochastic processes). Itô integral has the desired non-anticipating property: it integrates the integrand against the forward increments of the integrator, which makes economic sense. To see this, observe that, if the process a_t is simple (piecewise constant), then the stochastic integral is defined by

$$\int_0^t a_u dS_u = \sum_i a_{t_i}\big(S_{t_{i+1}\wedge t} - S_{t_i\wedge t}\big),$$

where $\big(S_{t_{i+1}\wedge t} - S_{t_i\wedge t}\big)$ is the forward increments of S_t. This means that the gain in value is obtained by multiplying the increments of the price process by the number of units of the asset held at the beginning of the relevant time interval. For general integrand, the stochastic integral is defined by (e.g. [53, p.134])

$$\int_0^t a_u dS_u = \int_0^t a(u)dS(u) = \lim_{n\to\infty}\int_0^t a_n(u)dS(u), \qquad 0 \le t \le T,$$

where $a_n(t)$ is a sequence of simple processes such that as $n \to \infty$ these processes converge to the process $a(t)$, i.e.,

$$\lim_{n\to\infty}\int_0^T \big|a_n(t) - a(t)\big|^2 dt = 0.$$

Similarly for the other integral if β_t is also stochastic.

The next theorem gives a criteria for a strategy to be self-financing in terms of the discounted price process.

Theorem 2.1.4 (Theorem 11.11 [34])**.** *A strategy π_t is self-financing if and only if the discounted value process $\frac{V_t^\pi}{\beta_t}$ is a stochastic integral with respect to the discounted price process*

$$\frac{V_t^\pi}{\beta_t} = V_0^\pi + \int_0^t a_u dZ_u, \tag{2.5}$$

where $Z_t = \frac{S_t}{\beta_t}$.

Now let $\mathcal{M}$ denotes the set of probability measures on $(\Omega, \mathcal{F})$ that is equivalent to P and under which the discounted price process $Z_t = \frac{S_t}{\beta_t}$ is a martingale. By Theorem 2.1.3, we know that $\mathcal{M}$ is not empty. An example of model that satisfies this assumption is the Black-Scholes model.

Example 2.1.1. (Example 11.11 [34])
For Black-Scholes model, $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$ and $\beta_t = e^{rt}$. Therefore, the discounted price process $Z_t = e^{-rt} S_t = S_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma B_t}$. When $\mu = r$, it is the exponential martingale of σB_t, thus is also a martingale. Otherwise, it is not a martingale. By stochastic differentiation, we can write $dS_t = \sigma S_t \big(\frac{\mu}{\sigma} dt + dB_t\big)$. Using change of measure for removing drift in diffusion, there is a unique measure Q and a Q-Brownian motion $\tilde{B}_t$ such that $\frac{\mu}{\sigma} dt + dB_t = \frac{r}{\sigma} dt + d\tilde{B}_t$. So, we have $\sigma B_t = rt + \sigma \tilde{B}_t - \mu t$. The discounted price process Z_t in terms of $\tilde{B}_t$ under Q is $Z_t = S_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma B_t} = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{B}_t}$, thus verifying that Q is an EMM.

Let there be an EMM $Q \in \mathcal{M}$. Then the discounted value of a replicating self-financing strategy $\frac{V_t^{\pi}}{\beta_t}$ in Theorem 2.1.4 is a Q-local martingale, since it is a stochastic integral with respect to the Q-martingale Z_t. But we would like it to be a martingale, because then the martingale property implies

$$\frac{V_t^{\pi}}{\beta_t} = \mathbf{E}_Q\left(\frac{V_T^{\pi}}{\beta_T} \middle| \mathcal{F}_t \right) = \mathbf{E}_Q\left(\frac{X}{\beta_T} \middle| \mathcal{F}_t \right), \tag{2.6}$$

where X is the claim (note that in the last equality, we have used the concept of *attainable* claim which will be defined later). The way to achieve this is to fix a reference EMM $Q \in \mathcal{M}$ and restricting attention to strategies $\hat{\pi}_t$ for which $\frac{V_t^{\hat{\pi}}}{\beta_t}$ is a martingale, not just a local martingale, under Q. This leads to the following definition for admissible strategy.

Definition 2.1.5. A predictable and self-financing strategy $\hat{\pi}_t = (\hat{a}_t, \hat{b}_t)$ is admissible if

$$\sqrt{\int_0^t \hat{a}_u^2 d[Z, Z]_u}, \quad 0 \le t \le T,$$

is finite and locally integrable under Q. Moreover, $Z_t = \frac{V_t^{\hat{\pi}}}{\beta_t}$ is a non-negative Q-martingale.

Remark 2.1. *There exists* `suicide strategy` *[34, p.300], a strategy with nonzero initial value but zero final value. Adding such a strategy to any* *other self-financing strategies will change the initial value but not the final value. Therefore, to exclude undesirable strategies from consideration, only martingale strategies are admissible.*

Pricing of Claims

On the other hand, a financial derivative, such as an option, has some value called *claim* or *payoff* at maturity T.

Definition 2.1.6. A claim X is a non-negative random variable. It is *attainable* if it is integrable, $\mathbf{E}(X) < \infty$, and there exist an admissible self-financing strategy $\hat{\pi}_t$ such that at maturity T, $V_T^{\hat{\pi}} = X$.

In this case, the admissible self-financing strategy $\hat{\pi}$ is said to replicate X. To avoid arbitrage, the price of the attainable claim X at any time $t < T$, denoted by C_t, must also be the same as that of the replicating strategy at time t, i.e., $C_t = V_t^{\hat{\pi}}$. Therefore, by martingale property of $\frac{V_t^{\hat{\pi}}}{\beta_t}$ and from (2.6) we have the following pricing formula with no arbitrage for the claim X.

Theorem 2.1.5 (Theorem 11.13 [34])**.** *The price C_t of an attainable claim X is given by the value of an admissible self-financing (replicating) strategy $V_t^{\hat{\pi}}$, which is equal to*

$$C_t = \mathbf{E}_Q\left(\frac{\beta_t}{\beta_T}X \middle| \mathcal{F}_t\right).$$

It is now easy to see that the price of a call option at time t, with claim at maturity given in (2.1), is equal to

$$C_t = e^{-r(T-t)}E_Q[(S_T - K)^+|\mathcal{F}_t], \tag{2.7}$$

where Q is an (not necessarily unique) EMM under which $e^{-rt}S_t$ is a martingale.

Remark 2.2. *The pricing formula* (2.7) *is also known as the risk-neutral pricing formula (e.g. [5, p.100]), since the price of the option is simply the expected payoff over a period $(T - t)$ discounted at the risk-free interest rate over that period.*

Remark 2.3. *The formula* (2.7) *is valid for any EMM's Q (see Chapter 4.1).*

Completeness of a Market Model

Definition 2.1.7. A market model is complete if any integrable claim is attainable.

The next theorem classify all claims that are attainable by using the predictable representation property of the discounted price process.

Theorem 2.1.6 (Theorem 11.14 [34])**.** *Let X be an attainable claim and let $M_t = \mathbf{E}_Q\big(\frac{X}{\beta_T}\big|\mathcal{F}_t\big)$, $t \in [0,T]$. Then X is attainable if and only if M_t admits an integral representation of the form*

$$M_t = M_0 + \int_0^t H_u dZ_u,$$

for some predictable process H_t. Moreover, $\frac{V_t^{\hat{\pi}}}{\beta_t} = M_t$ is the same for any admissible self-financing portfolio that replicates X.

In other words, in a complete market model, any claim can be replicated by an admissible self-financing portfolio and priced by no-arbitrage considerations, i.e., there exist an EMM Q under which $Z_t = \frac{S_t}{\beta_t}$ is a martingale. For a claim to be attainable, the martingale $M_t = \mathbf{E}_Q\big(\frac{X}{\beta_T}\big|\mathcal{F}_t\big)$ must have a predictable representation property with respect to the Q-martingale Z_t (Theorem 2.1.6). This implies that the martingale Z_t also has a predictable representation property, since a martingale has the predictable representation property if any other martingale can be represented as a stochastic integral with respect to it [34, p. 237]. The next theorem gives a full characterization of a complete market model which connects completeness to martingale measures and the predictable representation property of Z_t (e.g. [34, p. 302]).

Theorem 2.1.7. (Second Fundamental Theorem of Asset Pricing) *The following statements are equivalent:*

1. *The market model is complete.*

2. The martingale Z_t has the predictable representation property.

3. The EMM Q that makes $Z_t = \frac{S_t}{\beta_t}$ into a martingale is unique.

2.1.4 Change of Numeraire

In option pricing, it is sometimes convenient to change the numeraire as the choice of appropriate numeraire will provide the easiest calculations and the relevant hedging portfolio [24].

Definition 2.1.8. A numeraire is a price process β_t almost surely strictly positive for each $t \in [0, T]$.

An example of numeraire is the money account, i.e., $\beta_t = e^{rt}$, which we have let $\beta_0 = 1$. A change of numeraire does not change the self-financing portfolio of assets, and for every attainable claim in a given numeraire, it is also attainable in any other numeraire [24]. Hence, other probability measures that give prices of any security S relative to a numeraire of choice can be defined in a similar way as equivalent measures. In particular, the next theorem characterizes the change of numeraire (measure), which writes the stock price S_t as the numeraire, i.e., pricing through the reciprocal process (e.g. [34, p. 310])

Theorem 2.1.8. *Let Q and Q_1 be two probability measures defined on the same space, and let S_t/β_t, $0 \leq t \leq T$ be a positive Q-martingale. Define Q_1 by*

$$\frac{dQ_1}{dQ} = \Lambda_T = \frac{S_T/S_0}{\beta_T/\beta_0}. \tag{2.8}$$

Then β_t/S_t is a Q_1-martingale. Moreover, the price of an attainable claim X at time t is related under the different numeraire by the formula

$$C_t = \mathbf{E}_Q\left(\frac{\beta_t}{\beta_T}X\middle|\mathcal{F}_t\right) = \mathbf{E}_{Q_1}\left(\frac{S_t}{S_T}X\middle|\mathcal{F}_t\right). \tag{2.9}$$

2.2 Option Pricing with Log-symmetric Distribution in Discrete Time

This section contains a brief account of the works by Klebaner and Landsman [35].

2.2.1 Log-symmetric and Symmetric Distributions

In this subsection, we study the properties of log-symmetric and symmetric distributions that are used in [35] and also in this thesis later.

Definition 2.2.1. A random variable Y has a symmetric distribution if there is a number μ, called the location parameter, such that

$$-(Y-\mu) \stackrel{D}{=} (Y-\mu).$$

The characteristic function of a symmetric random variable Y with location μ and scale σ can be expressed in the form

$$\varphi_Y(u,\mu,\sigma,\psi) = e^{iu\mu}\psi\left(\frac{\sigma^2}{2}u^2\right), \tag{2.10}$$

where the function $\psi(u) : [0,\infty) \to \mathbb{R}$ is called the characteristic generator of the symmetric family (e.g. [20, p.32]).

Example 2.2.1. The normal family of distributions has the characteristic generator $\psi(u) = e^{-u}$.

In general, a member of the symmetric family of distributions need not have a density. But if the density exists, it takes the form

$$f_Y(y,\mu,\sigma,g) = c\frac{1}{\sigma}g\left(\frac{(y-\mu)^2}{2\sigma^2}\right),$$

where c is a constant, and the function $g(z)$ is known as the density generator of the symmetric family (e.g. [20], Chapter 2.2) that satisfies the condition

$$\int_0^\infty z^{-1/2} g(z) dz < \infty.$$

The normalizing constant c can be determined explicitly, which equals

$$c = \frac{1}{\sqrt{2}} \left(\int_0^\infty z^{-1/2} g(z) dz \right)^{-1}. \tag{2.11}$$

Example 2.2.2. For the normal family, the density generator is $g(z) = e^{-z}$ and the normalizing constant $c = \frac{1}{\sqrt{2}} \big(\Gamma(\frac{1}{2})\big)^{-1} = \frac{1}{\sqrt{2\pi}}$.

Moreover, if the density generator $g(z)$ satisfies the condition

$$\int_0^\infty g(z) dz < \infty,$$

then the symmetric distribution has a mean (e.g. [38]) which equals the location parameter μ, i.e., $\mathbf{E}(Y) = \mu$. If in addition the characteristic generator satisfies

$$|\psi'(0)| < \infty,$$

then the variance of the symmetric distribution exist and is equal to

$$Var(Y) = -\psi'(0)\sigma^2.$$

The characteristic generator can be chosen such that

$$\psi'(0) = -1,$$

then the variance becomes equal to the parameter σ^2, i.e., $Var(Y) = \sigma^2$. We shall always use such generators in this thesis.

The family of a symmetric distribution can be specified by the characteristic generator ψ or the density generator g. For a random variable Y from a symmetric family with mean μ, variance σ^2, we denote by $Y \sim S(\mu, \sigma^2, \psi)$ if the distribution is specified by the characteristic generator. If the density generator is used instead of the characteristic generator, then $Y \sim S(\mu, \sigma^2, g)$.

For the symmetric distributions considered in this thesis, we will specify the characteristic generator ψ due to its close relation with the Lévy-Khintchine representation.

Definition 2.2.2. Two symmetric distributions belong to the same family if and only if they have the same characteristic generator ψ.

The next two properties show that the symmetric family is closed under linear transformations.

Proposition 2.2.1. *All symmetric families are invariant under location shift.*

Proof. If $Y \sim S(\mu, \sigma^2, \psi)$ and c any constant, then $Y' = Y + c$ has characteristic function

$$\varphi_{Y'}(u) = \mathbf{E}\big(e^{iu(Y+c)}\big) = e^{iuc}\mathbf{E}\big(e^{iuY}\big) = e^{iu(\mu+c)}\psi\left(\frac{\sigma^2}{2}u^2\right),$$

which implies that $Y' \sim S(\mu + c, \sigma^2, \psi)$ from the same family. $\square$

Proposition 2.2.2. *Let $Y_1, \ldots, Y_n$ be n i.i.d random variables with each having symmetric distribution $S(\mu, \sigma^2, \psi)$, and $Z = Y_1 + \ldots + Y_n$. Then $Z \sim S(n\mu, n\sigma^2, \psi_n)$, where $\psi_n(u) = [\psi(\frac{u}{n})]^n$.*

Proof. By independence, the characteristic function of Z yields

$$\begin{aligned}
\varphi_Z(u) &= \mathbf{E}\big(e^{iu(Y_1+\ldots+Y_n)}\big) \\
&= \mathbf{E}\big(e^{iuY_1}\big)\cdots\mathbf{E}\big(e^{iuY_n}\big) \\
&= e^{iu\mu}\psi\left(\frac{\sigma^2}{2}u^2\right)\cdots e^{iu\mu}\psi\left(\frac{\sigma^2}{2}u^2\right) \\
&= e^{iu(n\mu)}\left[\psi\left(\frac{n\sigma^2}{2n}u^2\right)\right]^n \\
&= e^{iu(n\mu)}\psi_n\left(\frac{n\sigma^2}{2}u^2\right).
\end{aligned}$$

$\square$

For option pricing, we require that Y, which represents returns, to have finite first exponential moment, i.e.,

$$\mathbf{E}(e^{Y}) < \infty. \tag{2.12}$$

This implies that the moment generating function of Y, which can be obtained by

$$M_Y(u) = \mathbf{E}\big(e^{uY}\big) = \varphi_Y(-iu), \quad u \in \mathbb{R}.$$

exists and is finite for $u = 1$. As such, the condition (2.12) also extends the domain of the characteristic generator ψ of the symmetric distribution to the negative region (or at least to its subset), so the moment generating function of Y is given by (see (2.10))

$$M_Y(u) = e^{u\mu}\psi\left(-\frac{\sigma^2}{2}u^2\right). \tag{2.13}$$

Definition 2.2.3. A random variable X is said to have a log-symmetric distribution, denoted by $X \sim LS(\mu, \sigma^2, \psi)$, if the logarithm of X has a symmetric distribution, $Y = \ln(X) \sim S(\mu, \sigma^2, \psi)$.

If the moment generating function of Y exist for $u \leq 1$, then the mean of a log-symmetric distribution is finite [35] and is given by

$$\mathbf{E}(X) = e^{\mu}\psi\left(-\frac{\sigma^2}{2}\right). \tag{2.14}$$

The key observation useful for option pricing is the following property of log-symmetric distributions.

Proposition 2.2.3 (Proposition 3.1 [35])**.** *Let X be log-symmetric, $X \sim LS(\mu, \sigma^2, \psi)$. Then $\frac{1}{X}$ is also log-symmetric $LS(-\mu, \sigma^2, \psi)$.*

Proof. Let $X = e^Y$, where $Y \sim S(\mu, \sigma^2, \psi)$. Then $-Y \sim S(-\mu, \sigma^2, \psi)$ due to the definition of a symmetric distribution, $-(Y-\mu) \stackrel{D}{=} (Y-\mu) \sim S(0, \sigma^2, \psi)$. Hence, $\frac{1}{X} = e^{-Y} \sim LS(-\mu, \sigma^2, \psi)$.

□

2.2.2 Modified Option Pricing Formula in Discrete Time

In this subsection, we highlight some of the results in [35] that gave modifications to the Black-Scholes formula (1.1) for a variety of log-symmetric distributions in discrete time.

The model in [35] assumes a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ is a discrete filtration. The stock price process S_n is observed at discrete times $n = 0, 1, 2, \ldots, N$. The returns X_n, defined by

$$X_n = \frac{S_n}{S_{n-1}} \tag{2.15}$$

are strictly positive (prices are positive), independent and identically distributed with a log-symmetric distribution $LS(\mu, \sigma^2, \psi)$. The stock price at time N is given by the product of returns

$$S_N = S_0 \prod_{n=1}^{N} X_n.$$

While retaining the rest of the assumptions of the Black-Scholes model, the option price is then valued based on S_N by no-arbitrage approach. Their main result is an explicit formula for option pricing with log-symmetric distributions of the form

$$C_0(N) = S_0 Q_1\big(S_N > K\big) - e^{-rN} K Q\big(S_N > K\big),$$

where K is the strike, and Q and Q_1 are EMM's described below.

The choice of EMM is the one that keeps the returns in the same log-symmetric family of distributions. They called the EMM a *natural* EMM. Such EMM can be obtained by a natural change of measure, which is described in the next theorem with a proof. Adopting their notations, let us first denote by f_μ the density under probability measure P of the log-returns, $Y_n = \ln(X_n) \sim S(\mu, \sigma^2, \psi)$. The Density f is indexed only by the location parameter μ because it is a one-parameter family with σ and ψ fixed.

Theorem 2.2.1 (Theorem 3.1 [35]). *Let a measure Q be defined by*

$$\frac{dQ}{dP} = \Lambda_N = \prod_{n=1}^{N} \frac{f_{\mu^*}(Y_n)}{f_\mu(Y_n)}, \tag{2.16}$$

then Q is equivalent to P. Moreover, the returns $X_1, X_2, \dots, X_N$ remain independent and identically distributed under Q with the Q-density function of the returns in the same log-symmetric family with the location parameter μ replaced by μ^* and the density function f_{μ^*}.

Proof. This proof is due to Klebaner and Landsman [35]. Denote a set in $\mathbb{R}^N$ by $A = \{(u_1, \dots, u_N) : u_1 \le y_1, \dots, u_N \le y_N\}$ for some fixed $y_1, \dots, y_N \le \infty$, and for each $1 \le n \le N$, denote $A_n = \{u_n : u_n \le y_n\}$. The indicator function of A, $I_A\big((u_1, \dots, u_N)\big)$ is equal to 1 if $(u_1, \dots, u_N) \in A$ and 0 otherwise, has the property $I_A\big((u_1, \dots, u_N)\big) = \prod_{n=1}^N I_{A_n}(u_n)$. Consider the probability $Q(Y_1 \le y_1, \dots, Y_N \le y_N) = Q\big((Y_1, \dots, Y_N) \in A\big)$, we have

$$\begin{aligned}
Q(Y_1 \le y_1, \dots, Y_N \le y_N) &= \mathbf{E}_Q\Big(I_A\big((Y_1, \dots, Y_N)\big)\Big) \\
&= \mathbf{E}_p\Big(\Lambda_N I_A\big((Y_1, \dots, Y_N)\big)\Big) \\
&= \mathbf{E}_p\bigg(\prod_{n=1}^{N} \frac{f_{\mu^*}(Y_n)}{f_\mu(Y_n)} I_{A_n}(Y_n)\bigg).
\end{aligned}$$

The expectation can be obtained by integrating with respect to the joint P-density of $Y_1, \dots, Y_N$, which equals to $\prod_{n=1}^N f_\mu(u_n)$ by independence. Therefore we have

$$\begin{aligned}
Q(Y_1 \le y_1, \dots, Y_N \le y_N) &= \int \dots \int \prod_{n=1}^{N} \frac{f_{\mu^*}(u_n)}{f_\mu(u_n)} I_{A_n}(u_n) \prod_{n=1}^{N} f_\mu(u_n) du_1 \dots du_N. \\
&= \int \dots \int \prod_{n=1}^{N} f_{\mu^*}(u_n) I_{A_n}(u_n) du_1 \dots du_N. \\
&= \prod_{n=1}^{N} \int_{A_n} f_{\mu^*}(u_n) du_n.
\end{aligned}$$

Now taking all $y_i = \infty$ for $i \ne n$, we obtain that all the Y_n's are identically distributed in Q and have the density f_{μ^*} since

$$Q(Y_n \leq y_n) = \int_{A_n} f_{\mu^*}(u_n) du_n.$$

Putting this expression into the equation above, we obtain Q-independence

$$Q(Y_1 \leq y_1, \ldots, Y_N \leq y_N) = \prod_{n=1}^{N} Q(Y_n \leq y_n),$$

and this completes the proof. $\square$

It turns out that there are many equivalent measures that keep returns in the same log-symmetric family. However, with σ and ψ fixed, there is a unique EMM with the location parameter μ^* satisfying the following condition.

