Geometric Decompositions of Collective Motion

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Supplemental Calculations

The calculations in (a), (c), (d), (e), (f), (g) and (h) were adapted from [29] and streamlined whenever possible.

(a) Proof of Proposition 3.1

First, it is straightforward to show that vertical motions correspond to rigid translations of the collective: $d\pi_{\mathbb{R}^3}(\mathbf{v_r}) = [\mathbf{v_{r1}} - \mathbf{v_{com}}\; \mathbf{v_{r2}} - \mathbf{v_{com}} \dots \mathbf{v_{rn}} - \mathbf{v_{com}}]$ vanishes if and only if $\mathbf{v_{r1}} = \mathbf{v_{r2}} = \dots = \mathbf{v_{rn}} = \mathbf{v_{com}}$. Hence:

$$V_{\mathbf{r}} = \ker(d\pi_{\mathbb{R}^3})(\mathbf{r}) = \{\mathbf{v_r} \in T_{\mathbf{r}}\mathcal{R}\; s.t.\; \mathbf{v_r} = [\mathbf{v}\;\mathbf{v} \dots \mathbf{v}],\; \mathbf{v} \in \mathbb{R}^3\}. \tag{1}$$

Then it suffices to show that, $\forall \mathbf{v_r} \in T_{\mathbf{r}}\mathcal{R}$, the vertical motion $[\mathbf{v_{com}}\;\mathbf{v_{com}} \dots \mathbf{v_{com}}] \in V_{\mathbf{r}}$ and the residual $\mathbf{v_r} - [\mathbf{v_{com}}\;\mathbf{v_{com}} \dots \mathbf{v_{com}}]$ are orthogonal to each other:

$$\begin{aligned}
&\mathrm{tr}([\mathbf{v_{com}}\;\mathbf{v_{com}} \dots \mathbf{v_{com}}]M(\mathbf{v_r} - [\mathbf{v_{com}}\;\mathbf{v_{com}} \dots \mathbf{v_{com}}])^T) = \\
&\mathrm{tr}([m_1\mathbf{v_{com}}\; m_2\mathbf{v_{com}} \dots m_n\mathbf{v_{com}}][\mathbf{v_{r1}} - \mathbf{v_{com}}\;\mathbf{v_{r2}} - \mathbf{v_{com}} \dots \mathbf{v_{rn}} - \mathbf{v_{com}}]^T) = \\
&\sum_{i=1}^n m_i \mathrm{tr}(\mathbf{v_{com}}\mathbf{v_{ri}}^T - \mathbf{v_{com}}\mathbf{v_{com}}^T) = \sum_{i=1}^n m_i \mathbf{v_{ri}}^T\mathbf{v_{com}} - \sum_{i=1}^n m_i \mathbf{v_{com}}^T\mathbf{v_{com}} = \\
&m_{tot}\mathbf{v_{com}}^T\mathbf{v_{com}} - m_{tot}\mathbf{v_{com}}^T\mathbf{v_{com}} = 0,
\end{aligned}$$

where we have used the linearity of the trace operator and the property $\mathrm{tr}(\mathbf{a}\mathbf{b}^T) = \mathbf{a}^T\mathbf{b}\; \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^m$. By the uniqueness of the orthogonal decomposition $\mathbf{v_r} = A_{\mathbf{r}}(\mathbf{v_r}) + \mathrm{hor}(\mathbf{v_r})$, with $A_{\mathbf{r}}(\mathbf{v_r})$ belonging in $V_{\mathbf{r}}$ as in (1) and $\langle A_{\mathbf{r}}(\mathbf{v_r}), \mathrm{hor}(\mathbf{v_r})\rangle = 0$, we conclude that (3.4) and (3.5) must hold. □

(b) Derivation of E_{rel} in terms of individual velocities (cf. (3.6))

$$
\begin{aligned}
E_{rel} =& \frac{1}{2}\sum_{i=1}^{n} m_i |\mathbf{v_{ci}}|^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \left| \mathbf{v_{ri}} - \sum_{j=1}^{n} \frac{m_j}{m_{tot}} \mathbf{v_{rj}} \right|^2 \\
=& \frac{1}{2}\sum_{i=1}^{n} m_i \left| \sum_{j=1}^{n} \frac{m_j}{m_{tot}} \mathbf{v_{ri}} - \sum_{j=1}^{n} \frac{m_j}{m_{tot}} \mathbf{v_{rj}} \right|^2 = \frac{1}{2}\sum_{i=1}^{n} \frac{m_i}{m_{tot}^2} \left| \sum_{j=1}^{n} m_j (\mathbf{v_{ri}} - \mathbf{v_{rj}}) \right|^2 \\
=& \frac{1}{2}\sum_{i=1}^{n} \frac{m_i}{m_{tot}^2} \sum_{j=1}^{n}\sum_{k=1}^{n} m_j m_k (\mathbf{v_{ri}} - \mathbf{v_{rj}})^T (\mathbf{v_{ri}} - \mathbf{v_{rk}}) \\
=& \frac{1}{2m_{tot}^2} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j^2 (\mathbf{v_{ri}} - \mathbf{v_{rj}})^T (\mathbf{v_{ri}} - \mathbf{v_{rj}}) + \right. \\
& \left. \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j (\mathbf{v_{ri}} - \mathbf{v_{rj}})^T \sum_{k=1, k\neq i,j}^{n} m_k (\mathbf{v_{ri}} - \mathbf{v_{rk}}) \right) \\
=& \frac{1}{2m_{tot}^2} \left(\sum_{i<j} (m_i m_j^2 + m_j m_i^2) |\mathbf{v_{ri}} - \mathbf{v_{rj}}|^2 + \right. \\
& \left. \sum_{i<j} \sum_{k=1, k\neq i,j}^{n} (\mathbf{v_{ri}} - \mathbf{v_{rj}})^T ((\mathbf{v_{ri}} - \mathbf{v_{rk}}) - (\mathbf{v_{rj}} - \mathbf{v_{rk}})) m_i m_j m_k \right) \\
=& \frac{1}{2m_{tot}^2} \left(\sum_{i<j} m_i m_j \left(m_j + m_i + \sum_{k=1, k\neq i,j}^{n} m_k \right) |\mathbf{v_{ri}} - \mathbf{v_{rj}}|^2 \right) \\
=& \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{m_i m_j}{m_{tot}} |\mathbf{v_{ri}} - \mathbf{v_{rj}}|^2.
\end{aligned}
$$

(c) Derivation of (4.8)

Let K be any given matrix in $\mathbb{R}^{3\times 3}_{sym,>0}$. Then, there exist $Q \in SO(3)$ (matrix of eigenvectors of K) and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$ (eigenvalues of K) such that $QKQ^T = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Now $K = \mathbf{c}M\mathbf{c}^T \Leftrightarrow \Lambda = \tilde{\mathbf{c}}\tilde{\mathbf{c}}^T$ where $\tilde{\mathbf{c}} = Q\mathbf{c}M^{\frac{1}{2}}$, i.e. the three rows of $\tilde{\mathbf{c}}$ are orthogonal to each other and have norms $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$ respectively. Moreover, each $\mathbf{c} \in \mathcal{C}^{3