Theorem 2.2.2 (Theorem 3.2 [35]). *Let Q be defined by* (2.16). *For the discounted stock price process $\mathrm{e}^{-rn} S_n$, $n \leq N$, to be a martingale, it is necessary and sufficient that Q is risk-neutral, i.e.,*

$$\mu^* = r - \ln \psi(-\sigma^2/2). \tag{2.17}$$

Proof. By the properties of conditional expectation and (2.15), we have

$$\mathbf{E}_Q\big(e^{-r(n+1)} S_{n+1} \big| \mathcal{F}_n\big) = e^{-rn} S_n \mathbf{E}_Q\big(e^{-r} X_{n+1}\big) = r^{-rn} S_n \mathbf{E}_Q\big(e^{-r} X_1\big),$$

where the last equality is because the returns are identically distributed under Q. It can be seen that for $e^{-rn} S_n$ to be a Q-martingale, it is necessary and sufficient that

$$\mathbf{E}_Q(X_1) = e^r.$$

By (2.14), the claim follows.

$\square$

By a change of numeraire, another natural EMM Q_1 which is defined by

$$\frac{dQ_1}{dQ} = \Lambda = \frac{e^{-rN} S_N}{S_0}, \tag{2.18}$$

was introduced under which e^{rN}/S_N is a martingale. The necessary and sufficient condition for Q_1 to be a martingale measure is given in the theorem below.

Theorem 2.2.3 (Theorem 3.2 [35]). *Let Q_1 be defined by* (2.18)*. For the reciprocal process e^{rn}/S_n, $n \leq N$, to be a martingale, it is necessary and sufficient that*

$$\mu_1^* = r + \ln \psi(-\sigma^2/2). \tag{2.19}$$

Proof. Similar to Theorem 2.2.2, we obtain that the necessary and sufficient condition for e^{rn}/S_n to be a Q_1-martingale is

$$\mathbf{E}_{Q_1}\left(\frac{1}{X_1}\right) = e^{-r}.$$

By Proposition 2.2.3 and (2.14), the claim follows.

$\square$

The next theorem is the main result of [35], which gives the option pricing formula in terms of the natural EMM's Q and Q_1.

Theorem 2.2.4 (Theorem 3.3 [35]). *Let $X = e^Y \sim LS(\mu, \sigma^2, \psi)$ with $\mathbf{E}(X) < \infty$. Then the arbitrage-free price of a call option with N periods to expiration is given by*

$$C_0(N) = S_0 Q_1\big(S_N > K\big) - e^{-rN} K Q\big(S_N > K\big), \tag{2.20}$$

where the Q-distribution of Y is $S(\mu^*, \sigma^2, \psi)$ with $\mu^* = r - \ln \psi(-\sigma^2/2)$, and the Q_1-distribution of Y is $S(\mu_1^*, \sigma^2, \psi)$ with $\mu_1^* = r + \ln \psi(-\sigma^2/2)$.

Apply Theorems 2.2.2, 2.2.3 and the central limit theorem to $\ln(S_N)$ for Q and Q_1 in the formula (2.20), the modified option pricing formula that gives a correction to the Black-Scholes formula is obtained.

Proposition 2.2.4. *For large N,*

$$C_0(N) \approx S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \ln\psi(-\frac{\sigma^2}{2})\right)N}{\sigma\sqrt{N}}\right)$$
$$- e^{-rN} K \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \ln\psi(-\frac{\sigma^2}{2})\right)N}{\sigma\sqrt{N}}\right). \tag{2.21}$$

If returns follows a lognormal distribution where the characteristic generator for normal family is $\psi(u) = e^{-u}$, then the modified option pricing formula (2.21) is identical to the Black-Scholes formula (1.1) for any N.

Modified option pricing formulae for various other log-symmetric distributions were obtained in [35] as a direct application of (2.21). The following case is of particular importance.

Corollary 2.2.1. *Let returns follow a log-mixture of two normal distributions, $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, with contamination parameter $0 < \epsilon < 1$. Then the modified option pricing formula* (2.21) *is given by*

$$C_0(N) \approx S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \ln\left((1-\epsilon)e^{\sigma_1^2/2} + \epsilon e^{\sigma_2^2/2}\right)\right)N}{\sqrt{(1-\epsilon)\sigma_1^2 + \epsilon\sigma_2^2}\sqrt{N}}\right)$$
$$- e^{-rN} K \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \ln\left((1-\epsilon)e^{\sigma_1^2/2} + \epsilon e^{\sigma_2^2/2}\right)\right)N}{\sqrt{(1-\epsilon)\sigma_1^2 + \epsilon\sigma_2^2}\sqrt{N}}\right).$$

Numerical results showed that the Black-Scholes formula generally underprices options. Compared to the modified formula for log-mixture of normal distributions, the difference can be as much as 30% when the ratio $k = \frac{\sigma_2}{\sigma_1}$ increases even for small percentage of contamination ϵ.

2.3 Lévy Processes

In this section, we study some of the important properties of Lévy processes. The cases of symmetric Lévy processes, VG process, NIG process and Lévy processes in discrete time are given due focus with a subsection dedicated to each.

2.3.1 Definitions and Properties

A Lévy process is defined as the following (e.g. [10, p.69]).

Definition 2.3.1. (Lévy Process) A cádlág (right-continuous with left limits) stochastic process $Y_t,\ t \geq 0$, on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}$ such that $Y_0 = 0$ is called a Lévy process if it possesses the following properties:

1. Independent increments: for every increasing sequence of times $\{t_k\}$, $k = 0, 1, \ldots, n$, the random variables $Y_{t_0},\ Y_{t_1} - Y_{t_0}, \ldots,\ Y_{t_n} - Y_{t_{n-1}}$ are independent.

2. Stationary increments: the law of $Y_{t+h} - Y_t$ does not depend on t.

3. Stochastic continuity:
$$\forall \epsilon > 0,\ \lim_{h \to 0} P(|Y_{t+h} - Y_t| \geq \epsilon) = 0.$$

The third condition also means that the jump discontinuities happen at random times as the probability of seeing a jump at any particular time t is zero.

There is a close relationship between Lévy processes and *infinitely divisible* distributions.

Definition 2.3.2. (Infinite Divisibility) A probability distribution F on $\mathbb{R}$ is said to be infinitely divisible if for any integer $n \geq 2$, there exists n i.i.d random variables $\xi_1, \ldots, \xi_n$ such that the sum $\xi_1 + \ldots + \xi_n$ has distribution F.

Example 2.3.1. The normal distribution $N(\mu, \sigma^2)$ is infinitely divisible because the sum of n random variables $\xi_k \sim N(\mu/n, \sigma^2/n)$, $k = 1, \ldots, n$, has the original normal distribution, i.e.,

$$\xi_1 + \ldots + \xi_n \sim N(\mu/n + \ldots + \mu/n, \sigma^2/n + \ldots + \sigma^2/n) \stackrel{D}{=} N(\mu, \sigma^2).$$

The next proposition describes the relationship (e.g. [10, p.70]).

Proposition 2.3.1. *Let Y_t, $t \geq 0$, be a Lévy process. Then for every t, Y_t has an infinitely divisible distribution. Conversely, if F is an infinitely divisible distribution, then there exists a Lévy process Y_t such that the distribution of Y_1 is given by F.*

The characteristic function of infinitely divisible distributions is given by the celebrated Lévy-Khintchine representation (e.g. [49], Theorem 8.1). Due to the relationship given in Proposition 2.3.1, the characteristic function of a Lévy process has a specific form (e.g. [10, p.85]).

Theorem 2.3.1. (Lévy-Khintchine representation for Lévy process) *Let Y_t, $t \geq 0$ be a Lévy process on $\mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. There exist a triplet (a, c, ν) with $a \in \mathbb{R}$, $c \geq 0$ and ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |y|^2)\nu(dy) < \infty$, such that*

$$\mathbf{E}\left(e^{iuY_t}\right) = e^{t\Psi(u)}, \ u \in \mathbb{R},$$

where

$$\Psi(u) = iau - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\right)\nu(dy) \tag{2.22}$$

is called the characteristic exponent.

In Theorem 2.3.1, it is not hard to see that the characteristic function of Y_1 is $e^{\Psi(u)}$. Therefore, a Lévy process is fully determined by its initial value ($Y_0 = 0$), and the distribution of the increment over one unit of time, Y_1. The triplet (a, c, ν) is referred to as the *characteristic triplet* of the Lévy process Y_t, where a is the *drift*, c is the *Gaussian* or *diffusion coefficient* and ν is the *Lévy measure.*

Definition 2.3.3. The Lévy process is without a Brownian component if $c = 0$, i.e., a *pure jump* process.

Intuitively, the Lévy measure $\nu(A)$ gives the expected number of jumps whose size belongs to A in a time interval of unit length. Many useful information regarding the richness of the class of Lévy processes and the structure of the Lévy process can be derived from the integrability properties of the Lévy measure.

Definition 2.3.4. A Lévy process is said to have *finite activity* if, almost all paths have a finite number of jumps on every compact interval. On the other hand, if almost all paths have infinite number of jumps on every compact interval, then the Lévy process has *infinite activity.*

Proposition 2.3.2. *Let Y_t, $t \geq 0$, be a Lévy process with characteristic triplet (a, c, ν).*

(1) If $\nu(\mathbb{R}) < \infty$, then Y_t has finite activity.

(2) If $\nu(\mathbb{R}) = \infty$, then Y_t has infinite activity.

Proof. See [49], Theorem 21.3. □

Whether the variation of the sample paths of Y_t is finite or infinite also depends on the Lévy measure (and on the presence or absence of a Brownian component). Recall that the total variation of a function $f : [a, b] \to \mathbb{R}$ is defined by (e.g. [10, p.88])

$$\mathcal{V}_t(f) = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a, b]$.

Proposition 2.3.3. *Let Y_t, $t \geq 0$, be a Lévy process with characteristic triplet (a, c, ν). Then almost all paths of Y_t have*

(1) finite variation if $c = 0$ and $\int_{|y|\leq 1} |y|\nu(dy) < \infty$.

(2) infinite variation if $c \neq 0$ or $\int_{|y|\leq 1} |y|\nu(dy) = \infty$.

Proof. See e.g. [10], Proposition 3.9 or [49], Theorem 21.9. □

For option pricing, a necessary condition for the discounted stock price process $e^{-rt}S_t = e^{-rt}S_0e^{Y_t}$ to be a martingale is that the Lévy process Y_t has finite first exponential moment (which implies finite moments of all order)

$$\mathbf{E}\big(e^{Y_t}\big) < \infty.$$

In this case, $\mathbf{E}\big(e^{Y_t}\big) = e^{t\Psi(-i)}$ where Ψ is the characteristic exponent (2.22) (e.g [10, p.95]). The next proposition characterizes such Lévy processes in terms of their Lévy measure.

Proposition 2.3.4. *Let Y_t, $t \geq 0$, be a Lévy process with characteristic triplet (a, c, ν). Then*

(1) Y_t has finite p-th moment for $p \in \mathbb{R}^+$, i.e., $\mathbf{E}\big(|Y_t|^p\big) < \infty$, if and only if $\int_{|y|>1} |y|^p \nu(dy) < \infty$.

(2) Y_t has finite p-th exponential moment for $p \in \mathbb{R}$, i.e., $\mathbf{E}(e^{pY_t}) < \infty$, if and only if $\int_{|y|>1} e^{py}\nu(dy) < \infty$.

Proof. See [49], Theorem 25.3. $\square$

Consequently, Such Lévy processes have finite mean and variance of the following form (see e.g. [10], Proposition 3.13).

Proposition 2.3.5. *For a Lévy process with finite exponential moment, the mean and variance are, respectively,*

$$\mathbf{E}(Y_t) = t\Bigg(a + \int_{|y|>1} y\nu(dy)\Bigg), \tag{2.23}$$

$$Var(Y_t) = t\left(c^2 + \int_{\mathbb{R}} y^2\nu(dy)\right). \tag{2.24}$$

A Lévy process can be decomposed into four independent components: a linear drift, a Brownian motion, a compound Poisson process and a pure jump martingale (e.g. [10, p.81]).

Theorem 2.3.2. (Lévy-Itô decomposition) *Let Y_t be a Lévy process on $\mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and ν its Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |y|^2)\nu(dy) < \infty$. Denote by J the jump measure (Poisson random measure) of Y on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure $\nu(dy)dt$. There exists $a \in \mathbb{R}$ and a Brownian motion B_t such that*

$$
\begin{aligned}
Y_t = at + B_t + \int_0^t \int_{|y|>1} yJ(ds, dy) \\
+ \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon<|y|\leq 1} y\big(J(ds, dy) - \nu(dy)dt\big),
\end{aligned}
\tag{2.25}
$$

where the convergence in the last term is almost sure and uniform in $t \in [0, T]$.

Consequently, the canonical form of a Lévy process Y_t with characteristic triplet (a, c, ν) is

$$
\begin{aligned}
Y_t = at + cW_t + \int_0^t \int_{|y|>1} yJ(ds, dy) \\
+ \left(\int_0^t \int_{|y|\leq 1} yJ(ds, dy) - t \int_{|y|\leq 1} y\nu(dy) \right),
\end{aligned}
\tag{2.26}
$$

where W_t is the 1-dimensional standard Brownian motion (Wiener process). For a Lévy process with finite first moment, its canonical form is (e.g. [10, p.85]) given by

$$
Y_t = \mu t + cW_t + \left(\int_0^t \int_{\mathbb{R}} yJ(ds, dy) - t \int_{\mathbb{R}} y\nu(dy) \right),
\tag{2.27}
$$

where $\mu = a + \int_{|y|>1} y\nu(dy) = \mathbf{E}(Y_1)$ (see (2.23)). Its characteristic triplet becomes (μ, c, ν) and the corresponding Lévy-Khintchine formula takes the form $e^{t\Psi(u)}$ where

$$
\Psi(u) = i\mu u - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy)\nu(dy).
\tag{2.28}
$$

2.3.2 Constructing Lévy Processes

Several common methods of constructing a Lévy process are described below. Other methods can be found in [10] (Chapter 4). Naturally, some processes can be constructed by more than one methods.

(1) Specifying a Lévy triplet (μ, c, ν)

This method construct a Lévy process by specifying the presence or absence of a drift, a Brownian component and what is the Lévy measure.

Example 2.3.2. The Lévy triplet $(0, 1, 0)$ represents the standard Brownian motion, and the Lévy triplet $(0, 0, \lambda\delta_1)$, where δ_1 is the Dirac delta measure with unit mass at 1, represents the Poisson process with rate λ.

Example 2.3.3. A Lévy process with characteristic triplet $(a, 0, \nu)$ is a pure jump process. Moreover, if the drift $a = \frac{\delta}{\rho}\big(2\Phi(\rho) - 1\big)$, where $\delta, \rho > 0$ and Φ is the standard normal distribution function, and the Lévy measure

$$\nu(dy) = \frac{1}{\sqrt{2\pi(\delta y)^3}} e^{-\frac{1}{2}\rho^2 y} 1_{\{y>0\}} dy,$$

then it is the inverse Gaussian process [50, p. 53].

(2) Specifying the density of the increments at time scale 1

This method utilizes the fact that a Lévy process is fully determined by its initial value and the distribution of Y_1. A Lévy process is generated when it is taken along time intervals of length 1 with $Y_0 = 0$ and $Y_t - Y_{t-1} \stackrel{D}{=} Y_1$.

Example 2.3.4. If $Y_1 \sim N(\mu, \sigma^2)$, it generates a Lévy process Y_t, $t \geq 0$, that has marginal distribution $N(\mu t, \sigma^2 t)$ which is equivalent to a Brownian motion with drift, i.e., $Y_t = \mu t + \sigma W_t$.

Example 2.3.5. The normal inverse Gaussian (NIG) process introduced by Barndorff-Nielsen ([2], [3]) is a Lévy process generated by a NIG distribution at time scale 1, $Y_1 \sim NIG(\alpha, \beta, \delta, \mu)$. Hence, the NIG process $Y_t \sim NIG(\alpha, \beta, \delta t, \mu t)$ for all t. More details of this process in Subsection 2.3.5.

(3) Adding a linear drift

A linear drift does not effect the infinite divisibility property nor the self-decomposability of the marginal distribution of a Lévy process ([50], p.67). Therefore, we can construct a new Lévy process from a known Lévy process by altering its drift.

Example 2.3.6. Let X_t be a Lévy process with characteristic triplet (a, c, ν). Define the process

$$Y_t = X_t + mt,$$

then Y_t is a Lévy process with characteristic triplet $(a + m, c, \nu)$.

(4) Time-changing Brownian motion with an independent subordinator

This method is also known as subordination. In this method, a Lévy process is constructed by substituting the (calendar) time t of a Brownian motion B_t by an independent Lévy process called the *subordinator*.

Definition 2.3.5. A subordinator is an a.s. increasing Lévy process, $Y_t \geq 0$ for every $t > 0$. Equivalently, the characteristic triplet (a, c, ν) of a subordinator must satisfy $a \geq 0$, $c = 0$, $\int_{(-\infty,0)} \nu(dy) = 0$ and $\int_{(0,1]} y\nu(dy) < \infty$, i.e., subordinator has no diffusion component, only positive drift and positive jumps of finite variation.

Example 2.3.7. The inverse Gaussian process in Example 2.3.3 is a subordinator. This can be verified directly as it has no diffusion component, the drift is positive

$$a = \frac{\delta}{\rho}\big(2\Phi(\rho) - 1\big) > 0,$$

and the jumps are positive with finite variation since $\int_{(-\infty,0)} \nu(dy) = 0$, and

$$\begin{aligned}
\int_0^1 y\nu(dy) &< \int_0^\infty y \frac{1}{\sqrt{2\pi(\delta y)^3}} e^{-\frac{1}{2}\rho^2 y} dy \\
&= \frac{1}{\sqrt{2\pi\delta^3}} \int_0^\infty y^{-\frac{1}{2}} e^{-\frac{1}{2}\rho^2 y} dy \\
&= \frac{1}{\sqrt{2\pi\delta^3}} \frac{1}{\sqrt{\rho^2/2}} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt \\
&= \frac{1}{\sqrt{\pi\delta^3\rho^2}} \Gamma\left(\frac{1}{2}\right) \\
&< \infty.
\end{aligned}$$

Next is an example of a time-changed Brownian motion: NIG process.

Example 2.3.8. Let $Z(t;\delta,\rho)$ be an inverse Gaussian process with parameters δ and ρ, which is a subordinator as seen in Example 2.3.7. Let $B_t = B(t;\beta,1)$ denotes a Brownian motion with drift β and diffusion coefficient 1. Then the NIG process, denoted by $Y_t = Y(t;\alpha,\beta,\delta,\mu)$, can be obtained by time-changing a Brownian motion as follows [3]:

$$Y_t = B_{Z_t} + \mu t = \beta Z_t + W_{Z_t} + \mu t, \tag{2.29}$$

where W_t is the standard Brownian motion, and $Z_t = Z(t;\delta,\sqrt{\alpha^2-\beta^2})$ is an inverse Gaussian process with the given parameters, independent of B_t.

Remark 2.4. *The subordination method also applies to time-changing other Lévy processes (see [10], Theorem 4.2).*

2.3.3 Symmetric Lévy processes

There are many ways to define a symmetric Lévy process. For example, [18] defined symmetry as when a certain law of the Lévy process before and after the change of measure through Girsanov's theorem coincide. In our work, we assume the Lévy process has finite mean and take the following classical definition of symmetry for distributions.

Definition 2.3.6. A Lévy process is said to be symmetric if its marginal distributions are symmetric, i.e., $-(Y_1 - \mu) \stackrel{D}{=} (Y_1 - \mu)$ where $\mu = \mathbf{E}(Y_1)$.

This is equivalent to saying that a symmetric Lévy process has a symmetric Lévy measure (symmetrical about 0). Recall the characteristic triplet of a Lévy process with finite mean is (μ, c, ν) where $\mu = \mathbf{E}(Y_1)$.

Proposition 2.3.6. *Let* Y_t *is a Lévy process with characteristic triplet* (μ, c, ν) *where* μ *is the mean of* Y_1. *Then* Y_t *is symmetric if and only if its Lévy measure* ν *is a symmetric measure with* $\nu(-A) = \nu(A)$, *where* $-A = \{x \in \mathbb{R} : -x \in A\}$.

Proof. It is easy to see then $(Y_1 - \mu)$ is a Lévy process with characteristic triplet $(0, c, \nu)$, and $-(Y_1 - \mu)$ is a Lévy process with characteristic triplet $\big(0, c, \nu\big)$. By Lévy-Khintchine representation (2.28), the corresponding characteristic functions are

$$\mathbf{E}\left[e^{iu(Y_1-\mu)}\right] = \exp\left(-\frac{1}{2}c^2u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right)\nu(dy)\right),$$
$$\mathbf{E}\left[e^{iu[-(Y_1-\mu)]}\right] = \exp\left(-\frac{1}{2}c^2u^2 + \int_{\mathbb{R}} \left(e^{iu(-y)} - 1 - iu(-y)\right)\nu(dy)\right).$$

The two characteristic functions are the same (and therefore have the same distribution) if and only if $\nu(-dy) = \nu(dy)$.

□

Remark 2.5. *In ([49], p.263), a symmetric Lévy process is generated by a characteristic triplet of the form* $(0, c, \nu)$ *where* ν *is symmetric, i.e., both the process and its Lévy measure are symmetric about the origin. The symmetric Lévy process in Proposition 2.3.6 consists of a linear drift* μt *and another symmetric Lévy process with characteristic triplet* $(0, c, \nu)$ *for any time* t.

When ν is symmetric, the characteristic exponent of a Lévy process simplifies (e.g. [49], p.263):

Proposition 2.3.7. *The characteristic exponent of a symmetric Lévy process Y_t with characteristic triplet (μ, c, ν) can be written as*

$$\Psi(u) = iu\mu - \frac{1}{2}c^2u^2 - 2\int_0^\infty (1 - \cos uy)\nu(dy). \tag{2.30}$$

Proof. For a symmetric Lévy process with characteristic triplet (μ, c, ν), the first two terms of its characteristic exponent are straight forward. The third term can be simplified as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\right)\nu(dy) \\
&= \int_{\mathbb{R}^-} \left(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\right)\nu(dy) + \int_{\mathbb{R}^+} \left(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\right)\nu(dy) \\
&= \int_{\mathbb{R}^+} \left(e^{-iuy} - 1 + iuy1_{\{|y|\leq 1\}}\right)\nu(-dy) + \int_{\mathbb{R}^+} \left(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\right)\nu(dy) \\
&= \int_{\mathbb{R}^+} \left(e^{iuy} + e^{-iuy} - 2\right)\nu(dy) \\
&= -2\int_0^\infty (1 - \cos uy)\nu(dy).
\end{aligned}$$

□

2.3.4 Variance Gamma Process

There are several ways of describing a Variance Gamma (VG) process. The one we present below is by time-changing a Brownian motion with drift by a gamma subordinator [39]. Alternatively, a VG process can be described as the difference of two independent increasing gamma processes ([39], [41]), or by specifying an alternative form of Lévy measure for the VG process [40].

Gamma Subordinator

The Gamma process with mean rate α and variance rate κ, denoted by $\gamma(t; \alpha, \kappa)$, is the process of independent gamma increments over non-overlapping time intervals $(t, t + h)$. The increment $g = \gamma(t + h; \alpha, \kappa) - \gamma(t; \alpha, \kappa)$ has a gamma distribution with mean αh and variance κh whose density, denoted

$f_h(g)$, is given by [39]

$$f_h(g) = \left(\frac{\alpha}{\kappa}\right)^{\beta} \frac{g^{\beta-1}e^{-\frac{\alpha}{\kappa}g}}{\Gamma(\beta)}, \quad g > 0, \tag{2.31}$$

where $\beta = \frac{\alpha^2 h}{\kappa}$ and $\Gamma(\beta)$ is the gamma function. The characteristic function of gamma density is given by

$$\varphi_{\gamma(t)}(u) = \mathbf{E}(e^{iu\gamma(t;\alpha,\kappa)}) = \left(\frac{1}{1 - iu\frac{\kappa}{\alpha}}\right)^{\frac{\alpha^2 t}{\kappa}}.$$

The Lévy measure of gamma process [39] is given explicitly by

$$\nu_\gamma(g)dg = \frac{\alpha^2 \exp\left(-\frac{\alpha}{\kappa}g\right)}{\kappa g} 1_{\{g>0\}}dg.$$

The Lévy measure has infinite mass and therefore the gamma process has an infinite arrival rate of (positive) jumps, mostly small, as indicated by the concentration of the Lévy measure at the origin.

VG Process as Time-changed Brownian Motion

Let $B(t;\theta,\sigma)$ denote a Brownian motion with drift θ and volatility σ, i.e.,

$$B(t,\theta,\sigma) = \theta t + \sigma W_t,$$

where W_t is a standard Brownian motion. A VG process $Y(t;\theta,\sigma,\kappa)$ is obtained by time-changing a Brownian motion $B(t;\theta,\sigma)$ with a gamma subordinator of unit mean rate $\gamma(t;1,\kappa)$ [39]:

$$Y(t;\theta,\sigma,\kappa) = B\big(\gamma(t;1,\kappa);\theta,\sigma\big) = \theta\gamma(t;1,\kappa) + \sigma B\big(\gamma(t;1,\kappa);0,1\big). \tag{2.32}$$

The process provides two additional dimensions of control on the distribution over and above that of the volatility: the control over skewness via θ and over kurtosis with κ. The density function for the marginal distribution of VG process at time t [39] can be obtained by first expressing the conditional density conditioned on the realization of the gamma time change g as a Normal density function, then integrating out g using the density of gamma distribution (2.31). This gives the density function for the marginal distribution of VG process Y_t as

$$f_{Y_t}(y) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(y-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\kappa}-1} e^{-\frac{g}{\kappa}}}{\kappa^{\frac{t}{\kappa}}\Gamma\left(\frac{t}{\kappa}\right)} dg. \tag{2.33}$$

The characteristic function of the VG process Y_t is given by

$$\mathbf{E}(e^{iuY_t}) = \left(\frac{1}{1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u} \right)^{\frac{t}{\kappa}}. \tag{2.34}$$

The VG process is a pure jump process with characteristic triplet $(\theta, 0, \nu)$, where the Lévy measure ν, in terms of (θ, σ, κ), is given by [39]

$$\nu(y) = \frac{C}{|y|} e^{-\alpha_-|y|} 1_{\{y<0\}} + \frac{C}{y} e^{-\alpha_+ y} 1_{\{y>0\}}, \tag{2.35}$$

where $C = \frac{1}{\kappa}$, $\alpha_- = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2} + \frac{\theta}{\sigma^2}$ and $\alpha_+ = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2} - \frac{\theta}{\sigma^2}$.

Explicit expression of mean, variance, skewness and kurtosis of the VG process at time $t = 1$ are given in [50, p.58]. Specifically,

$$\text{Mean} = \theta \tag{2.36}$$

$$\text{Variance} = \sigma^2 + \kappa\theta^2 \tag{2.37}$$

$$\text{Skewness} = \kappa\theta(3\sigma^2 + 2\kappa\theta^2)(\sigma^2 + \kappa\theta^2)^{-3/2} \tag{2.38}$$

$$\text{Kurtosis} = 3\big(1 + 2\kappa - \kappa\sigma^4(\sigma^2 + \kappa\theta^2)^{-2}\big). \tag{2.39}$$

When $\theta = 0$, it is clear from (2.36) and (2.38) that there is no skewness and the VG process is symmetrical about zero. We will largely work with symmetric VG process in our model in Chapter 5. The characteristics of a symmetric VG process and its marginals are given in Section 5.1.

2.3.5 Normal Inverse Gaussian Process

The Lévy process with normal inverse Gaussian (NIG) marginal distributions is known as a NIG Lévy process or motion. The NIG distribution was first introduced by Barndorff-Nielsen [2] as a subclass of the Generalized Hyperbolic distribution with parameter $\lambda = -\frac{1}{2}$. Denote a random variable with NIG distribution by $Y \sim NIG(\alpha, \beta, \delta, \mu)$ where μ is the location parameter, δ is the scale parameter, α is the shape parameter and β is for skewness, the density of Y is given by [3]

$$f_Y(y) = \frac{\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(y-\mu)} \frac{K_1\Big(\alpha\delta\sqrt{1+(\frac{y-\mu}{\delta})^2}\Big)}{\sqrt{1+(\frac{y-\mu}{\delta})^2}}, \tag{2.40}$$

where K_1 is the modified Bessel function of the third kind, $y, \mu \in \mathbb{R}$, $\delta > 0$ and $0 \leq |\beta| < \alpha$. The characteristic function of NIG distribution is [3]

$$\varphi_Y(u) = e^{iu\mu} \frac{e^{\delta\sqrt{\alpha^2-\beta^2}}}{e^{\delta\sqrt{\alpha^2-(\beta+iu)^2}}}, \tag{2.41}$$

and the mean, variance, skewness and kurtosis of NIG distribution are (e.g. [50, p.60] or [47])

$$\text{Mean} = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \tag{2.42}$$

$$\text{Variance} = \frac{\delta\alpha^2}{\big(\sqrt{\alpha^2-\beta^2}\big)^3}. \tag{2.43}$$

$$\text{Skewness} = \frac{3\beta}{\alpha\big(\delta\sqrt{\alpha^2-\beta^2}\big)^{\frac{1}{2}}} \tag{2.44}$$

$$\text{Kurtosis} = 3\left(1 + \frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right). \tag{2.45}$$

The NIG distribution is infinitely divisible and closed under convolution [3]. In Example 2.3.5, we have seen that a Lévy process taken along time intervals of length 1 with $Y_0 = 0$ and $Y_t - Y_{t-1} \stackrel{D}{=} Y_1 \sim NIG(\alpha, \beta, \delta, \mu)$ generates a NIG Lévy process $Y_t \sim NIG(\alpha, \beta, \delta t, \mu t)$, and its characteristic function

$$\varphi_{Y_t}(u) = \left[\varphi_{Y_1}(u)\right]^t = e^{iu\mu t}\frac{e^{\delta t\sqrt{\alpha^2-\beta^2}}}{e^{\delta t\sqrt{\alpha^2-(\beta+iu)^2}}}.$$

Alternatively, a NIG process can be constructed by time-changing a Brownian motion with drift by an IG subordinator (see Example 2.3.8 or [3]). The NIG process is also a pure jump process, and its Lévy measure is given by [3]

$$\nu(dy) = \frac{\alpha\delta}{\pi|y|}e^{\beta y}K_1\big(\alpha|y|\big)dy.$$

When $\beta = 0$ (skewness (2.44) is zero), the NIG process is symmetric about μ. The characteristics of a symmetric NIG process and its marginals are given in Section 6.1 where the symmetric NIG model is considered.

2.3.6 Lévy Processes in Discrete Time

In the discrete time setting, suppose we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ in discrete time.

Definition 2.3.7. A discrete time Lévy process Y_n, $n \in \mathbb{N}$ is a process adapted to $\mathbb{F}$ with stationary independent increments.

Note that the notion of right-continuous with left limits has no significance here. A discrete time Lévy process can be expressed as the cumulative sum of all the increments (e.g. [31], p.93)

$$Y_N = \sum_{n=1}^{N} \Delta Y_n,$$

where $\Delta Y_n = Y_n - Y_{n-1}$ denotes the n-th increment. All ΔY_n, $n = 1, \dots, N$ are independent and have the same distribution as the distribution of ΔY_1. If $\Delta Y_1 \sim S(\mu, \sigma^2, \psi)$, then the Lévy process in discrete time Y_N is symmetric. Denote by $f(y_n)$ the density function of ΔY_n. By independence, the joint density of $\Delta Y_1, \dots, \Delta Y_N$ is

$$f(y_1, \dots, y_N) = \prod_{n=1}^{N} f(y_n). \tag{2.46}$$

Remark 2.6. *Intuitively, a discrete time Lévy process can be seen as a discrete time series drawn from a continuous time Lévy process at equidistant time-points. It is a special case contained in the continuous time case. A discrete time model is important for applications, as the empirical data of stock prices is essentially a discrete time series. It can also serve as an approximation to the continuous time model.*

Chapter 3

The Natural Change of Measure

This chapter addresses the issue of finding a natural equivalent martingale measure for option pricing with log-symmetric Lévy price processes. We first investigate the continuous time model followed by the discrete time model. Our approach to construct a natural change of measure starts with the Girsanov theorem for Lévy processes which gives the characteristics of the Lévy processes under equivalent measures. Then for symmetric Lévy processes, additional constrains on these characteristics are imposed to ensure the Lévy processes remain symmetric and have a law that remains in the same family of symmetric distributions. The main results for the natural change of measure are Theorems 3.2.3 and 3.2.4 for the continuous time case, and Theorems 3.3.3 and 3.3.4 for the discrete time case.

3.1 Symmetric Lévy Processes and Marginal Distributions

In this section, we identify the relations between the characteristic triplet (μ, c, ν) of a symmetric Lévy process and the parameters of the symmetric distribution $S(\mu, \sigma^2, \psi)$.

Recall that a symmetric Lévy process Y_t has symmetric marginals, i.e.,

$-(Y_1-\mu) \stackrel{D}{=} (Y_1-\mu)$ (see Definition 2.3.6). In what follows we consider Y_1 with finite mean and variance (in fact with finite exponential moments). By the properties of Lévy process (Proposition 2.3.5), it can be seen that the mean of Y_1 is μ,

$$\mathbf{E}(Y_1) = \mu, \tag{3.1}$$

and the variance is σ^2 given by

$$\sigma^2 = \mathrm{Var}(Y_1) = c^2 + \int_{\mathbb{R}} y^2 \nu(dy). \tag{3.2}$$

The marginal distribution for Y_1 is symmetric $S(\mu, \sigma^2, \psi)$ (see Section 2.2.1 for properties of symmetric distributions). The following proposition gives the relations between the two sets of parameters.

Proposition 3.1.1. *The relations between* (μ, c, ν) *and the parameters in distribution* $S(\mu, \sigma^2, \psi)$ *of* Y_1 *are given by equations* (3.1) *and* (3.2), *and* (3.3) *below*

$$\psi(v) = \exp\left\{ -\frac{c^2 v}{\sigma^2} - 2\int_0^\infty \left(1-\cos(y\sqrt{2v}/\sigma)\right)\nu(dy) \right\}, \tag{3.3}$$

where $v = \frac{\sigma^2 u^2}{2}$. *Furthermore, the distribution of* Y_t *is also symmetric from the family* $S(\mu t, \sigma^2 t, \psi_t)$, *with*

$$\psi_t(v) = \left[\psi(\frac{v}{t})\right]^t. \tag{3.4}$$

Proof. Follows from the characteristic functions (2.10) and (2.30). The form of ψ_t is due to

$$\mathbf{E}(e^{iuY_t}) = \left[\mathbf{E}(e^{iuY_1})\right]^t = \varphi^t_{Y_1}(u) = e^{iu\mu t}\psi^t\left(\frac{\sigma^2 t}{2t}u^2\right).$$

□

3.2 Equivalent Change of Measure for Lévy processes

In general, a Lévy process under an equivalent measure need not remain a Lévy process, as independence of increments is not preserved (e.g. [10, p.322]). However, there is a class of equivalent measures under which the process remains a Lévy process. The following theorem gives such measures and shows how the characteristic triplet is transformed.

Theorem 3.2.1. *Let Y_t be a Lévy process on $\mathbb{R}$ with characteristic triplet (μ, c, ν) under P, $\eta \in \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\int_{\mathbb{R}} (e^{\phi(y)/2} - 1)^2 \nu(dy) < \infty.$$

Then the following statements hold true.

1. *The limit*

$$\lim_{\epsilon \downarrow 0} \left(\sum_{s \le t,\ |\Delta Y_s| > \epsilon} \phi(\Delta Y_s) - t \int_{|y|>\epsilon} \left(e^{\phi(y)} - 1\right) \nu(dy) \right)$$

exists (uniformly in t on any bounded interval).

2. *The process*

$$\begin{aligned} D_t = \eta Y_t^c - \frac{\eta^2 c^2 t}{2} - \eta \mu t \\ + \lim_{\epsilon \downarrow 0} \left(\sum_{s \le t,\ |\Delta Y_s| > \epsilon} \phi(\Delta Y_s) - t \int_{|y|>\epsilon} \left(e^{\phi(y)} - 1\right) \nu(dy) \right), \end{aligned}$$

where Y_t^c is the continuous part of Y_t, defines a probability measure Q equivalent to P by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = e^{D_t}. \tag{3.5}$$

3. *The process Y_t remains a Lévy process under Q with characteristic*

triplet $(\tilde{\mu}, c, \tilde{\nu})$*, where*

$$\tilde{\mu} = \mu + \int_{-1}^{1} y(\tilde{\nu} - \nu)(dy) + c^2\eta$$

and

$$\tilde{\nu}(dy) = e^{\phi(y)}\nu(dy).$$

4. *Conversely, any probability measure equivalent to* P *under which* Y_t *remains a Lévy process must be of the form* (3.5) *and the characteristic triplet must be* $(\tilde{\mu}, c, \tilde{\nu})$ *as specified above.*

Proof. This theorem is paraphrasing Lemma 33.6 and Theorems 33.1 and 33.2 of Sato [49] to suit our purposes. 1. and 2. follow from Lemma 33.6. The Lévy property of the process Y_t under Q is a consequence of Theorem 33.2. The form of its characteristic triplet is given by Theorem 33.1. The converse (4.) is given by Theorems 33.1 and 33.2.

□

Consider now a symmetric Lévy process Y_t, i.e., its Lévy measure ν is symmetric under P. In order for the Lévy process to remain symmetric under an equivalent measure Q, it is necessary and sufficient that $\phi(y)$ is an even function. Transformation of the parameters of the symmetric family is given in the following result.

Theorem 3.2.2. *Let* Q *be an equivalent change of measure for a symmetric Lévy process under which it remains a symmetric Lévy process. The Lévy measure* $\tilde{\nu}$ *is symmetric if and only if* $\phi(-y) = \phi(y)$ ν*-a.e. The* Q*-distribution of* Y_1 *is* $S(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\psi})$ *where*

$$\tilde{\mu} = \mu + c^2\eta, \tag{3.6}$$

$$\tilde{\sigma}^2 = c^2 + \int_{\mathbb{R}} y^2 \tilde{\nu}(dy), \tag{3.7}$$

$$\tilde{\psi}(v) = \exp\left\{-\frac{c^2 v}{\tilde{\sigma}^2} - 2\int_0^\infty \left(1 - \cos(y\sqrt{2v}/\tilde{\sigma})\right)\tilde{\nu}(dy)\right\}. \tag{3.8}$$

Proof. If $\phi(y)$ is even then $\tilde{\nu}$ is symmetric,

$$\tilde{\nu}(-dy) = e^{\phi(-y)}\nu(-dy) = e^{\phi(y)}\nu(dy) = \tilde{\nu}(dy).$$

Conversely, if $\tilde{\nu}$ is symmetric, then for all $\tilde{\nu}$-measurable (thus also ν-measurable) function g,

$$\int g(y)\tilde{\nu}(dy) = \int g(-y)\tilde{\nu}(dy).$$

Expending,

$$\int g(y)e^{\phi(y)}\nu(dy) = \int g(-y)e^{\phi(y)}\nu(dy) = \int g(y)e^{\phi(-y)}\nu(dy).$$

Hence,

$$e^{\phi(y)}\nu(dy) = e^{\phi(-y)}\nu(dy)$$

by the uniqueness of Radon-Nikodym derivative,

$$\phi(y) = \phi(-y) \quad \nu\text{-a.e.}$$

Furthermore, if $\tilde{\nu}$ and ν are both symmetric, then $\int_{-1}^{1} y(\tilde{\nu} - \nu)(dy) = 0$. Hence, $\tilde{\mu} = \mu + c^2\eta$. The rest of the parameters of the symmetric family of distributions are obtained from Proposition 3.1.1, namely (3.1), (3.2) and (3.3).

□

3.2.1 The Natural Change of Measure

The purpose of this Section is to state and prove a result that gives equivalent changes of measure under which a symmetric Lévy process remains Lévy with marginals from the same symmetric family.

Consider a symmetric Lévy process Y_t with characteristic triplet (μ, c, ν) and the P-distribution of Y_1 from $S(\mu, \sigma^2, \psi)$. We define "natural" in the change of measure as the following.

Definition 3.2.1. An equivalent measure $Q \sim P$ is a natural change of measure if it keeps the distribution of Y_1 in the same family of symmetric distributions, i.e., the Q-distribution of Y_1 has the same ψ.

Since we consider equivalent measures Q that preserve Lévy property and symmetry, we denote the characteristic triplet under Q with $(\tilde{\mu}, c, \tilde{\nu})$. By Theorems 3.2.1 and 3.2.2, we know that $\tilde{\mu}$ is (3.6) and

$$\tilde{\nu}(dy) = e^{\phi(y)}\nu(dy), \tag{3.9}$$

for some even function ϕ.

The next result gives necessary conditions that must hold when the change of measure is a natural equivalent measure. It shows that for processes with a Brownian component a natural change of measure results only in the change of the drift μ, and for processes without Brownian component it results only in the change of the Lévy measure ν.

Theorem 3.2.3. *Let Y_t be a symmetric Lévy process with characteristic triplet (μ, c, ν) and the P-distribution of Y_1 from $S(\mu, \sigma^2, \psi)$, and Q a natural change of measure.*

(1) If $c \neq 0$, a Brownian component is present, then under Q, the characteristic triplet becomes $(\tilde{\mu}, c, \nu)$, where $\tilde{\mu} = \mu + c^2\eta$, for some $\eta \in \mathbb{R}$, while c and ν do not change.

(2) If $c = 0$, a Brownian component is absent, then under Q the characteristic triplet becomes $(\mu, 0, \tilde{\nu})$ where $\tilde{\nu}(A) = \int 1_{\{A\}}(\beta y)\nu(dy)$, for some $\beta > 0$, while μ (and c) do not change.

Proof. The proof of the Theorem uses Theorems 3.2.1 and 3.2.2 and an analytical lemma.

Using the expression for ψ from (3.8) and (3.3), we can see that Q is natural (i.e., $\psi = \tilde{\psi}$) if and only if for all $v > 0$, the even function ϕ in (3.9) satisfies the following integral equation

$$\int_0^\infty \Big[\big(1-\cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)} - \big(1-\cos(y\sqrt{2v}/\sigma)\big)\Big]\nu(dy) + \frac{c^2 v}{2}\Big(\frac{1}{\tilde{\sigma}^2} - \frac{1}{\sigma^2}\Big) = 0. \tag{3.10}$$

In Lemma 3.2.1 we will show that

$$\lim_{v\to\infty}\int_0^\infty \frac{1}{v}\Big[\big(1-\cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)}-\big(1-\cos(y\sqrt{2v}/\sigma)\big)\Big]\nu(dy)=0. \quad (3.11)$$

Hence by dividing by v and taking limit in (3.10) we must have

$$\frac{c^2}{2}\Big(\frac{1}{\tilde{\sigma}^2}-\frac{1}{\sigma^2}\Big)=0. \quad (3.12)$$

(1). Consider first the case $c\neq 0$. It then follows from (3.12) that $\tilde{\sigma}^2=\sigma^2$. With $\omega=\frac{\sqrt{2v}}{\sigma}=\frac{\sqrt{2v}}{\tilde{\sigma}}>0$, rewrite (3.10) using (3.12) as

$$\int_0^\infty \big(1-\cos(\omega y)\big)\tilde{\nu}(dy)=\int_0^\infty \big(1-\cos(\omega y)\big)\nu(dy), \quad \forall\omega>0. \quad (3.13)$$

We shall show that (3.13) implies $\tilde{\nu}=\nu$.

Since ν and $\tilde{\nu}$ are not probability measures, extra work is required to see that they are equal a.e. given that they have the same transform above. To show it, we derive that the Mellin transform (see appendix A.2) for their associated probability distributions are the same. Consider Laplace transform of (3.13) in ω

$$\int_0^\infty e^{-\lambda\omega}\Big(\int_0^\infty \big(1-\cos(\omega y)\big)\tilde{\nu}(dy)\Big)d\omega = \int_0^\infty e^{-\lambda\omega}\Big(\int_0^\infty \big(1-\cos(\omega y)\big)\nu(dy)\Big)d\omega.$$

By Fubini's theorem, we obtain

$$\int_0^\infty \Big(\int_0^\infty e^{-\lambda\omega}\big(1-\cos(\omega y)\big)d\omega\Big)\tilde{\nu}(dy) = \int_0^\infty \Big(\int_0^\infty e^{-\lambda\omega}\big(1-\cos(\omega y)\big)d\omega\Big)\nu(dy). \quad (3.14)$$

Now, consider the inner integral of (3.14), by using complex exponential

$(\cos(\omega y) = \mathrm{Re}(e^{i\omega y}))$, we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda\omega}\big(1-\cos(\omega y)\big)d\omega &= \frac{1}{\lambda} - \int_0^\infty e^{-\lambda\omega}\cos(\omega y)d\omega \\
&= \frac{1}{\lambda} - \mathrm{Re}\Big\{\int_0^\infty e^{(-\lambda+iy)\omega}d\omega\Big\} \\
&= \frac{1}{\lambda} - \mathrm{Re}\Big\{\Big[\frac{1}{-\lambda+iy}e^{(-\lambda+iy)\omega}\Big]_0^\infty\Big\} \\
&= \frac{1}{\lambda} - \mathrm{Re}\Big\{-\frac{1}{-\lambda+iy}\Big\} \\
&= \frac{y^2}{\lambda(\lambda^2+y^2)} \qquad (3.15)
\end{aligned}$$

in which we have used the fact $\frac{-1}{-\lambda+iy} = \frac{\lambda+iy}{\lambda^2+y^2}$. Hence (3.14) yields from (3.15) and after multiplying both sides by λ,

$$\int_0^\infty \frac{y^2}{\lambda^2+y^2}\tilde{\nu}(dy) = \int_0^\infty \frac{y^2}{\lambda^2+y^2}\nu(dy), \quad \forall\lambda > 0. \qquad (3.16)$$

Now let

$$\begin{aligned}
\tilde{\kappa} &= \int_0^\infty y^2\tilde{\nu}(dy) < \infty, \\
\kappa &= \int_0^\infty y^2\nu(dy) < \infty.
\end{aligned}$$

Since $\tilde{\sigma}^2 = \sigma^2$, we have from (3.7) and (3.2) that $\tilde{\kappa} = \kappa$. Therefore, we can rewrite (3.16) as

$$\int_0^\infty \frac{y^2}{\lambda^2+y^2}\frac{\tilde{\nu}(dy)}{\tilde{\kappa}} = \int_0^\infty \frac{y^2}{\lambda^2+y^2}\frac{\nu(dy)}{\kappa}. \qquad (3.17)$$

Let now the measures on $(0,\infty)$, $\tilde{n}(dy) = \frac{y^2}{\tilde{\kappa}}\tilde{\nu}(dy)$ and $n(dy) = \frac{y^2}{\kappa}\nu(dy)$, such that

$$\begin{aligned}
\int_0^\infty h(y^2)y^2\frac{\tilde{\nu}(dy)}{\tilde{\kappa}} &= \int_0^\infty h(y)\tilde{n}(dy), \\
\int_0^\infty h(y^2)y^2\frac{\nu(dy)}{\kappa} &= \int_0^\infty h(y)n(dy).
\end{aligned}$$

The measures $\tilde{n}$ and n are probability measures since

$$\int_0^\infty \tilde{n}(dy) = \int_0^\infty y^2 \frac{\tilde{\nu}(dy)}{\tilde{\kappa}} = 1,$$

and similarly for n. Applying $\tilde{n}$ and n to (3.17) yields

$$\int_0^\infty \frac{1}{\lambda + y} \tilde{n}(dy) = \int_0^\infty \frac{1}{\lambda + y} n(dy), \quad \forall \lambda > 0. \tag{3.18}$$

Next, let $\tilde{\xi}$ and ξ be two random variables that have the laws $\tilde{n}$ and n respectively. Let $s \in [0,1]$, and denote the probability generating functions of $\tilde{\xi}$ and ξ by

$$\begin{aligned} \tilde{f}(s) &= \mathbf{E}\big(s^{\tilde{\xi}}\big) = \int_0^\infty s^y \tilde{n}(dy), \\ f(s) &= \mathbf{E}\big(s^{\xi}\big) = \int_0^\infty s^y n(dy). \end{aligned}$$

Then the Mellin transformation of $\tilde{f}$ and f are

$$\begin{aligned} \mathcal{M}\tilde{f}(\lambda) = \int_0^1 s^{\lambda-1} \tilde{f}(s) ds &= \int_0^1 s^{\lambda-1} \left(\int_0^\infty s^y \tilde{n}(dy) \right) ds \\ &= \int_0^\infty \left(\int_0^1 s^{\lambda-1} s^y ds \right) \tilde{n}(dy) \\ &= \int_0^\infty \left(\int_0^1 s^{\lambda+y-1} ds \right) \tilde{n}(dy) \\ &= \int_0^\infty \left[\frac{1}{\lambda+y} s^{\lambda+y} \right]_0^1 \tilde{n}(dy) \\ &= \int_0^\infty \frac{1}{\lambda+y} \tilde{n}(dy), \end{aligned}$$

and similarly,

$$\begin{aligned} \mathcal{M}f(\lambda) = \int_0^1 s^{\lambda-1} f(s) ds &= \int_0^1 s^{\lambda-1} \left(\int_0^\infty s^y n(dy) \right) ds \\ &= \int_0^\infty \left(\int_0^1 s^{\lambda+y-1} ds \right) n(dy) \\ &= \int_0^\infty \frac{1}{\lambda+y} n(dy). \end{aligned}$$

Therefore, it follows from (3.18) that, $\forall\lambda > 0$,

$$\mathcal{M}\tilde{f}(\lambda) = \mathcal{M}f(\lambda).$$

By the uniqueness of Mellin transformation, we have $\tilde{f} = f$. This implies that the probability measures $\tilde{n} = n$. Consequently, by the definitions of the probability measures

$$\tilde{n}(dy) = \frac{y^2\tilde{\nu}(dy)}{\tilde{\kappa}} = \frac{y^2\nu(dy)}{\kappa} = n(dy),$$

which implies that $\tilde{\nu}(dy) = \nu(dy)$ since $\tilde{\kappa} = \kappa$.

(2). Consider now the case $c = 0$. With $\beta = \frac{\tilde{\sigma}}{\sigma}$ and $\lambda = \frac{\sqrt{2v}}{\tilde{\sigma}}$ we have from (3.10) and using (3.12)

$$\int_0^\infty \big(1 - \cos(\lambda y)\big)\tilde{\nu}(dy) = \int_0^\infty \big(1 - \cos(\beta\lambda y)\big)\nu(dy), \quad \forall\lambda > 0. \tag{3.19}$$

Denote the measure $\nu(\frac{1}{\beta}dy) = \nu_\beta(dy)$ such that

$$\int h(\beta y)\nu(dy) = \int h(y)\nu_\beta(dy).$$

Then (3.19) becomes

$$\int_0^\infty \big(1 - \cos(\lambda y)\big)\tilde{\nu}(dy) = \int_0^\infty \big(1 - \cos(\lambda y)\big)\nu_\beta(dy), \ \ \forall\lambda > 0.$$

But we have seen in the proof above in the first part (see (3.13)) that this implies $\tilde{\nu} = \nu_\beta$.

$\square$

Lemma 3.2.1.

$$\lim_{v\to\infty} \int_0^\infty \frac{1}{v}\Big[\big(1 - \cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)} - \big(1 - \cos(y\sqrt{2v}/\sigma)\big)\Big]\nu(dy) = 0.$$

Proof. Let

$$f_v(y) = \frac{1}{v}\Big[\big(1 - \cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)} - \big(1 - \cos(y\sqrt{2v}/\sigma)\big)\Big].$$

From (3.10), it is clear that $\{f_v\}$ are integrable functions on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu\big)$. For any fixed y, $\big(1 - \cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)} - \big(1 - \cos(y\sqrt{2v}/\sigma)\big)$ is a bounded function of v, and so

$$\lim_{v\to\infty} f_v(y) = 0.$$

Using

$$1 - \cos(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots \leq \frac{x^2}{2},$$

and the triangular inequality, we obtain

$$\begin{aligned}|f_v(y)| &\leq \frac{1}{v}\Bigg[\big(1 - \cos(y\sqrt{2v}/\tilde{\sigma})\big)e^{\phi(y)} + \big(1 - \cos(y\sqrt{2v}/\sigma)\big)\Bigg] \\ &\leq \frac{y^2}{\tilde{\sigma}^2}e^{\phi(y)} + \frac{y^2}{\sigma^2} = G(y).\end{aligned}$$

The function $G(y) \geq 0$ is integrable with respect to ν, since the Lévy measures $\tilde{\nu}$ and ν satisfy

$$\int_{\mathbb{R}} (1 \wedge y^2)\tilde{\nu}(dy) < \infty, \qquad \text{and} \qquad \int_{\mathbb{R}} (1 \wedge y^2)\nu(dy) < \infty,$$

and the existence of variance implies

$$\int_{|y|>1} y^2\tilde{\nu}(dy) < \infty, \qquad \text{and} \qquad \int_{|y|>1} y^2\nu(dy) < \infty.$$

Hence, we have

$$\int_{\mathbb{R}} y^2\tilde{\nu}(dy) < \infty, \qquad \text{and} \qquad \int_{\mathbb{R}} y^2\nu(dy) < \infty.$$

Therefore,

$$\begin{aligned}\int_0^\infty G(y)\nu(dy) &= \int_0^\infty \left(\frac{y^2}{\tilde{\sigma}^2}e^{\phi(y)} + \frac{y^2}{\sigma^2}\right)\nu(dy) \\ &= \frac{1}{\tilde{\sigma}^2}\int_0^\infty y^2\tilde{\nu}(dy) + \frac{1}{\sigma^2}\int_0^\infty y^2\nu(dy) \\ &< \infty.\end{aligned}$$

The result follows by dominated convergence theorem (see appendix A.1).

□

The next result shows that natural change of measure exists.

Theorem 3.2.4. *Let Y_t be a symmetric Lévy process with characteristic triplet (μ, c, ν), and the P-distribution of Y_1 from $S(\mu, \sigma^2, \psi)$.*

(1) Let $c \neq 0$. Then for any $\eta \in \mathbb{R}$, there is a natural change of measure Q such that the characteristic triplet becomes $(\tilde{\mu}, c, \nu)$, where $\tilde{\mu} = \mu + c^2\eta$. The Q-distribution of Y_1 is $S(\tilde{\mu}, \sigma^2, \psi)$.

(2) Let $c = 0$. Then for any $\beta > 0$, there is a natural change of measure Q such that the characteristic triplet becomes $(\mu, 0, \tilde{\nu})$, where $\tilde{\nu}(A) = \int 1_{\{A\}}(\beta y)\nu(dy)$. The Q-distribution of Y_1 is $S(\mu, \tilde{\sigma}^2, \psi)$, where $\tilde{\sigma}^2 = \int_{\mathbb{R}} y^2\tilde{\nu}(dy) = \beta^2\sigma^2$, $(\beta = \frac{\tilde{\sigma}}{\sigma})$.

Proof. The existence of a natural change of measure Q follows by using Theorems 3.2.1, 3.2.2 and 3.2.3, where the transformation of the characteristic triplet are given by Theorem 3.2.3. The Q-distribution of Y_1 are as claimed following Theorems 3.2.2 and 3.2.3.

□

The explicit Radon-Nikodym derivative for the natural change of measure is given below (consequence of Theorems 3.2.1, 3.2.2 and 3.2.3).

Corollary 3.2.1. *Let Y_t be a symmetric Lévy process on $\mathbb{R}$ with characteristic triplet (μ, c, ν), and the P-distribution of Y_1 from $S(\mu, \sigma^2, \psi)$. A measure Q defined by*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = e^{D_t}. \tag{3.20}$$

is a natural change of measure if D_t is in one of the following forms.

(1) If $c \neq 0$, then

$$D_t = \eta Y_t^c - \frac{\eta^2 c^2 t}{2} - \eta \mu t \tag{3.21}$$

for any $\eta \in \mathbb{R}$. In this case, Q is a natural change of measure that changes only the μ while c and ν do not change.

(2) If $c = 0$, then

$$D_t = \lim_{\epsilon \downarrow 0} \left(\sum_{s \leq t,\ |\Delta Y_s| > \epsilon} \phi(\Delta Y_s) - t \int_{|y|>\epsilon} \big(e^{\phi(y)} - 1\big) \nu(dy) \right) \tag{3.22}$$

for any even function ϕ satisfying $\int_{\mathbb{R}} (e^{\phi(y)/2} - 1)^2 \nu(dy) < \infty$, and $\phi(y) = \ln \frac{\tilde{\nu}(dy)}{\nu(dy)}$ where $\tilde{\nu}(A) = \int 1_{\{A\}}(\beta y)\nu(dy)$, $\beta > 0$. In this case, Q is a natural change of measure that changes only the ν, while μ (and c) do not change.

3.2.2 Natural Equivalent Martingale Measures

Let now $S_t = S_0 e^{Y_t}$ be a model for stock prices, where Y_t is a symmetric Lévy process. According to the Fundamental Theorems of Mathematical Finance (see Section 2.1.3), options on stock are priced by using an equivalent martingale measure (EMM) Q, under which the discounted stock price process $e^{-rt}S_t,\ 0 < t \leq T$ is a martingale.

Definition 3.2.2. A natural change of measure is an EMM, called natural EMM, if it makes the discounted price $e^{-rt}S_t$ into a martingale.

The next theorem characterizes all the natural EMM's obtained by natural change of measure.

Theorem 3.2.5. *1. Let Q be a natural EMM for a symmetric Lévy process, then the following relation must hold between the parameters of the Q-distribution of Y_1,*

$$\tilde{\mu} + \ln \psi\Big(- \frac{\tilde{\sigma}^2}{2} \Big) = r. \tag{3.23}$$

Hence

2. If a Brownian component is present ($c \neq 0$) and Q is a natural EMM, then

$$\tilde{\mu} = r - \ln \psi \left(-\frac{\sigma^2}{2} \right). \tag{3.24}$$

Further, such Q exists and is unique.

3. If a Brownian component is absent ($c = 0$) and Q is a natural EMM, then $\tilde{\sigma}^2$ is a root of the equation

$$\ln \psi \Big(-\frac{\tilde{\sigma}^2}{2} \Big) = r - \mu. \tag{3.25}$$

Further, such Q exists if and only if the $\mu < r$, and when it exists it is unique.

Proof. The result follows from the martingale property of $e^{-rt} S_t$ by using symmetry and independent and stationary increments properties of Y_t. We have, for $0 < \tau < t$,

$$\begin{aligned} \mathbf{E}_Q \left(e^{-rt} S_t | \mathcal{F}_\tau \right) &= e^{-r\tau} S_0 e^{Y_\tau} \mathbf{E}_Q \left(e^{-r(t-\tau)} e^{Y_t - Y_\tau} | \mathcal{F}_\tau \right) \\ &= e^{-r\tau} S_\tau \mathbf{E}_Q \left(e^{-r(t-\tau)} e^{Y_{t-\tau}} \right). \end{aligned}$$

So Q is an EMM if and only if $\mathbf{E}_Q \left(e^{-r(t-\tau)} e^{Y_{t-\tau}} \right) = 1$, i.e.,

$$\mathbf{E}_Q \left(e^{Y_{t-\tau}} \right) = e^{r(t-\tau)}.$$

Since under natural EMM Q, Y_t is from the same symmetric family $Y_t \sim S(\tilde{\mu} t, \tilde{\sigma}^2 t, \psi_t)$, with $\psi_t(u) = \left[\psi(\frac{u}{t}) \right]^t$, we have

$$\mathbf{E}_Q \left(e^{Y_{t-\tau}} \right) = e^{\tilde{\mu}(t-\tau)} \psi^{t-\tau} \Big(-\frac{\tilde{\sigma}^2}{2} \Big).$$

Therefore, the martingale property holds if and only if

$$e^{\tilde{\mu}(t-\tau)} \psi^{t-\tau} \Big(-\frac{\tilde{\sigma}^2}{2} \Big) = e^{r(t-\tau)},$$

or by taking logarithms equals

$$\tilde{\mu} + \ln \psi\Big(-\frac{\tilde{\sigma}^2}{2}\Big) = r.$$

By the natural change of measure (Theorem 3.2.3), if $c \neq 0$, only μ can be changed, consequently we obtain (3.24). If $c = 0$, only σ^2 can be changed, hence we obtain (3.25).

When $c = 0$, we have by Jensen's inequality $\mathbf{E}_Q\big(e^{Y_1}\big) \geq e^{\mathbf{E}_Q(Y_1)} = e^{\mu}$. But by (2.10) (or (2.13)), $\mathbf{E}_Q\big(e^{Y_1}\big) = e^{\mu}\psi\Big(- \frac{\tilde{\sigma}^2}{2}\Big)$. Hence $\psi\Big(- \frac{\tilde{\sigma}^2}{2}\Big) \geq 1$. Thus equation (3.25) has a positive solution for $\tilde{\sigma}^2$ if and only if $\mu < r$.

□

3.2.3 Dichotomy in the Natural Change of Measure

Here, we investigate the canonical decomposition of Lévy processes under natural EMM's to explain why when $c = 0$ (no Brownian component), only the Lévy measure changes but not the mean; and when $c \neq 0$ (with Brownian component), only the mean changes and not the Lévy measure.

Recall that the canonical decomposition of a Lévy process Y_t with characteristic triplet $(a, 0, \nu)$ is (see (2.26))

$$Y_t = at + \int_0^t \int_{|y|>1} yJ(dy, ds) + \int_0^t \int_{|y|\leq 1} y\Big(J(dy, ds) - \nu(dy)ds\Big).$$

Observe that if Y_t is symmetric, then the process

$$\int_0^t \int_{|y|>1} yJ(dy, ds)$$

is a martingale, since $\int_{|y|>1} y\nu(dy) = 0$ (ν is symmetric). In particular, $a = \mu = \mathbf{E}(Y_1)$. This also implies that Y_t is a special martingale and its canonical decomposition is $Y_t = M_t + A_t$, where $A_t = at$ is predictable and

$$M_t = \int_0^t \int_{|y|>1} yJ(dy, ds) + \int_0^t \int_{|y|\leq 1} y\Big(J(dy, ds) - \nu(dy)ds\Big)$$

is a martingale under P.

Under a natural equivalent measure Q, the canonical decomposition of Y_t is

$$\begin{aligned} Y_t &= at + \int_0^t \int_{|y|>1} yJ(dy,ds) + \int_0^t \int_{|y|\le 1} y\Big(J(dy,ds) - e^{\phi(y)}\nu(dy)ds\Big) \\ &= at + \int_0^t \int_{|y|>1} yJ(dy,ds) + \int_0^t \int_{|y|\le 1} y\Big(J(dy,ds) - \nu(dy)ds\Big) \\ &\qquad + t\int_{|y|\le 1} y\Big(\nu(dy) - e^{\phi(y)}\nu(dy)\Big). \end{aligned} \tag{3.26}$$

By symmetry of the Lévy measures, the last term in (3.26) is

$$\int_{|y|\le 1} y\Big(\nu(dy) - e^{\phi(y)}\nu(dy)\Big) = \int_{|y|\le 1} y\Big(\nu(dy) - \tilde{\nu}(dy)\Big) = 0.$$

Thus, M_t is also a Q-martingale, and that $M_t e^{D_t}$ is a P-martingale, or equivalently, $< M, e^D >_t = 0$ under P. This can also be checked directly. Recall (Theorem 3.2.1) that

$$D_t = \int_0^t \int_{\mathbb{R}} \Big(\phi(y)J(dy,ds) - \big(e^{\phi(y)} - 1\big)\nu(dy)ds\Big).$$

So, $\Delta D_t = \phi(\Delta Y_t)$, and

$$\Delta e^{D_t} = e^{D_t} - e^{D_{t-}} = e^{D_{t-}}\big(e^{\Delta D_t} - 1\big) = e^{D_{t-}}\big(e^{\phi(\Delta Y_t)} - 1\big).$$

The quadratic covariation between M_t and e^{D_t} is

$$[M, e^D]_t = \sum_{s\le t} e^{D_{s-}}\Delta Y_s\big(e^{\phi(\Delta Y_s)} - 1\big) = \int_0^t e^{D_{s-}} \int_{\mathbb{R}} y\big(e^{\phi(y)} - 1\big)J(dy,ds).$$

Hence,

$$< M, e^D >_t = \int_0^t e^{D_{s-}} \int_{\mathbb{R}} y\big(e^{\phi(y)} - 1\big)\nu(dy)ds = 0,$$

because ν is symmetry and $\phi(y)$ is even, i.e., $[M, e^D]_t$ is also a martingale.

Therefore, M_t is a P-martingale and a Q-martingale. So, there is no change in the martingale under Q. What really changes is the compensator of $[M, M]_t$, since it depends on the measure. We have, under P,

$$< M, M >_t = t \int_{\mathbb{R}} y^2 \nu(dy),$$

whereas under Q,

$$< M, M >_t = t \int_{\mathbb{R}} y^2 e^{\phi(y)} \nu(dy),$$

which is the second moment (variance) (see (2.24)).

In the case when Brownian component is present, the natural change of measure is equivalent to the change of measure for Brownian motion in the classical case. In particular, $< M, M >_t$ does not change under Q. But the martingale changes, where $M_t = B_t$ is the martingale under P, and $M_t = B_t - c^2 \eta t$ is the martingale under Q.

3.3 Change of Measure for Symmetric Lévy Processes in Discrete time

Let $Y_N = \sum_{n=1}^{N} \Delta Y_n$ be a Lévy process in discrete time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. It is a sum of independent and identically distributed (i.i.d) random variables $\Delta Y_1, \ldots, \Delta Y_N$. The change of measure for Lévy processes in discrete time can be characterized in terms of the joint densities of $\Delta Y_1, \ldots, \Delta Y_N$ (e.g. [17]). The process Y_N remains Lévy under equivalent measures if $\Delta Y_1, \ldots, \Delta Y_N$ remain i.i.d. The next result gives the class of equivalent measures under which $\Delta Y_1, \ldots, \Delta Y_N$ remain i.i.d.

Theorem 3.3.1. *Let f and $\tilde{f}$ be two probability densities, and $f(y_n)$ is the P-density of the increments ΔY_n, $n = 1, \ldots, N$. Define a measure Q by*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_N} = \prod_{n=1}^{N} \frac{\tilde{f}(\Delta Y_n)}{f(\Delta Y_n)}, \tag{3.27}$$

where $\mathcal{F}_N$ is a filtration generated by $\Delta Y_1, \ldots, \Delta Y_N$[1]*. Then Q is equivalent* *to P. The increments $\Delta Y_1, \ldots, \Delta Y_N$ remain independent and identically distributed under Q with the Q-density function of ΔY_n becomes $\tilde{f}(y_n)$.*

Proof. Denote a set in $\mathbb{R}^N$ by $A = \{(u_1, \ldots, u_N) : u_1 \leq y_1, \ldots, u_N \leq y_N\}$ for some fixed $y_1, \ldots, y_N \leq \infty$, and for each $1 \leq n \leq N$, denote $A_n = \{u_n : u_n \leq y_n\}$. The indicator function of A, $I_A\big((u_1, \ldots, u_N)\big)$ is equal to 1 if $(u_1, \ldots, u_N) \in A$ and 0 otherwise, has the property $I_A\big((u_1, \ldots, u_N)\big) = \prod_{n=1}^N I_{A_n}(u_n)$. Consider the probability $Q(\Delta Y_1 \leq y_1, \ldots, \Delta Y_N \leq y_N) = Q\big((\Delta Y_1, \ldots, \Delta Y_N) \in A\big)$, we have

$$\begin{aligned}
Q(\Delta Y_1 \leq y_1, \ldots, \Delta Y_N \leq y_N) &= \mathbf{E}_Q\Big(I_A\big((\Delta Y_1, \ldots, \Delta Y_N)\big)\Big) \\
&= \mathbf{E}_p\bigg(\prod_{n=1}^N \frac{\tilde{f}(\Delta Y_n)}{f(\Delta Y_n)} I_A\big((\Delta Y_1, \ldots, \Delta Y_N)\big)\bigg) \\
&= \mathbf{E}_p\bigg(\prod_{n=1}^N \frac{\tilde{f}(\Delta Y_n)}{f(\Delta Y_n)} I_{A_n}(\Delta Y_n)\bigg).
\end{aligned}$$

The expectation can be obtained by integrating the function of a random vector with respect to the joint P-density of $\Delta Y_1, \ldots, \Delta Y_N$, which equals to $\prod_{n=1}^N f(u_n)$ by independence. Therefore we have

$$\begin{aligned}
Q(\Delta Y_1 \leq y_1, \ldots, \Delta Y_N \leq y_N) &= \int \cdots \int \prod_{n=1}^N \frac{\tilde{f}(u_n)}{f(u_n)} I_{A_n}(u_n) \prod_{n=1}^N f(u_n) du_1 \ldots du_N. \\
&= \int \cdots \int \prod_{n=1}^N \tilde{f}(u_n) I_{A_n}(u_n) du_1 \ldots du_N. \\
&= \prod_{n=1}^N \int_{A_n} \tilde{f}(u_n) du_n.
\end{aligned}$$

Now taking all $y_i = \infty$ for $i \neq n$, we obtain that all the ΔY_n's are identically

[1]As usual, one may let $\frac{\tilde{f}(\Delta Y_n)}{f(\Delta Y_n)} = 1$ (or any other value) whenever $f(\Delta Y_n) = 0$, which has P-probability 0 of occurring.

distributed in Q and have the density $\tilde{f}$ since

$$Q(\Delta Y_n \leq y_n) = \int_{A_n} \tilde{f}(u_n) du_n.$$

Putting this expression into the equation above, we obtain Q-independence

$$Q(\Delta Y_1 \leq y_1, \ldots, \Delta Y_N \leq y_N) = \prod_{n=1}^{N} Q(\Delta Y_n \leq y_n),$$

and this completes the proof.

□

In Theorem 3.3.1, any density function $\tilde{f}$ used will produce an equivalent measure Q, under which the increments $\Delta Y_1, \ldots, \Delta Y_N$ remain i.i.d.

For a symmetric Lévy process Y_N with the P-density of ΔY_n, denoted $f(y_n)$, from a symmetric family $S(\mu, \sigma^2, \psi)$. It remains a symmetric Lévy process under Q if all ΔY_n's remain i.i.d with Q-density $\tilde{f}(y_n)$ that belongs to a symmetric family (not necessary the same as the symmetric family under P).

Theorem 3.3.2. *In the same setting as Theorem 3.3.1, let the P-density f belongs to the symmetric family $S(\mu, \sigma^2, \psi)$. For any density $\tilde{f}$ that belongs to a symmetric family $S(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\psi})$, the measure Q is equivalent to P. The increments $\Delta Y_1, \ldots, \Delta Y_N$ remain independent and identically distributed under Q with the Q-density of ΔY_n becomes $\tilde{f}(y_n)$.*

Proof. Follows from Theorem 3.3.1 with $\tilde{f}$ belongs to the symmetric family $S(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\psi})$.

□

3.3.1 Natural Change of Measure

Consider now a symmetric Lévy process $Y_N = \sum_{n=1}^{N} \Delta Y_n$, where $\Delta Y_n \sim S(\mu, \sigma^2, \psi)$ with P-density $f(y_n)$. A natural change of measure preserves i.i.d and the symmetric family of the distributions of ΔY_n. The next result describes all natural changes of measures. It states that they are obtained by changing the mean and variance of ΔY_n.

Theorem 3.3.3. *In the same setting as Theorems 3.3.1 and 3.3.2, if f and $\tilde{f}$ belongs to the same symmetric family ($\psi = \tilde{\psi}$), then the measure $Q \sim P$ is a natural change of measure. The increments ΔY_n, $n = 1, \dots, N$ remain independent and identically distributed under Q and $\Delta Y_n \sim S(\tilde{\mu}, \tilde{\sigma}^2, \psi)$ with Q-density $\tilde{f}(y_n)$.*

Proof. Follows from Theorem 3.3.1 and 3.3.2 with $\tilde{\psi} = \psi$. $\square$

Remark 3.1. *In [35] it was assumed that, in addition to having same ψ, the variance (scale parameter) σ^2 is also the same under natural change of measures. Such is a subclass of natural change of measure under which $\Delta Y_1 \sim S(\tilde{\mu}, \sigma^2, \psi)$.*

3.3.2 Natural Equivalent Martingale Measure

Let now $S_N = S_0 e^{Y_N}$ be a model for the stock prices, where $Y_N = \sum_{n=1}^{N} \Delta Y_n$ is a symmetric Lévy process in discrete time and $\Delta Y_1 \sim S(\mu, \sigma^2, \psi)$. For option pricing, it is required that Q is an EMM, under which the discounted stock price $e^{-rn} S_n$, $n \leq N$ is a martingale. The next result gives a necessary and sufficient condition for Q to be a natural EMM. The condition is an equation of $\tilde{\mu}$ and $\tilde{\sigma}$, similar to that of the continuous time case but without the dichotomy.

Theorem 3.3.4. *Let Q be a natural change of measure defined by (3.27). For the process $e^{-rn} S_n, n \leq N$ to be a martingale, it is necessary and sufficient that $\tilde{\mu}$ and $\tilde{\sigma}$ satisfy*

$$\tilde{\mu} + \ln \psi\Big(\frac{-\tilde{\sigma}^2}{2}\Big) = r. \tag{3.28}$$

Proof. For $e^{-rn} S_n$ to be a Q-martingale, we must have

$$\mathbf{E}_Q\big(e^{-r(n+1)} S_{n+1} \big| \mathcal{F}_n\big) = e^{-r(n+1)} S_n \mathbf{E}_Q\big(e^{\Delta Y_{n+1}}\big) = r^{-rn} S_n,$$

which implies that

$$\mathbf{E}_Q\big(e^{\Delta Y_{n+1}}\big) = e^r.$$

Since the increments are identically distributed under Q, $\Delta Y_{n+1} \stackrel{D}{=} \Delta Y_1 \sim S(\tilde{\mu}, \tilde{\sigma}^2, \psi)$. Using the expression for the mean of log-symmetric random variable (2.14), the claim follows.

□

The equation (3.28) admits infinitely many solutions of $\tilde{\mu}$ and $\tilde{\sigma}^2$, which implies that the natural EMM for discrete time model is not unique. However, if we consider the subclass of natural change of measures that only changes μ, but not σ^2 and ψ (see Remark 3.1 or [35]), then the natural EMM is unique with the location parameter $\tilde{\mu}$ satisfying the following condition.

Proposition 3.3.1. *Let Q be the subclass of natural change of measures under which $\Delta Y_1 \sim S(\tilde{\mu}, \sigma^2, \psi)$. For the process $e^{-rn}S_n, n \leq N$ to be a martingale, it is necessary and sufficient that*

$$\tilde{\mu} = r - \ln \psi\Big(\frac{-\sigma^2}{2}\Big). \tag{3.29}$$

Remark 3.2. *In this thesis, we shall always use this subclass of natural change of measure when a discrete time model is considered for option pricing. Therefore, in discrete time, a natural EMM always exists and it is unique. It is obtained by changing only the location parameter (mean) μ, while σ^2 and ψ do not change.*

3.3.3 Difference Between Discrete and Continuous Time Cases

In this subsection, we use the canonical decomposition of Lévy process to explain why the dichotomy in the natural change of measure in continuous time does not occur in discrete time.

Recall the canonical decomposition of a symmetric Lévy process Y_t with characteristic triplet (μ, c, ν) is (see (2.27))

$$Y_t = \mu t + cW_t + \int_0^t \int_{\mathbb{R}} y\Big(J(dy, ds) - \nu(dy)ds\Big).$$

For $t = n \in \mathbb{N}$, the increments ΔY_n can be represented as a Lévy process on a unit time interval

$$\begin{aligned}\Delta Y_n &= Y_n - Y_{n-1} \\ &= \mu + c\big(W_n - W_{n-1}\big) + \int_{n-1}^n \int_{\mathbb{R}} y\Big(J(dy, ds) - \nu(dy)ds\Big).\end{aligned}$$

Using the properties of Lévy process, we have

$$\mathbf{E}(\Delta Y_n) = \mathbf{E}(\Delta Y_1) = \mu, \tag{3.30}$$

$$Var(\Delta Y_n) = Var(\Delta Y_1) = c^2 + \int_{\mathbb{R}} y^2 \nu(dy). \tag{3.31}$$

The distribution of ΔY_n is symmetric $S(\mu, \sigma^2, \psi)$. Let f be the density of the distribution of ΔY_n, we have

$$\mu = \mathbf{E}(\Delta Y_n) = \int_{\mathbb{R}} y f(y) dy, \tag{3.32}$$

$$\sigma^2 = Var(\Delta Y_n) = \int_{\mathbb{R}} (y - \mu)^2 f(y) dy. \tag{3.33}$$

From (3.32) and (3.33), it can be seen that the density function f of the random variable ΔY_n has control over both the mean μ and variance σ^2. Since the natural change of measure in discrete time changes the density function f, it leads to a change in μ and σ^2, regardless of the value of the parameter c. Therefore, the natural EMM is a measure from the family $S(\tilde{\mu}, \tilde{\sigma}^2, \psi)$, where $\tilde{\mu}$ and $\tilde{\sigma}^2$ satisfy (3.28). Moreover, if σ^2 is fixed, we obtain the subclass of natural EMM under which $\Delta Y_n \sim S(\tilde{\mu}, \sigma^2, \psi)$, where $\tilde{\mu}$ satisfying (3.29) regardless of the value of c.

However, this is not the case for the continuous time. Clearly, (3.30) shows that only the drift term μ has control over the mean, and (3.31) shows that the Lévy measure ν only has control over the variance. Therefore, when

$c = 0$, the natural change of measure changes the Lévy measure ν, which in turn changes only the σ^2. In this case, the unique natural EMM is the one associated with $S(\mu, \tilde{\sigma}^2, \psi)$. When $c \neq 0$, the natural change of measure changes the drift μ, which is also the mean. In this case, the unique natural EMM is the one associated with $S(\tilde{\mu}, \sigma^2, \psi)$.

Chapter 4

Option Pricing with a Natural Equivalent Martingale Measure

In this chapter, we explore option pricing with a natural EMM for log-symmetric Lévy price processes.

4.1 Derivation of Option Pricing Formula

Let there be a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. According to pricing by no arbitrage approach, the price of options is given by

$$C_t = e^{-r(T-t)}\mathbf{E}_Q[(S_T - K)^+|\mathcal{F}_t], \tag{4.1}$$

where Q is an EMM such that $e^{-rt}S_t$ is a Q-martingale. We propose that Q be a natural EMM.

Firstly, observe that under natural EMM's, a symmetric Lévy process (which represents returns) remains to be a symmetric Lévy process. By independent of increments, it is Markov. Using Markov's property, the conditional expectation for a function g of S_T becomes

$$\mathbf{E}\big(g(S_T)|\mathcal{F}_t\big) = \mathbf{E}\big(g(S_T)|S_t\big) = \mathbf{E}\big(g(S_{T-t})|S_0\big).$$

Hence, it is enough to derive the pricing formula at time $t = 0$,

$$C_0 = e^{-rT}\mathbf{E}_Q[(S_T - K)^+]. \tag{4.2}$$

The formula for $t > 0$ given by (4.1) can be recovered by identifying t with 0 and T with $T - t$.

Secondly, pricing of options by formula (4.2) (or (4.1)) using a natural EMM Q is arbitrage-free. This is because the formula is valid for any EMM $Q \sim P$ (e.g. [52, p.398] or [5, p.148]). In fact, denote by $\mathcal{M}$, the class of probability measures on $(\Omega, \mathcal{F})$ that is equivalent to P, under which $e^{-rt}S_t$ is a martingale, there is an interval of option prices at time 0 that does not allow for arbitrage opportunities [52, 398]

$$C_0(Q) \in \Big(\inf_{Q\in\mathcal{M}} C_0(Q), \sup_{Q\in\mathcal{M}} C_0(Q) \Big).$$

In particular, [13] proved the following statements that gave a range of the arbitrage-free prices of options on stock, using all possible EMM's for the valuation, when the underlying model for stock price S_t is a pure jump Lévy process.

Theorem 4.1.1 (Theorem 2 [13])**.** *Let $\mathcal{M}$ be the class of measures equivalent to P, under which $e^{-rt}S_t$ is a martingale. Suppose the Lévy measure ν of the Lévy process Y_t under P has the following properties:*

(i) $\nu\big((-\infty, a]\big) > 0 \quad \forall a \in \mathbb{R}$.

(ii) *ν has no atom and satisfies* $\int_{[-1,0)} |y|\nu(dy) = \int_{(0,1]} y\nu(dy) = \infty$.

Then $\mathcal{M}$ is not empty, and for any EMM $Q \in \mathcal{M}$, the price of options

$$C_0(Q) = e^{-rT}\mathbf{E}_Q\big[g(S_T)\big] \in \big(e^{-rT}g(e^{rT}S_0), S_0\big).$$

where $g(S_T)$ is the payoff function of the option.

For the class of EMM that preserves Lévy processes, $\mathcal{M}' \subset \mathcal{M}$, the range of option prices is the full interval $\big(e^{-rT}g(e^{rT}S_0), S_0\big)$ (see [13], Remark 3).

Leaving the specificity of the measure Q to the next section, we write the option pricing formula (4.2) using change of numeraire (see [24], or [34], section 11.5), which proceeds as follows

$$\begin{aligned} C_0 &= e^{-rT}\mathbf{E}_Q\big[(S_T - K)^+\big] \\ &= e^{-rT}\mathbf{E}_Q\big[S_T I(S_T > K)\big] - e^{-rT}K\mathbf{E}_Q\big[I(S_T > K)\big] \\ &= S_0 Q_1\big(S_T > K\big) - e^{-rT}KQ\big(S_T > K\big), \end{aligned} \tag{4.3}$$

where Q_1, defined by

$$\frac{dQ_1}{dQ} = \frac{S_T e^{-rT}}{S_0} = e^{Y_T - rT}, \tag{4.4}$$

is the measure under which the process e^{rt}/S_t is a martingale.

In the remaining of this chapter, we will determine the measures Q and Q_1 for option pricing with log-symmetric Lévy price processes and derive the corresponding option pricing formulae. Three cases are considered: Continuous time log-symmetric Lévy model with Brownian component, without Brownian component, and discrete time model.

4.2 Log-symmetric Lévy Model with Brownian Component

Let now $S_t = S_0 e^{Y_t}$, $t > 0$ be a stock price process where Y_t is a symmetric Lévy process with characteristic triplets (μ, c, ν), $c \neq 0$, and the distribution of Y_1 belongs to $S(\mu, \sigma^2, \psi)$ under P.

We first obtain the EMM Q by a natural change of measure which shifts only the mean. According to Theorems 3.2.3 and 3.2.5, Y_t remains a symmetric Lévy process with characteristic triplet $(\tilde{\mu}, c, \nu)$ and the Q-distribution of Y_1 belongs to $S(\tilde{\mu}, \sigma^2, \psi)$, where $\tilde{\mu} = r - \ln\psi\big(-\frac{\sigma^2}{2}\big)$. Hence, Y_t has marginals from $S(\tilde{\mu}t, \sigma^2 t, \psi_t)$ under Q (see Proposition 3.1.1).

For the second EMM Q_1, the next result shows that it is also a natural EMM when Brownian component is present in Y_t, and gives the condition such that $S_t^{-1}e^{rt}$ is a Q_1-martingale.

Theorem 4.2.1. *Let Y_t be a symmetric Lévy process with characteristic triplet $(\tilde{\mu}, c, \nu)$, $c \neq 0$, under Q and the Q-distribution of Y_1 belongs to $S(\tilde{\mu}, \sigma^2, \psi)$. The measure Q_1 defined in (4.4) is a unique natural EMM if and only if the Q_1-distribution of Y_1 is $S(\tilde{\mu}_1, \sigma^2, \psi)$, where*

$$\tilde{\mu}_1 = r + \ln \psi\Big(-\frac{\sigma^2}{2}\Big). \tag{4.5}$$

Proof. Notice that $S_t^{-1}e^{rt} = S_0^{-1}e^{-Y_t+rt}$ is also a log-symmetric Lévy process, because $-Y_t$ is a Lévy process with characteristic triplets $(-\tilde{\mu}, c, \nu)$, and the distribution of $-Y_1 \sim S(-\tilde{\mu}, \sigma^2, \psi)$ (see Proposition 2.2.3). For $S_t^{-1}e^{rt}$ to be a Q_1-martingale, its expectation under Q_1 is constant which equals S_0^{-1}. Taking $t = 1$, we obtain the necessary condition

$$\mathbf{E}_{Q_1}\big(e^{-Y_1}\big) = e^{-r}.$$

The exponential moment of $-Y_1$ is given by

$$\mathbf{E}\big(e^{-Y_1}\big) = e^{-\tilde{\mu}}\psi\Big(-\frac{\sigma^2}{2}\Big).$$

Solving these two equations for $\tilde{\mu}$, we obtain the Q_1-distribution of Y_1 belongs to the symmetric family $S(\tilde{\mu}_1, \sigma^2, \psi)$ where

$$\tilde{\mu}_1 = r + \ln \psi\Big(-\frac{\sigma^2}{2}\Big).$$

By the uniqueness of $\tilde{\mu}_1$, Q_1 is unique.

□

Hence under Q_1, Y_t is a symmetric Lévy process with characteristic triplets $(\tilde{\mu}_1, c, \nu)$ and has marginals from the family $S(\tilde{\mu}_1 t, \sigma^2 t, \psi_t)$.

Observe that the distribution of the standardized variable $\frac{Y_T - E(Y_T)}{\sigma\sqrt{T}}$ is the same for all three probability measures P, Q and Q_1. This can be verified from its characteristic function, which is, under all probabilities, given by (see Proposition 3.1.1) $\psi^T(\frac{u^2}{2T})$. Denote by $F_T(y)$ the corresponding cumulative distribution function of the standardized variable, then

$$F_T(y) = P\left(\frac{Y_T - \mu T}{\sigma\sqrt{T}} \leq y\right) = Q\left(\frac{Y_T - \tilde{\mu} T}{\sigma\sqrt{T}} \leq y\right) = Q_1\left(\frac{Y_T - \tilde{\mu}_1 T}{\sigma\sqrt{T}} \leq y\right).$$

Consequently, we obtain the following pricing formula.

Theorem 4.2.2. *For stock price process $S_t = S_0 e^{Y_t}$ where Y_t is a symmetric Lévy process with characteristic triplet (μ, c, ν), $c \neq 0$, and the P-distribution of Y_1 belongs to $S(\mu, \sigma^2, \psi)$, the option pricing formula is given by*

$$\begin{aligned} C_0 = S_0 F_T &\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \ln\psi(-\frac{\sigma^2}{2})\right)T}{\sigma\sqrt{T}}\right) \\ &- e^{-rT} K F_T\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \ln\psi(-\frac{\sigma^2}{2})\right)T}{\sigma\sqrt{T}}\right). \end{aligned} \tag{4.6}$$

Proof. Since F_T is symmetric about zero, it holds that $1 - F_T(a) = F_T(-a)$. From Theorem 3.2.5 and (3.4), we obtain

$$\tilde{\mu} T = rT - \ln\psi_T\left(-\frac{\sigma^2 T}{2}\right) = rT - T\ln\psi\left(-\frac{\sigma^2}{2}\right).$$

Therefore,

$$\begin{aligned} Q(Y_T > a) = Q\left(\frac{Y_T - \tilde{\mu} T}{\sigma\sqrt{T}} > \frac{a - \tilde{\mu} T}{\sigma\sqrt{T}}\right) &= F_T\left(\frac{-a + \tilde{\mu} T}{\sigma\sqrt{T}}\right) \\ &= F_T\left(\frac{-a + \left(r - \ln\psi(-\frac{\sigma^2}{2})\right)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Similarly for Q_1, equations (4.5) and (3.4) give

$$\tilde{\mu}_1 T = rT + \ln\psi_T\left(-\frac{\sigma^2 T}{2}\right) = rT + T\ln\psi\left(-\frac{\sigma^2}{2}\right).$$

Therefore we obtain

$$Q_1(Y_T > a) = F_T\left(\frac{-a + \left(r + \ln\psi(-\frac{\sigma^2}{2})\right)T}{\sigma\sqrt{T}}\right).$$

Finally, using $a = \ln\left(\frac{K}{S_0}\right)$, we obtain the formula.

$\square$

4.3 Log-symmetric Lévy Model without Brownian Component

Consider now a stock price process $S_t = S_0 e^{Y_t}$, $t > 0$ where Y_t is a symmetric Lévy process with characteristic triplets $(\mu, 0, \nu)$, and the distribution of Y_1 belongs to $S(\mu, \sigma^2, \psi)$ under P.

The natural EMM Q changes the Lévy measure which in turn changes the variance of the marginal distribution yielding $\tilde{\sigma}^2$, which is the root of the equation

$$\ln\psi\left(\frac{-\tilde{\sigma}^2}{2}\right) = r - \mu, \tag{4.7}$$

and exists only if $\mu < r$. Hence, Y_t remains a symmetric Lévy process with characteristic triplet $(\mu, 0, \tilde{\nu})$ and the Q-distribution of Y_t belongs to $S(\mu t, \tilde{\sigma}^2 t, \psi_t)$.

For the second EMM Q_1, the next observation shows that it is not a natural EMM when Y_t has no Brownian component due to symmetry not preserved.

Proposition 4.3.1. *Let Y_t be a symmetric Lévy process with characteristic triplet $(\mu, 0, \tilde{\nu})$ under a natural EMM Q, and $\phi : \mathbb{R} \to \mathbb{R}$ satisfy $\int_{\mathbb{R}} (e^{\phi(y)/2} - 1)^2 \tilde{\nu}(dy) < \infty$. Then under the equivalent measure Q_1 defined in (4.4), the process Y_t is a Lévy process with characteristic triplet $(a_1, 0, \tilde{\nu}_1)$, where the drift*

$$a_1 = \mu + \int_{-1}^{1} y \tilde{\nu}_1(dy). \tag{4.8}$$

and the Lévy measure $\tilde{\nu}_1(dy) = e^{\phi(y)} \tilde{\nu}(dy)$ is not symmetric.

Proof. The proof uses Theorems 3.2.1 and 3.2.2. Let

$$\left.\frac{dQ_1}{dQ}\right|_{\mathcal{F}_t} = \frac{e^{-rt}S_t}{S_0} = e^{Y_t - rt} = e^{U_t}. \tag{4.9}$$

We have $\mathbf{E}_Q(e^{U_t}) = e^{-rt}\mathbf{E}_Q(e^{Y_t}) = e^{-rt}e^{t\Psi(-i)} = 1$ if and only if $\Psi(-i) = r$.

Using the canonical form (see (2.27)) of a Lévy process Y_t (finite mean) with characteristic triplet $(\mu, 0, \tilde{\nu})$, we can write

$$\begin{aligned}
U_t &= Y_t - rt \\
&= (-r+\mu)t + \lim_{\epsilon \downarrow 0}\left(\sum_{s\le t,\ |\Delta Y_s|>\epsilon} \Delta Y_s - t\int_{|y|>\epsilon} y\tilde{\nu}(dy)\right) \\
&= \Big(-r + \mu + \int_{\mathbb{R}} \big(e^y - 1 - y\big)\tilde{\nu}(dy)\Big)t \\
&\qquad + \lim_{\epsilon \downarrow 0}\left(\sum_{s\le t,\ |\Delta Y_s|>\epsilon} \Delta Y_s - t\int_{|y|>\epsilon} \big(e^y - 1\big)\tilde{\nu}(dy)\right) \\
&= \lim_{\epsilon \downarrow 0}\left(\sum_{s\le t,\ |\Delta Y_s|>\epsilon} \Delta Y_s - t\int_{|y|>\epsilon} \big(e^y - 1\big)\tilde{\nu}(dy)\right),
\end{aligned}$$

in which we have used $\Psi(-i) = \mu + \int_{\mathbb{R}}\big(e^y - 1 - y\big)\tilde{\nu}(dy) = r$ in the last equality.

Clearly, U_t conforms with D_t in Theorem 3.2.1 for the case $\eta = 0$. Hence, (4.9) is a change of measure that preserves Lévy process. However, the function $\phi(y) = y$ is not an even function. Therefore, by Theorem 3.2.2, the Lévy measure $\tilde{\nu}_1$ is not symmetric, which implies that the Lévy process Y_t is asymmetric under Q_1. The expressions of the characteristic triplet $(a_1, 0, \tilde{\nu}_1)$ under Q_1 follows from Theorem 3.2.1 for the case $c = 0$ and $\tilde{\nu}$ symmetric. □

Corollary 4.3.1. *Under Q_1, the asymmetric Lévy process Y_t has finite mean* *and variance. They are $\mu_1 t$ and $\tilde{\sigma}_1^2 t$ respectively, where (see Proposition 2.3.5)*

$$\mu_1 = \mathbf{E}_{Q_1}(Y_1) = a_1 + \int_{|y|>1} y\tilde{\nu}_1(dy) = \mu + \int_{\mathbb{R}} y\tilde{\nu}_1(dy),$$
$$\tilde{\sigma}_1^2 = Var_{Q_1}(Y_1) = \int_{\mathbb{R}} y^2\tilde{\nu}_1(dy).$$

Although Q_1 is not a natural EMM, it is useful for option pricing. The Q_1-distribution of Y_t can be determined using the following observation.

Proposition 4.3.2. *Denote by $f_{Y_t}^{Q}$ the Q-density of Y_t. Then the Q_1-density of Y_t is given by*

$$f_{Y_t}^{Q_1}(y) = e^{y-rt} f_{Y_t}^{Q}(y). \tag{4.10}$$

Proof. By the definition of equivalent measure Q_1, we have $\mathbf{E}_Q\big(e^{Y_t - rt}\big) = 1$, and for $A = \{y : y \le u\}$,

$$Q_1(Y_t \in A) = \mathbf{E}_{Q_1}\big(I(Y_t \in A)\big) = \mathbf{E}_Q\big(e^{Y_t - rt} I(Y_t \in A)\big) = \int_A e^{y-rt} f_{Y_t}^{Q}(y)dy.$$

$\square$

In specific cases considered in this thesis, namely Variance Gamma and Normal Inverse Gaussian (see Chapters 5 and 6, respectively), we are able to identify the distributions using (4.10). In such cases, denote by $F_{Y_T}^{Q_1}$ the cumulative distribution function of the standardized random variable $(Y_T - \mu_1 T)/\tilde{\sigma}_1\sqrt{T}$ under Q_1, and by $F_{Y_T}^{Q}$ the cumulative distribution function of the standardized random variable $(Y_T - \mu T)/\tilde{\sigma}\sqrt{T}$ under Q, we obtain the exact option pricing formula

$$\begin{aligned} C_0 &= S_0 Q_1\big(S_T > K\big) - e^{-rT} K Q\big(S_T > K\big) \\ &= S_0\left[1 - F_{Y_T}^{Q_1}\left(-\frac{\ln\left(\frac{S_0}{K}\right) + \mu_1 T}{\tilde{\sigma}_1\sqrt{T}}\right)\right] \\ &\qquad - e^{-rT} K F_{Y_T}^{Q}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \mu T}{\tilde{\sigma}\sqrt{T}}\right). \end{aligned} \tag{4.11}$$

4.4 Log-symmetric Lévy Model in Discrete Time

Let now $S_N = S_0 e^{Y_N}$ be a model for stock prices, where $Y_N = \sum_{n=1}^{N} \Delta Y_n$ is a symmetric Lévy process in discrete time, and ΔY_n, $n = 1, \dots, N$ are i.i.d with a symmetric distribution from $S(\mu, \sigma^2, \psi)$ under P.

The natural EMM Q always exist (see Remark 3.2), which can be obtained by changing only the location parameter (mean) μ. The Q-distribution of ΔY_n belongs to $S(\tilde{\mu}, \sigma^2, \psi)$, where $\tilde{\mu} = r - \ln \psi\big(-\frac{\sigma^2}{2}\big)$. Hence, Y_N has distribution $S(\tilde{\mu}N, \sigma^2 N, \psi_N)$ for all N under Q.

By Theorem 2.2.3 (see also [35], Theorem 3.2), the measure Q_1 defined by

$$\left.\frac{dQ_1}{dQ}\right|_{\mathcal{F}_N} = \frac{S_N e^{-rN}}{S_0}, \tag{4.12}$$

is a unique natural EMM. The Q_1-distribution of ΔY_n belongs to $S(\tilde{\mu}_1, \sigma^2, \psi)$ where $\tilde{\mu}_1 = r + \ln \psi\big(-\frac{\sigma^2}{2}\big)$. Hence, Y_N has distribution $S(\tilde{\mu}_1 N, \sigma^2 N, \psi_N)$ for all N under Q_1.

Notice that the distribution of the standardized variable $\frac{Y_N - E(Y_N)}{\sigma\sqrt{N}}$ is the same for all three probability measures P, Q and Q_1. Denote by $F_N(y)$ the cumulative distribution function of the standardized variable, then

$$F_N(y) = P\left(\frac{Y_N - \mu N}{\sigma\sqrt{N}} \le y\right) = Q\left(\frac{Y_N - \tilde{\mu} N}{\sigma\sqrt{N}} \le y\right) = Q_1\left(\frac{Y_N - \tilde{\mu}_1 N}{\sigma\sqrt{N}} \le y\right).$$

Theorem 4.4.1. *Let stock price $S_N = S_0 e^{Y_N}$ where Y_N is a symmetric Lévy process in discrete time, and the P-distribution of the ΔY_n, $n = 1, \dots, N$, belongs to $S(\mu, \sigma^2, \psi)$. The option pricing formula is given by*

$$\begin{aligned} C_0 = S_0 F_N &\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \ln\psi(-\frac{\sigma^2}{2})\right)N}{\sigma\sqrt{N}}\right) \\ &- e^{-rN} K F_N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \ln\psi(-\frac{\sigma^2}{2})\right)N}{\sigma\sqrt{N}}\right). \end{aligned} \tag{4.13}$$

Proof. Since F_N is symmetric about zero, it holds that $1 - F_N(\alpha) = F_N(-\alpha)$. By theorem 3.2.5 and (3.4), we obtain

$$\tilde{\mu}N = rN - \ln \psi_N \left(-\frac{\tilde{\sigma}^2 N}{2}\right) = rN - N \ln \psi \left(-\frac{\tilde{\sigma}^2}{2}\right).$$

Therefore,

$$\begin{aligned} Q(Y_N > \alpha) = Q\left(\frac{Y_N - \tilde{\mu}N}{\sigma\sqrt{N}} > \frac{\alpha - \tilde{\mu}N}{\sigma\sqrt{N}}\right) &= F_N\left(\frac{-\alpha + \tilde{\mu}N}{\sigma\sqrt{N}}\right) \\ &= F_N\left(\frac{-\alpha + \big(r - \ln \psi(-\frac{\sigma^2}{2})\big)N}{\sigma\sqrt{N}}\right). \end{aligned}$$

Similarly for Q_1, we obtain

$$Q_1(Y_N > \alpha) = F_N\left(\frac{-\alpha + \big(r + \ln \psi(-\frac{\sigma^2}{2})\big)N}{\sigma\sqrt{N}}\right).$$

Using $\alpha = \ln\left(\frac{K}{S_0}\right)$, we obtain the formula.

□

Chapter 5

Log-symmetric Variance Gamma Model

This chapter contains application of the natural EMM approach to option pricing with log-symmetric VG price process.

5.1 Symmetric Variance Gamma Process and Distribution

The marginal distributions of the VG process was originally given in [39] in terms of special functions involving the modified Bessel function of the second kind and the degenerate hypergeometric function. In the special case of symmetric processes, the marginals turn out to be symmetric Bessel distribution. A brief introduction of the Bessel distribution is given in Appendix B.

Denote by $Bessel(\mu, \sigma^2, \lambda)$ the Bessel distribution with mean μ, variance σ^2 and shape parameter λ. A random variable X with symmetric Bessel distribution has mean $\mu = 0$, and has characteristic function (B.9) of the form

$$\mathbf{E}(e^{iuX}) = \left(\frac{1}{1 + \frac{u^2\sigma^2}{2\lambda}}\right)^{\lambda}.$$

The density function of the symmetric Bessel distribution is given by (B.8)

$$f_X(x) = \sqrt{\frac{2\lambda}{\pi\sigma^2}} \left(\sqrt{\frac{\lambda x^2}{2\sigma^2}} \right)^{\lambda - \frac{1}{2}} \frac{1}{\Gamma(\lambda)} K_{\lambda - \frac{1}{2}} \left(2\sqrt{\frac{\lambda x^2}{2\sigma^2}} \right), \tag{5.1}$$

where $K_w(.)$ is the modified Bessel function of the second kind. Therefore, X shifted by a constant μ has a symmetric Bessel distribution with mean μ, i.e., $Y = X + \mu \sim Bessel(\mu, \sigma^2, \lambda)$. The characteristic function of a shifted symmetric Bessel random variable Y is given by

$$\mathbf{E}(e^{iuY}) = e^{iu\mu} \left(\frac{1}{1 + \frac{u^2\sigma^2}{2\lambda}} \right)^{\lambda}, \tag{5.2}$$

and the density function of Y is given by

$$f_Y(y) = f_X(y - \mu),$$

where $f_X(\cdot)$ is defined in (5.1). We always consider symmetric Bessel distribution of this kind but we will drop the word "shifted" for brevity. The symmetric Bessel distribution $Bessel(\mu, \sigma^2, \lambda)$ belongs to the family of symmetric distributions $S(\mu, \sigma^2, \psi)$ with characteristic generator

$$\psi(v) = \left(\frac{1}{1 + \frac{v}{\lambda}} \right)^{\lambda}. \tag{5.3}$$

This can be easily checked from the characteristic function (5.2). Note that the kurtosis of symmetric Bessel distribution is $3 + \frac{3}{\lambda}$ and hence, the shape parameter λ is related to the excess kurtosis by $\lambda = \frac{3}{\gamma}$ (since the excess kurtosis of a random variable Y is $\gamma = \frac{\mathbf{E}[(Y-\mu)^4]}{\sigma^4} - 3$. For Normal distribution, $\gamma = 0$).

A symmetric VG process Z_t also has zero mean (see Section 2.3.4). The characteristic function of the marginal distribution of a symmetric VG process at time t is given by [39]

$$\mathbf{E}(e^{iuZ_t}) = \left(\frac{1}{1 + \frac{u^2\sigma^2\kappa}{2}} \right)^{\frac{t}{\kappa}},$$

where σ^2 is the variance of Z_1 and κ is the variance rate of the gamma subordinator. By adding a drift μt, the Lévy process $Y_t = Z_t + \mu t$ has characteristic function

$$\mathbf{E}(e^{iuY_t}) = e^{iu\mu t}\left(\frac{1}{1+\frac{u^2\sigma^2\kappa}{2}}\right)^{\frac{t}{\kappa}} = e^{iu\mu t}\left(\frac{1}{1+\frac{u^2\sigma^2 t}{2\lambda t}}\right)^{\lambda t}, \tag{5.4}$$

in which we have employed $\lambda = \frac{1}{\kappa}$. Again, for brevity, we will refer to a symmetric VG process with drift as just symmetric VG process.

In the following theorem, we identify the marginals of a symmetric VG process for any time t.

Theorem 5.1.1. *The marginals of a symmetric Variance Gamma process Y_t is a symmetric Bessel distribution with mean μt, variance $\sigma^2 t$ and shape parameter λt, i.e., $Y_t \sim Bessel(\mu t, \sigma^2 t, \lambda t)$, which belongs to the family of symmetric distributions $S(\mu t, \sigma^2 t, \psi_t)$ where the characteristic generator is given by*

$$\psi_t(v) = [\psi(v/t)]^t = \left(\frac{1}{1+\frac{v}{\lambda t}}\right)^{\lambda t}. \tag{5.5}$$

Proof. The proof follows immediately upon comparing the characteristic function of a symmetric VG process (5.4) and the characteristic function of a symmetric Bessel distribution (5.2). By using (3.4) and (5.3), we obtain the characteristic generator (5.5).

□

Remark 5.1. *In [39], the density function for the marginal distributions of VG process was known, and has the form* (2.33)*. However, they did not link them to the class of symmetric Bessel distributions.*

Remark 5.2. *The marginals of a VG process is not Bessel distribution in general. Only when the VG process is symmetric, then it has a symmetric Bessel distribution for all time t.*

5.2 Option Pricing with Log-symmetric Variance Gamma Process in Continuous Time

Let now $S_t = S_0 e^{Y_t}$ be a stock price process, where Y_t is a symmetric VG process with characteristic triplets $(\mu, 0, \nu)$ under P. Since the VG process has no Brownian component, the derivation of the option pricing formula follows the arguments presented in Sections 4.3. Assuming the condition $\mu < r$ is met, it remains to determine the distributions of Y_t under the EMM's Q (natural) and Q_1 (not natural).

By Theorems 3.2.3 and 3.2.5, the Q-distribution of Y_1 is symmetric Bessel distribution $Bessel(\mu, \tilde{\sigma}^2, \psi)$ where $\tilde{\sigma}^2$ is a solution of the equation

$$\ln \psi\Big(-\frac{\tilde{\sigma}^2}{2}\Big) = r - \mu. \tag{5.6}$$

For symmetric Bessel distribution, this solution exist and it is unique. Using (5.3) we get from (5.6)

$$\tilde{\sigma}^2 = 2\lambda(1 - e^{-(r-\mu)/\lambda}). \tag{5.7}$$

Hence, the Q-distribution Y_t is $Bessel(\mu t, \tilde{\sigma}^2 t, \lambda t)$ where $\tilde{\sigma}^2$ is given in (5.7).

Under Q_1, the distribution of Y_t is identified in the following result.

Proposition 5.2.1. *Denote by $f^Q_{Y_t}$ the Q-density of $Y_t \sim Bessel(\mu t, \tilde{\sigma}^2 t, \lambda t)$. Then the Q_1-density of Y_t, given by $e^{y-rt} f^Q_{Y_t}(y)$, is the density function of an asymmetric Bessel distribution.*

Proof. Using (5.1), the Q_1-density of Y_t is give by

$$\begin{aligned} &e^{y-rt} f^Q_{Y_t}(y) \\ &= e^{y-rt}\sqrt{\frac{2\lambda}{\pi\tilde{\sigma}^2}}\Bigg(\sqrt{\frac{\lambda(y-\mu t)^2}{2\tilde{\sigma}^2}}\Bigg)^{\lambda t - \frac{1}{2}} \frac{1}{\Gamma(\lambda t)} K_{\lambda t - \frac{1}{2}}\Bigg(2\sqrt{\frac{\lambda(y-\mu t)^2}{2\tilde{\sigma}^2}}\Bigg). \end{aligned} \tag{5.8}$$

For any time t, (5.6) gives

$$\ln \psi_t\Big(-\frac{\tilde{\sigma}^2 t}{2}\Big) = rt - \mu t.$$

Using the characteristic generator (5.5) for the symmetric Bessel distribution, it follows that

$$e^{y-rt} = e^{y-\mu t}\left(1 - \frac{\tilde{\sigma}^2}{2\lambda}\right)^{\lambda t}. \tag{5.9}$$

Now, apply (5.9) and let $y^* = y - \mu t$, the expression in the second line of (5.8) becomes

$$e^{y^*}\left(1 - \frac{\tilde{\sigma}^2}{2\lambda}\right)^{\lambda t}\sqrt{\frac{2\lambda}{\pi\tilde{\sigma}^2}}\left(\sqrt{\frac{\lambda(y^*)^2}{2\tilde{\sigma}^2}}\right)^{\lambda t-\frac{1}{2}}\frac{1}{\Gamma(\lambda t)}K_{\lambda t-\frac{1}{2}}\left(\sqrt{\frac{2\lambda(y^*)^2}{\tilde{\sigma}^2}}\right).$$

Subsequently, let $m = \lambda t - \frac{1}{2}$, $a = -\sqrt{\frac{\tilde{\sigma}^2}{2\lambda}}$ and $b = \sqrt{\frac{\tilde{\sigma}^2}{2\lambda}}$, we obtain

$$\begin{aligned} e^{y-rt}f^Q_{Y_t}(y) &= e^{y^*}\left(1-a^2\right)^{m+\frac{1}{2}}\frac{1}{b\sqrt{\pi}}\left(\frac{|y^*|}{2b}\right)^m\frac{1}{\Gamma(m+\frac{1}{2})}K_m\left(\frac{|y^*|}{b}\right) \\ &= \frac{(1-a^2)^{m+\frac{1}{2}}|y^*|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+\frac{1}{2})}e^{y^*}K_m\left(\left|\frac{y^*}{b}\right|\right). \end{aligned} \tag{5.10}$$

On closer observation, we immediately recognize that the density written in the form (5.10) is the density of an asymmetric Bessel distribution (see Appendix B, eq.(B.2) or [32], p.50), where $a = -\sqrt{\frac{\tilde{\sigma}^2}{2\lambda}}$, $b = -a$ and $m = \lambda t - \frac{1}{2}$.

□

Let the auxiliary process $Y^*_t = Y_t - \mu t$, it follows from (5.10) that Y^*_t has asymmetric Bessel distribution, denoted $Bessel^1(\mu_1 t, \tilde{\sigma}_1^2 t, \lambda t)$, where the mean $\mu_1 t$ and variance $\tilde{\sigma}_1^2 t$ are given by (B.4) and (B.5) respectively. In particular,

$$\mathbf{E}(Y^*_1) = \mu_1 = 2\lambda\left(e^{(r-\mu)/\lambda} - 1\right), \tag{5.11}$$

$$Var(Y^*_1) = \tilde{\sigma}_1^2 = 2\lambda\left(e^{(r-\mu)/\lambda} - 1\right)\left(2e^{(r-\mu)/\lambda} - 1\right), \tag{5.12}$$

in which we have employed (5.7). Then under Q_1, $Y_t = Y^*_t + \mu t \sim Bessel^1(\mu t + \mu_1 t, \tilde{\sigma}_1^2 t, \lambda t)$ for all time t.

Finally, denote by $B_{\lambda t}(y)$ the cumulative distribution function of the standardized symmetric Bessel random variable $\frac{Y_t - \mu t}{\tilde{\sigma}\sqrt{t}} \sim Bessel(0, 1, \lambda t)$ under Q, and by $B^1_{\lambda t}(y)$ the cumulative distribution function of the standardized asymmetric Bessel random variable $\frac{Y_t-\mu t-\mu_1 t}{\tilde{\sigma}_1\sqrt{t}} \sim Bessel^1(0,1,\lambda t)$ under Q_1. The exact option pricing formula is given next.

Theorem 5.2.1. *Let the stock price $S_t = S_0 e^{Y_t}$ where Y_t is a symmetric VG process with marginals $Bessel(\mu t, \sigma^2 t, \lambda t)$, and $\mu < r$. Then the arbitrage-free price using natural EMM of a call option with time to expiration T is given by*

$$C_0 = S_0\left[1 - B^1_{\lambda T}\left(-\frac{\ln\left(\frac{S_0}{K}\right)+\mu T+\mu_1 T}{\tilde{\sigma}_1\sqrt{T}}\right)\right] - e^{-rT}KB_{\lambda T}\left(\frac{\ln\left(\frac{S_0}{K}\right)+\mu T}{\tilde{\sigma}\sqrt{T}}\right), \tag{5.13}$$

where $\mu_1 = 2\lambda\big(e^{(r-\mu)/\lambda}-1\big)$, $\tilde{\sigma}_1^2 = 2\lambda\big(e^{(r-\mu)/\lambda}-1\big)\big(2e^{(r-\mu)/\lambda}-1\big)$ and $\tilde{\sigma}^2 = 2\lambda(1-e^{-(r-\mu)/\lambda})$.

Note that formulae for option pricing with VG process were given in ([40], eq. 6.7) and ([39], eq. 25), where a non-symmetric case was also included. They derived it by analytical methods and used the Normal density function integrated with respect to a gamma density. For comparison, we reproduce below, the closed form option pricing formula given in [39]. Specifically, denote by $Y(t;\theta,\sigma,\kappa)$ the VG process obtained by gamma time-changed Brownian motion [39], where θ and σ are, respectively, the drift and the volatility of the Brownian motion, and κ is the variance rate of the gamma time change. Then the option price on stock when the risk neutral dynamics of the stock price is governed by a VG process (with risk-neutral parameters $\theta,\ \sigma,\ \kappa$) is given by

$$C_0 = S_0\Upsilon\left(d\sqrt{\frac{1-l_1}{\kappa}}, (\zeta+1)s\sqrt{\frac{\kappa}{1-l_1}}, \frac{T}{\kappa}\right) - Ke^{-rT}\Upsilon\left(d\sqrt{\frac{1-l_2}{\kappa}}, \zeta s^2\sqrt{\frac{\kappa}{1-l_2}}, \frac{T}{\kappa}\right),$$

where

$$
\begin{aligned}
d &= \frac{1}{s}\left[\ln\left(\frac{S_0}{K}\right) + rT + \frac{T}{\kappa}\ln\left(\frac{1-l_1}{1-l_2}\right)\right],\\
\zeta &= -\frac{\theta}{\sigma^2},\\
s &= \sigma\left[1+\left(\frac{\theta}{\sigma}\right)^2\frac{\kappa}{2}\right]^{-\frac{1}{2}},\\
l_1 &= \frac{1}{2}\kappa(\zeta+1)^2 s^2,\\
l_2 &= \frac{1}{2}\kappa(\zeta s)^2,
\end{aligned}
$$

and the function Υ is defined by

$$
\begin{aligned}
\Upsilon(a,b,\gamma) = {} & \frac{\alpha^{\gamma+\frac{1}{2}} e^{\operatorname{sign}(a)\alpha}(1+u)^{\gamma}}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \times \\
& \quad K_{\gamma+\frac{1}{2}}(\alpha)\Theta\left(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)\alpha(1+u)\right) \\
& - \operatorname{sign}(a)\frac{\alpha^{\gamma+\frac{1}{2}} e^{\operatorname{sign}(a)\alpha}(1+u)^{1+\gamma}}{\sqrt{2\pi}\Gamma(\gamma)(1+\gamma)} \times \\
& \quad K_{\gamma-\frac{1}{2}}(\alpha)\Theta\left(1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)\alpha(1+u)\right) \\
& + \operatorname{sign}(a)\frac{\alpha^{\gamma+\frac{1}{2}} e^{\operatorname{sign}(a)\alpha}(1+u)^{\gamma}}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \times \\
& \quad K_{\gamma-\frac{1}{2}}(\alpha)\Theta\left(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\operatorname{sign}(a)\alpha(1+u)\right),
\end{aligned}
$$

where $\alpha = |a|\sqrt{2+b^2}$, $u = \frac{b|a|}{\alpha}$, K_ω is the modified Bessel function of the second kind of order ω, and Θ is the degenerate hypergeometric function of two variables which has the integral representation

$$
\Theta(\lambda,\delta,\rho;x,y) = \frac{\Gamma(\rho)}{\Gamma(\lambda)\Gamma(\rho-\lambda)}\int_0^1 u^{\lambda-1}(1-u)^{\rho-\lambda-1}(1-ux)^{-\delta}e^{uy}du.
$$

As we can see, for symmetric case, the formula (5.13) is much simpler and elegant.

5.3 Option Pricing with Log-symmetric Variance Gamma Process in Discrete Time

Let now $S_N = S_0 e^{Y_N}$ be a model for stock prices, where $Y_N = \sum_{n=1}^{N} \Delta Y_n$ is a symmetric VG process in discrete time under P. The ΔY_n, $n = 1, \dots, N$ are i.i.d with symmetric Bessel distribution $Bessel(\mu, \sigma^2, \lambda)$.

It is possible to choose both Q and Q_1 as natural EMM's that shift only the location parameter (see Section 4.4). Hence, the Q-distribution of ΔY_n is $Bessel(\tilde{\mu}, \sigma^2, \lambda)$ with $\tilde{\mu} = r - \ln\psi\big(-\frac{\sigma^2}{2}\big)$. The Q_1-distribution of ΔY_n is $Bessel(\tilde{\mu}_1, \sigma^2, \lambda)$ with $\tilde{\mu}_1 = r + \ln\psi\big(-\frac{\sigma^2}{2}\big)$. Apply (5.3), we obtain

$$\ln\psi\Big(-\frac{\sigma^2}{2}\Big) = -\lambda \ln\Big(1 - \frac{\sigma^2}{2\lambda}\Big). \tag{5.14}$$

Denote by $B_{\lambda N}(y)$ the cumulative distribution function of the standardized symmetric Bessel random variable $\frac{Y_N - \mu N}{\sigma\sqrt{N}}$, we obtain the following result for option pricing formula.

Theorem 5.3.1. *Let ΔY_n follow a symmetric Bessel distribution $Bessel(\mu, \sigma^2, \lambda)$, then the arbitrage-free price of a call option with N periods to expiration is given by*

$$\begin{aligned} C_0 = {} & S_0 B_{\lambda N}\left(\frac{\ln\big(\frac{S_0}{K}\big) + \big(r - \lambda\ln(1 - \frac{\sigma^2}{2\lambda})\big)N}{\sigma\sqrt{N}}\right) \\ & - e^{-rN} K B_{\lambda N}\left(\frac{\ln\big(\frac{S_0}{K}\big) + \big(r + \lambda\ln(1 - \frac{\sigma^2}{2\lambda})\big)N}{\sigma\sqrt{N}}\right). \end{aligned} \tag{5.15}$$

5.4 Numerical Comparisons

For comparisons, we approximate the distributions of the standardized Bessel random variables $(Y_T - \mu T)/\sigma\sqrt{T}$ (the time T is replaced by N in the discrete case) by the standard Normal, in other words, $B_{\lambda T}$ by Φ. We also approximate the standardized asymmetric Bessel random variable that arises in the continuous time case by the standard Normal, i.e., $B^1_{\lambda T}$ by Φ, because its distribution is only slightly negatively skewed and therefore it is negligible.

We will assume this is the case (skewness is small) in our approximation. Moreover, recall that the shape parameter λ and the excess kurtosis γ of the symmetric Bessel distribution are related by $\lambda = \frac{3}{\gamma}$. Thus, for each of the continuous time and discrete time cases, we obtain an easy to use Black-Scholes type formula for option pricing which gives correction that accounts for the access kurtosis.

In the continuous time case, the generalized or modified Black-Scholes formula for log-symmetric VG model (VG-C) is given by

$$C_0 \approx S_0 \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \mu T + \mu_1 T}{\tilde{\sigma}_1 \sqrt{T}}\right) - e^{-rT} K \Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \mu T}{\tilde{\sigma}\sqrt{T}}\right), \qquad (5.16)$$

where $\mu_1 = \frac{6}{\gamma}\big(e^{(r-\mu)\gamma/3} - 1\big)$, $\tilde{\sigma}_1^2 = \frac{6}{\gamma}\big(e^{(r-\mu)\gamma/3} - 1\big)\big(2e^{(r-\mu)\gamma/3} - 1\big)$ and $\tilde{\sigma}^2 = \frac{6}{\gamma}(1 - e^{-(r-\mu)\gamma/3})$. Note that the Black-Scholes formula is a special case of the generalized version (VG-C) (5.16) when $\gamma \to 0$ due to the followings:

$$\begin{aligned}
\mu_1 &= \frac{6}{\gamma}\big(e^{(r-\mu)\gamma/3} - 1\big) \to 2(r-\mu), \\
\tilde{\sigma}_1^2 &= \frac{6}{\gamma}\big(e^{(r-\mu)\gamma/3} - 1\big)\big(2e^{(r-\mu)\gamma/3} - 1\big) \to 2(r-\mu), \\
\tilde{\sigma}^2 &= \frac{6}{\gamma}\big(1 - e^{-(r-\mu)\gamma/3}\big) \to 2(r-\mu).
\end{aligned}$$

And if $2(r-\mu) = \sigma^2$, which is a constant as in the Black-Scholes model (Recall that under the risk-neutral measure Q, the mean $\mu = r - \frac{\sigma^2}{2}$ and the volatility σ is a constant), then by using these results and some simple manipulations, it is not hard to see that the generalized formula (VG-C) (5.16) is the exact Black-Scholes formula.

In the discrete time case, the modified Black-Scholes formula for log-symmetric VG model (VG-D) is given by

$$C_0 \approx S_0 \Phi \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{3}{\gamma}\ln(1 - \frac{\gamma\sigma^2}{6})\right)N}{\sigma\sqrt{N}} \right)$$
$$- e^{-rN} K \Phi \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{3}{\gamma}\ln(1 - \frac{\gamma\sigma^2}{6})\right)N}{\sigma\sqrt{N}} \right). \quad (5.17)$$

It can be seen that the Black-Scholes formula is a limit of the generalized version (VG-D) (5.17) for every N when $\gamma \to 0$ due to

$$\frac{3}{\gamma}\ln\left(1 - \frac{\gamma\sigma^2}{6}\right) \to -\frac{\sigma^2}{2}.$$

The classical Black-Scholes formula (BS) is considered robust in the sense that for small values of excess kurtosis γ, it coincides with the modified Black-Scholes formulae in both continuous time and discrete time cases. Note that the robustness of Black-Scholes formula was discussed in [16] and [46]. In their papers, the robustness is with respect to stochastic volatility. More specifically, [16] provided conditions under which the Black-Scholes formula is robust when there is a misspecification of volatility in a two-asset market model for option pricing. While [46] gave some sufficient conditions for the reduction of the Black-Scholes-Barenblatt equation, which is a nonlinear partial differential equation that arises in multi-asset market model for European options, to a linear Black-Scholes equation. In our case, the robustness of Black-Scholes formula is with respect to the heavy tail probability of the symmetric distributions. However, even for the moderate values of excess kurtosis γ, the distinction between the modified Black-Scholes formulae (VG-C and VG-D) and BS is noticeable (see Figure 5.1), with the disagreement between VG-C and BS formulae being greater than the disagreement between VG-D and BS. The exact prices and percentage differences are represented in Table 5.1.

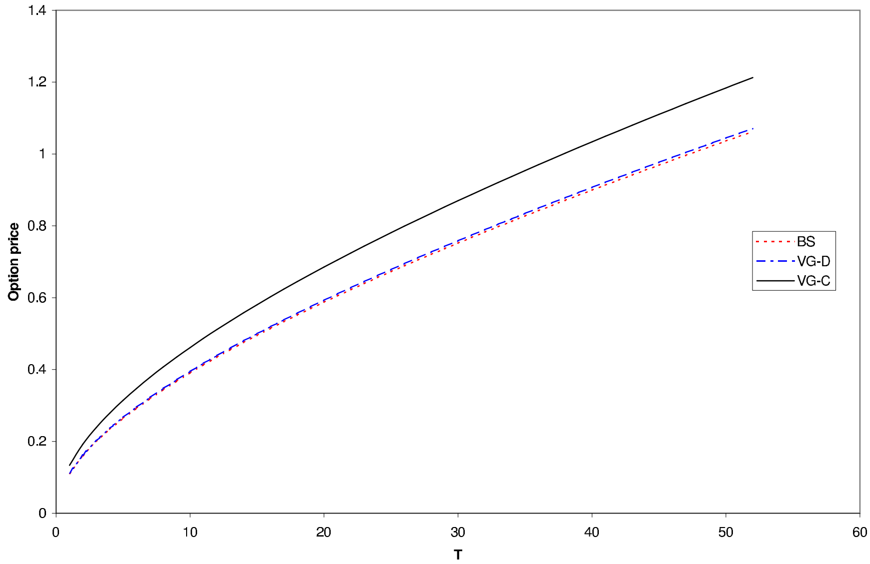

Figure 5.1: Option prices obtained by VG-C, VG-D and BS formulae for log-symmetric Bessel distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

Time to maturity (weeks)	2	12	22	32	42	52
BS formula	0.1603	0.4344	0.6221	0.7823	0.9273	1.0622
VG-D formula	0.1621	0.4388	0.6279	0.7892	0.9351	1.0707
Percentage difference (%)	1.13	1.01	0.94	0.88	0.84	0.80
VG-C formula	0.1921	0.5112	0.7245	0.9040	1.0647	1.2127
Percentage difference (%)	19.85	17.67	16.46	15.55	14.81	14.17

Table 5.1: Option prices and percentage differences obtained by VG-C, VG-D and BS formulae for log-symmetric Bessel distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

Chapter 6

Log-symmetric Normal Inverse Gaussian Model

This chapter contains application of the natural EMM approach to option pricing for log-symmetric NIG price process.

6.1 Symmetric NIG Process and Distribution

Recall from Section 2.3.5 that a NIG process is a pure jump Lévy process with NIG marginal distributions, denoted by $NIG(\alpha, \beta, \delta, \mu)$. Therefore, a symmetric NIG Lévy process has symmetric NIG marginal distributions, i.e., when $\beta = 0$. The density of a symmetric NIG (see (2.40)) distribution is

$$f_Y(y) = \frac{\alpha}{\pi} e^{\alpha\delta} \frac{K_1\Big(\alpha\delta\sqrt{1 + (\frac{y-\mu}{\delta})^2}\Big)}{\sqrt{1 + (\frac{y-\mu}{\delta})^2}}, \tag{6.1}$$

where K_1 is the modified Bessel function of the third kind. The characteristic function for symmetric NIG (see (2.41)) is

$$\varphi(u) = e^{iu\mu} e^{\alpha\delta\left(1-\sqrt{1+(\frac{u}{\alpha})^2}\right)}. \tag{6.2}$$

It follows from equations (2.42), (2.43) and (2.45), respectively, that μ is the mean, variance is $\frac{\delta}{\alpha}$ and kurtosis is $3 + \frac{3}{\alpha\delta}$. We will denote the distribution of a symmetric NIG by $SNIG(\alpha, 0, \delta, \mu)$.

Proposition 6.1.1. *The characteristic generator of the symmetric NIG family of distributions is given by*

$$\psi(v) = e^{\zeta\left(1-\sqrt{1+\frac{2v}{\zeta}}\right)}, \tag{6.3}$$

where $\zeta = \alpha\delta$.

Proof. Denote by σ^2, the variance of symmetric NIG, then

$$\sigma^2 = \frac{\delta}{\alpha} = \frac{\delta^2}{\alpha\delta}.$$

The characteristic function of symmetric NIG can be written as

$$\begin{aligned}\varphi(u) = e^{iu\mu} e^{\alpha\delta\left(1-\sqrt{1+(\frac{u}{\alpha})^2}\right)} &= e^{iu\mu} e^{\alpha\delta\left(1-\sqrt{1+(\frac{u\delta}{\alpha\delta})^2}\right)} \\ &= e^{iu\mu} e^{\zeta\left(1-\sqrt{1+\frac{u^2\sigma^2}{\zeta}}\right)},\end{aligned}$$

where $\zeta = \alpha\delta$. Thus, the characteristic generator is

$$\psi(v) = e^{\zeta\left(1-\sqrt{1+\frac{2v}{\zeta}}\right)},$$

where $v = \frac{\sigma^2 u^2}{2}$, and it is not hard to verify that $\psi'(0) = -1$.

□

Thus, the marginals of a symmetric NIG process Y_t is $SNIG(\alpha, 0, \delta t, \mu t)$ which belongs to the family $S(\mu t, \sigma^2 t, \psi_t)$ with $\sigma^2 = \frac{\delta}{\alpha}$, and

$$\psi_t(v) = \left[\psi(v/t)\right]^t = e^{\zeta t\left(1-\sqrt{1+\frac{2v}{\zeta t}}\right)}. \tag{6.4}$$

Note that the parameter ζ is related to the excess kurtosis γ by $\gamma = \frac{3}{\zeta}$.

6.2 Option Pricing with Log-symmetric NIG Process in Continuous Time

Let $S_t = S_0 e^{Y_t}$ be the stock price process where Y_t is a symmetric NIG Lévy process with characteristic triplets $(\mu, 0, \nu)$ under P. Since the NIG process has no Brownian component, the derivation of the option pricing formula follows the arguments presented in Section 4.3. Assume $\mu < r$, we determine the distributions of Y_t under the EMM's Q (natural) and Q_1 (not natural).

By Theorems 3.2.3 and 3.2.5, the Q-distribution of Y_1 is symmetric NIG distribution $SNIG(\alpha, 0, \tilde{\delta}, \mu)$, where $\frac{\tilde{\delta}}{\alpha} = \tilde{\sigma}^2$ solves the equation

$$\ln \psi\Big(-\frac{\tilde{\sigma}^2}{2}\Big) = r - \mu. \tag{6.5}$$

For symmetric NIG, this solution exist and it is unique. Using (6.3) we get from (6.5)

$$\tilde{\sigma}^2 = 2(r-\mu) - \left(\frac{r-\mu}{\alpha\sigma}\right)^2. \tag{6.6}$$

Consequently, $Y_t \sim SNIG(\alpha, 0, \tilde{\delta}t, \mu t)$ under Q with mean μt and variance $\tilde{\sigma}^2 t = \frac{\tilde{\delta}t}{\alpha}$.

Under Q_1, the distribution of Y_t is identified in the following result.

Proposition 6.2.1. *Denote by $f^Q_{Y_t}$ the Q-density of $Y_t \sim SNIG(\alpha, 0, \tilde{\delta}t, \mu t)$. Then the Q_1-density of Y_t, given by $e^{y-rt} f^Q_{Y_t}(y)$, is the density function of an asymmetric NIG distribution, i.e., $Y_t \sim NIG(\alpha, 1, \tilde{\delta}t, \mu t)$ under Q_1.*

Proof. For any time t, (6.5) gives

$$\ln \psi_t\Big(-\frac{\tilde{\sigma}^2 t}{2}\Big) = rt - \mu t.$$

Using the characteristic generator (6.4) for the symmetric NIG distribution, we can write

$$\begin{aligned} rt &= \mu t + \ln \psi_t\Big(-\frac{\tilde{\sigma}^2 t}{2}\Big) \\ &= \mu t + \alpha\tilde{\delta}t \left(1 - \sqrt{1 - \frac{\tilde{\sigma}^2}{\alpha\tilde{\delta}}}\right) \\ &= \mu t + \alpha\tilde{\delta}t \left(1 - \sqrt{1 - \frac{\tilde{\delta}^2}{(\alpha\tilde{\delta})^2}}\right) \\ &= \mu t + \alpha\tilde{\delta}t - \tilde{\delta}t\sqrt{\alpha^2 - 1}, \end{aligned}$$

in which we have applied $\tilde{\sigma}^2 = \frac{\tilde{\delta}}{\alpha} = \frac{\tilde{\delta}^2}{\alpha\tilde{\delta}}$. It then follows that

$$\begin{aligned} e^{y-rt} f^{Q}_{Y_t}(y) &= e^{y-\mu t-\alpha\tilde{\delta}t+\tilde{\delta}t\sqrt{\alpha^2-1}}\frac{\alpha}{\pi}e^{\alpha\tilde{\delta}t}\frac{K_1\Big(\alpha\tilde{\delta}t\sqrt{1+\big(\frac{y-\mu t}{\tilde{\delta}t}\big)^2}\Big)}{\sqrt{1+\big(\frac{y-\mu t}{\tilde{\delta}t}\big)^2}} \\ &= \frac{\alpha}{\pi}e^{y-\mu t+\tilde{\delta}t\sqrt{\alpha^2-1}}\frac{K_1\Big(\alpha\tilde{\delta}t\sqrt{1+\big(\frac{y-\mu t}{\tilde{\delta}t}\big)^2}\Big)}{\sqrt{1+\big(\frac{y-\mu t}{\tilde{\delta}t}\big)^2}}. \end{aligned} \tag{6.7}$$

One can verify that (6.7) is the density function of an asymmetric NIG distribution (see (2.40)) with parameters α (unchange), $\beta = 1$, $\mu = \mu t$ and $\delta = \tilde{\delta}t$.

□

The mean and variance of $Y_t \sim NIG(\alpha, 1, \tilde{\delta}t, \mu t)$ are given by (2.42) and (2.43), respectively. In particular,

$$\mathbf{E}(Y_1) = \mu_1 = \mu + \sqrt{\frac{\alpha^2}{\alpha^2 - 1}}\tilde{\sigma}^2, \tag{6.8}$$

$$Var(Y_1) = \tilde{\sigma}_1^2 = \left(\sqrt{\frac{\alpha^2}{\alpha^2 - 1}}\right)^3 \tilde{\sigma}^2, \tag{6.9}$$

where $\tilde{\sigma}^2$ is given by (6.6).

Denote by $F_{SNIG}(y)$, the cumulative distribution function of the standardized symmetric NIG random variable $\frac{Y_t - \mu t}{\tilde{\sigma}\sqrt{t}} \sim SNIG(\alpha, 0, \alpha, 0)$ under

Q, and denote by $F_{NIG}(y)$ the cumulative distribution function of the standardized asymmetric NIG random variable $\frac{Y_t-\mu_1 t}{\tilde{\sigma}_1\sqrt{t}} \sim NIG(\alpha,1,\alpha,0)$ under Q_1, we obtain the explicit option pricing formula for log-symmetric NIG Lévy price process.

Theorem 6.2.1. *Let the stock price $S_t = S_0 e^{Y_t}$ where Y_t is a symmetric NIG Lévy process with marginals $SNIG(\alpha,0,\delta t,\mu t)$ under P, and $\mu < r$. Then the arbitrage-free price using natural EMM of a call option with time to expiration T is given by*

$$C_0 = S_0\left[1 - F_{NIG}\left(-\frac{\ln\left(\frac{S_0}{K}\right)+\mu_1 T}{\tilde{\sigma}_1\sqrt{T}}\right)\right] - e^{-rT}KF_{SNIG}\left(\frac{\ln\left(\frac{S_0}{K}\right)+\mu T}{\tilde{\sigma}\sqrt{T}}\right). \tag{6.10}$$

where $\mu_1 = \mu + \sqrt{\frac{\alpha^2}{\alpha^2-1}}\tilde{\sigma}^2$, $\tilde{\sigma}_1^2 = \left(\sqrt{\frac{\alpha^2}{\alpha^2-1}}\right)^3\tilde{\sigma}^2$ *and* $\tilde{\sigma}^2 = 2(r-\mu) - \left(\frac{r-\mu}{\alpha\sigma}\right)^2$.

6.3 Option Pricing with Log-symmetric NIG Process in Discrete Time

Consider now the stock price process $S_N = S_0 e^{Y_N}$ where $Y_N = \sum_{n=1}^{N}\Delta Y_n$ is a symmetric NIG Lévy process in discrete time, and ΔY_n, $n = 1,\ldots,N$ are i.i.d with symmetric NIG distribution $SNIG(\alpha,0,\delta,\mu)$ under P.

Recall that for fixed σ^2 (and ψ), we can obtain two natural EMM's Q and Q_1 by changing only the location parameter μ (see Section 4.4) so that Y_N remains a symmetric NIG Lévy process. The Q-distribution of ΔY_n is $SNIG(\alpha,0,\delta,\tilde{\mu})$ where $\tilde{\mu} = r - \ln\psi\big(-\frac{\sigma^2}{2}\big)$, and the Q_1-distribution of ΔY_n is $SNIG(\alpha,0,\delta,\tilde{\mu}_1)$ with $\tilde{\mu}_1 = r + \ln\psi\big(-\frac{\sigma^2}{2}\big)$. By (6.3) and using $\zeta = \alpha\delta = \alpha^2\sigma^2$, we obtain

$$\ln\psi\Big(-\frac{\sigma^2}{2}\Big) = \zeta\left(1-\sqrt{1-\frac{\sigma^2}{\zeta}}\right) = \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{\alpha^2-1}. \tag{6.11}$$

Therefore, we obtain the following result for the exact option pricing formula with log-symmetric NIG process in discrete time.

Theorem 6.3.1. *Let ΔY_n follow a symmetric NIG distribution $SNIG(\alpha, 0, \delta, \mu)$, then the arbitrage-free price of a call option with N periods to expiration is given by*

$$C_0 = S_0 F_{SNIG}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{\alpha^2-1}\right)N}{\sigma\sqrt{N}}\right) - e^{-rN} K F_{SNIG}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \alpha^2\sigma^2 + \alpha\sigma^2\sqrt{\alpha^2-1}\right)N}{\sigma\sqrt{N}}\right). \quad (6.12)$$

6.4 Numerical Comparisons

For comparisons, we approximate the standardized symmetric NIG distribution by the standard normal, in other words, F_{SNIG} by Φ. We also approximate the standardized asymmetric NIG distribution that arises in the continuous time case by the standard Normal, i.e., F_{NIG} by Φ, since it is only slightly positively skewed. Therefore, we assume that the skewness is negligible. Moreover, recall that the shape parameter $\zeta = \alpha\delta = \alpha^2\sigma^2$ and the excess kurtosis γ of the symmetric NIG distribution are related by $\gamma = \frac{3}{\zeta}$. Thus, for each of the continuous time and discrete time cases, we obtain an easy to use Black-Scholes type formula for option pricing which gives correction that accounts for the access kurtosis.

In the continuous time case, the generalized or modified Black-Scholes formula for the log-symmetric NIG model (NIG-C) is given by

$$C_0 \approx S_0\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \mu_1 T}{\tilde{\sigma}_1\sqrt{T}}\right) - e^{-rT}K\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \mu T}{\tilde{\sigma}\sqrt{T}}\right), \quad (6.13)$$

where $\mu_1 = \mu + \sqrt{\frac{3}{3-\gamma\sigma^2}}\tilde{\sigma}^2$, $\tilde{\sigma}_1^2 = \left(\sqrt{\frac{3}{3-\gamma\sigma^2}}\right)^3 \tilde{\sigma}^2$ and $\tilde{\sigma}^2 = 2(r-\mu) - \frac{\gamma}{3}(r-\mu)^2$, in which we have applied the fact that

$$\frac{\alpha^2}{\alpha^2-1} = \frac{\alpha^2\sigma^2}{\alpha^2\sigma^2 - \sigma^2} = \frac{\alpha\delta}{\alpha\delta - \sigma^2} = \frac{3}{3-\gamma\sigma^2}.$$

Observe that, when $\gamma \to 0$, we have $\tilde{\sigma}^2 \to 2(r-\mu)$ and $\frac{3}{3-\gamma\sigma^2} \to 1$. Consequently, the Black-Scholes formula is a special case of the generalized version (NIG-C) (6.13) when $\gamma \to 0$ because

$$\begin{aligned}
\mu_1 &= \mu + \sqrt{\frac{3}{3-\gamma\sigma^2}}\tilde{\sigma}^2 \to \mu + 2(r-\mu), \\
\tilde{\sigma}_1^2 &= \left(\sqrt{\frac{3}{3-\gamma\sigma^2}}\right)^3 \tilde{\sigma}^2 \to 2(r-\mu).
\end{aligned}$$

And let $2(r-\mu) = \sigma^2$, which is a constant as in the Black-Scholes model (Recall that under the risk-neutral measure Q, the mean $\mu = r - \frac{\sigma^2}{2}$ and the volatility σ is a constant), then by using these results and some simple manipulations, it is not hard to see that the generalized formula (NIG-C) (6.13) is the exact Black-Scholes formula.

In the discrete time case, the modified Black-Scholes formula for log-symmetric NIG model (NIG-D) is given by

$$\begin{aligned}
C_0 \approx S_0\Phi &\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{3}{\gamma}\left(1 - \sqrt{1 - \frac{\gamma\sigma^2}{3}}\right)\right)N}{\sigma\sqrt{N}} \right) \\
&- e^{-rN}K\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{3}{\gamma}\left(1 - \sqrt{1 - \frac{\gamma\sigma^2}{3}}\right)\right)N}{\sigma\sqrt{N}} \right). \qquad (6.14)
\end{aligned}$$

It can be seen that the Black-Scholes formula is a limit of the generalized version (NIG-D) (6.14) for every N when $\gamma \to 0$ due to

$$\frac{3}{\gamma}\left(1 - \sqrt{1 - \frac{\gamma\sigma^2}{3}}\right) \to \frac{\sigma^2}{2}.$$

An example of the option price formulae plotted against the expiration time T using the similar set of parameter values as in the log-symmetric VG model (Chapter 5) is given below (see Figure 6.1). Again, it is evident that the distinction between the modified Black-Scholes formulae (NIG-C and NIG-D) and the Black-Scholes formula (BS) is noticeable even for this moderate values of γ (see Figure 6.1). As in the log-symmetric VG model, the disagreement between NIG-C and BS formulae is greater than the disagreement between NIG-D and BS. The exact prices and percentage differences are represented in Table 6.1.

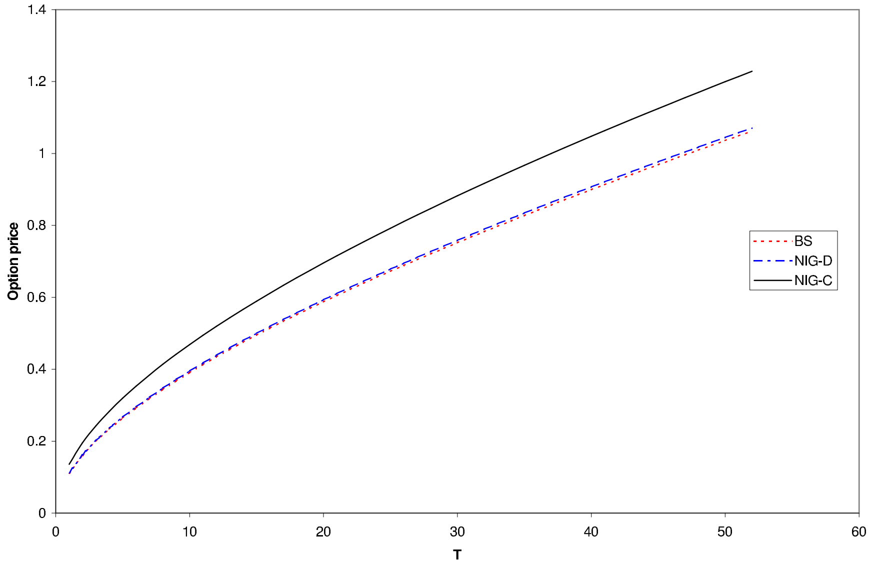

Figure 6.1: Option prices obtained by NIG-C, NIG-D and BS formulae for log-NIG distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

Time to maturity (weeks)	2	12	22	32	42	52
BS formula	0.1603	0.4344	0.6221	0.7823	0.9273	1.0622
NIG-D formula	0.1621	0.4388	0.6279	0.7893	0.9352	1.0708
Percentage difference (%)	1.14	1.01	0.94	0.89	0.85	0.81
NIG-C formula	0.1954	0.5192	0.7352	0.9167	1.0791	1.2286
Percentage difference (%)	21.91	19.52	18.18	17.18	16.36	15.66

Table 6.1: Option prices and percentage differences obtained by NIG-C, NIG-D and BS formulae for log-NIG distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

Chapter 7

Conclusions and Discussions

We have generalized the natural change of measure suggested by Klebaner and Landsman [35] for symmetric Lévy processes in continuous time (Theorem 3.2.3). The natural change of measure preserves symmetric Lévy processes and keeps the marginals in the same family of symmetric distributions.

In continuous time, if the symmetric Lévy process has a Brownian component, the natural change of measure changes the drift which leads to a unique natural EMM that changes only the location (mean) parameter μ, while the scale (variance) σ^2 and the characteristic generator ψ do not change. If the symmetric Lévy process has no Brownian component, the natural change of measure changes the Lévy measure. If, in addition, $\mu < r$, then a unique natural EMM exists that changes only the scale (variance) σ^2, while μ and ψ do not change (Theorems 3.2.3, 3.2.4 and 3.2.5).

In discrete time, the natural change of measure changes the location μ and scale σ^2 parameters of the symmetric distribution, regardless of the presence of Brownian component in the Lévy process. Thus, the natural EMM always exist but not unique (Theorem 3.3.3). However, if we fixed the variance σ^2, in addition to the characteristic generator ψ, we obtain unique natural EMM's that changes only the location parameter μ (Proposition 3.3.1). These results complement [35]. The reason we choose this unique natural EMM as our pricing measure for discrete time models is in accordance to the Black-Scholes model (volatility σ is constant) and [35]. However, we agree that this may not be the best practice for choosing the pricing measure. The final choice of pricing measure may require further procedures, such as calibration using historical option quotes, or other analytical methods similar to the optimization of the relative entropy or some other utility functions considered in the literature (e.g. [17]). This requires further analysis in future.

We considered option pricing with a natural EMM and derived explicit option pricing formulae for log-symmetric Lévy models of price process. These option pricing formulae are arbitrage-free. New option pricing formulae are derived for log-symmetric VG and log-symmetric NIG models (Chapters 5 and 6, respectively). In the VG case, the new option pricing formula is much simpler and elegant than the formula given in [39].

For continuous time models, two important information are required for our derivation: the structure of the underlying Lévy process, i.e, whether or not the Brownian component is present; and the marginal distributions of the underlying Lévy process. This is because the natural change of measure depends on the presence of Brownian component, and the corresponding natural EMM do not change the family of the marginal distributions. This poses a challenge for us to provide an explicit example for option pricing using natural EMM for log-symmetric Lévy processes with Brownian component other than the geometric Brownian motion with drifts which has a lognormal distribution. Pricing of options with geometric Brownian motions using natural EMM will lead to the Black-Scholes formula. The classes of Lévy process with Brownian component, other than the Brownian motion with drifts, is the jump-diffusion Lévy processes. However, to the best knowledge of the author, there is no known distribution for the marginals of the jump-diffusion processes.

Another shortcoming of the natural EMM approach in continuous time is that $\mu < r$ for log-symmetric Lévy models without Brownian component. This is overcome by using the discrete time, where natural EMM exists also when $\mu \geq r$.

The discrete time models do not require the knowledge of the presence of Brownian component in the underlying Lévy process. For any symmetric Levy process with known marginal distributions, it is always possible to derive a formula for pricing options under a natural EMM.

In Chapters 5 and 6, We obtained modified Black-scholes formulae, in continuous time as well as in discrete time, that gives correction for the kurtosis by approximating the respective Bessel distributions and NIG distributions with normal distribution. The modified Black-Scholes formulae can be used for any distribution with a positive excess kurtosis, in the same way as the Black-Scholes formula is used without fitting a distribution to the returns. In particular, for the log-symmetric VG model, the suggested modified Black-Scholes formulae for continuous time (5.16) and and discrete time (5.17) present a good alternative to the option pricing formula given by [40] and [39], since their formula requires numerical integration.

For future works, it would be interesting to investigate the sensitivity of the option prices obtained by the new formulae in this thesis to small change in the underlying parameters. The *delta*, $\Delta = \frac{\partial C}{\partial S}$, is the sensitivity of the option price C with respect to the underlying stock price S; the *vega*, $\mathcal{V} = \frac{\partial C}{\partial \sigma}$, is the sensitivity with respect to the volatility of the underlying stock; the *theta*, $\Theta = \frac{\partial C}{\partial t}$, is the sensitivity with respect to the time t; the *rho*, $\rho = \frac{\partial C}{\partial r}$, is the sensitivity with respect to the interest rate r; and *gamma*, $\Gamma = \frac{\partial \Delta}{\partial S}$, measures the rate of change of delta Δ with respect to the changes in the underlying stock price S. These are some of the common sensitivity measures (Greeks) that are vital for risk management, especially to the derivatives traders who seek to hedge their portfolios from adverse changes in market conditions. In particular, the delta Δ of an option is closely related to the self-financing strategy in the sense that it reveals the number of shares to be held in order to replicate the option. However, the process of delta hedging requires constant monitoring and rebalancing of the hedge over time.

We also suggest to extend the use of our approach in this thesis, i.e., a log-symmetric Lévy market model and the concept of a natural change of measure, to pricing American options, barrier options, Asian options and other exotic options. Note that it is possible to obtain closed form formula for these options under the Black-Scholes framework (e.g. ([50], Chapter 9) and [25]). However, in the Lévy framework, finding explicit formula becomes a challenge. Descriptions of how prices of some exotic options, including barrier options, can in principle be calculated is available in ([50], Chapter 9), but the analytical calculations are very involved. A survey of results in the literature is also provided. In most cases, results are obtained with some specific model restrictions. It is our desire to see some simplifications in the calculations under the assumptions made in this thesis that lead to new and elegant formula for exotic options.

We notice that both the VG process and the NIG process are subclasses of the Generalized Hyperbolic (GH) Lévy process ([8], [15]). The GH Lévy process constitute a large subclass of Lévy processes that are generated by the GH distributions (the GH distributions is infinitely divisible [15] and therefore generate a Lévy process Y_t such that the distribution of Y_1 is given by the GH distribution). The GH distributions were first introduced in [1] in relation with geology, but was picked up by a number of authors for finance (e.g. [14], [48]). Other important subclasses include the hyperbolic distributions (H) and the generalized inverse Gaussian (GIG) distributions [8]. For future works, we suggest to use GH distributions in our approach for option pricing using natural change of measure. However, one problem surfaces as GH distribution is not closed under convolution in general. Only two subclasses, namely the VG and NIG distributions are closed under convolution ([8], [15]), which is an important property for option pricing. Therefore, further generalization of the natural change of measure is required for the GH model.

Appendices

A Some Mathematical Tools

A.1 Dominated Convergence Theorem

Dominated convergence theorem is one of the most important theorem on Lebesgue integral. We recall that a real-valued measurable function f is said to be *integrable* or *summable* on a measure space (Ω, Σ, μ) if $\int_\Omega f d\mu < \infty$.

Theorem A.1. Dominated Convergence Theorem
Let $\{f_n\}$ be a sequence of integrable functions on (Ω, Σ, μ). Suppose that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for every $x \in \mathbb{R}$, and there exist a μ-integrable function $G(x) \geq 0$ such that, for each $x \in \mathbb{R}$,

$$|f_n(x)| \leq G(x) \quad \forall n.$$

Then

$$|f(x)| \leq G(x),$$

and

$$\lim_{n\to\infty} \int_\Omega f_n(x) d\mu(x) = \int_\Omega f(x) d\mu(x).$$

A.2 Mellin Transform

The Mellin transform, name after Finnish mathematician Hjalmar Mellin, is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. It is closely related to Laplace transform and Fourier transform, and the theory of the gamma function and allied special functions. The Mellin transform of a function f is

$$\mathcal{M}f(s) = \int_0^\infty x^{s-1} f(x) dx.$$

B Bessel Function Distribution

Let X_1 and X_2 be mutually independent random variables, each distributed as Chi-squared χ^2 with d degrees of freedom. The Bessel function distributions ([32], p.50) can be obtained as distributions of $X_1\sigma_1^2 \pm X_2\sigma_2^2$. The first form, $Z = X_1\sigma_1^2 + X_2\sigma_2^2$ has probability density function

$$f_Z(z) = \frac{|1-a^2|^{m+\frac{1}{2}}|z|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma\left(m+\frac{1}{2}\right)} e^{-\frac{az}{b}} I_m\left(\left|\frac{z}{b}\right|\right), \quad z > 0, \tag{B.1}$$

with

$$\begin{aligned} a &= (\sigma_1^2 + \sigma_2^2)(\sigma_1^2 - \sigma_2^2)^{-1} > 1, \\ b &= 4\sigma_1^2\sigma_2^2(\sigma_1^2 - \sigma_2^2)^{-1}, \\ m &= 2d + 1, \end{aligned}$$

where $I_m(.)$ is the modified Bessel function of the first kind of order m. The second form, $Y = X_1\sigma_1^2 - X_2\sigma_2^2$ has probability density function

$$f_Y(y) = \frac{|1-a^2|^{m+\frac{1}{2}}|y|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma\left(m+\frac{1}{2}\right)} e^{-\frac{ay}{b}} K_m\left(\left|\frac{y}{b}\right|\right), \tag{B.2}$$

with

$$\begin{aligned} a &= -(\sigma_1^2 - \sigma_2^2)(\sigma_1^2 + \sigma_2^2)^{-1}, \quad |a| < 1, \\ b &= 4\sigma_1^2\sigma_2^2(\sigma_1^2 + \sigma_2^2)^{-1}, \\ m &= 2d + 1, \end{aligned}$$

where $K_m(.)$ is the modified Bessel function of the second kind of order m.

For both kinds of distributions (B.1) and (B.2), the moment generating function ([32], p.50) can be written as

$$M_W(t) = \left(\frac{1-a^2}{1-(a-tb)^2}\right)^{m+\frac{1}{2}}$$

with the corresponding values of a and b for the kind of distribution W being either Y or Z. Replacing $t = iu$ in the moment generating function, we obtain, for both kinds of distributions with appropriate values of a and b, the characteristic function of Bessel function distribution

$$\mathbf{E}(e^{iuW}) = M_W(iu) = \left(\frac{1-a^2}{1-(a-iub)^2}\right)^{m+\frac{1}{2}}. \tag{B.3}$$

Explicit expression of mean, variance, skewness and kurtosis of Bessel function distribution are given in ([32], p.51). Specifically,

$$\text{Mean} = (2m+1)ba(a^2-1)^{-1} \tag{B.4}$$

$$\text{Variance} = (2m+1)b^2(a^2+1)(a^2-1)^{-2} \tag{B.5}$$

$$\text{Skewness} = 2a(a^2+3)(2m+1)^{-1/2}(a^2+1)^{-3/2} \tag{B.6}$$

$$\text{Kurtosis} = 3+6(a^4+6a^2+1)(2m+1)^{-1}(a^2+1)^{-2}. \tag{B.7}$$

In Chapter 5, we primarily work with the Bessel function distribution of the second kind (B.2) with $|a| < 1$, $b > 0$ and $m > 0$. In particular, the symmetric version of this form (when $a = 0$) is symmetrical about the origin (zero mean) with variance and kurtosis equal to, respectively, $(2m+1)b^2$ and $3+6(2m+1)^{-1}$.

We denote the Bessel function as a function of three parameters, $Y \sim Bessel(\mu, \sigma^2, \lambda)$ where μ is the mean, σ^2 is the variance and the shape parameter $\lambda = m + \frac{1}{2}$. Then a symmetric Bessel function distribution has mean $\mu = 0$, variance $\sigma^2 = 2\lambda b^2$ and the kurtosis $3 + \frac{3}{\lambda}$. Consequently, (B.2) yields the density of a symmetric Bessel function distribution (see also [32], p.50)

$$\begin{aligned} f_Y(y) &= \frac{|y|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma\left(m+\frac{1}{2}\right)} K_m\left(\left|\frac{y}{b}\right|\right) \\ &= \sqrt{\frac{2\lambda}{\pi\sigma^2}}\left(\sqrt{\frac{\lambda y^2}{2\sigma^2}}\right)^{\lambda-\frac{1}{2}} \frac{1}{\Gamma(\lambda)} K_{\lambda-\frac{1}{2}}\left(2\sqrt{\frac{\lambda y^2}{2\sigma^2}}\right), \end{aligned} \tag{B.8}$$

and (B.3) yields the characteristic function of a symmetric Bessel function distribution (see also [32], p.51)

$$\mathbf{E}(e^{iuY}) = \left(\frac{1}{1+u^2b^2}\right)^{m+\frac{1}{2}} = \left(\frac{1}{1+\frac{u^2\sigma^2}{2\lambda}}\right)^{\lambda}. \tag{B.9}$$

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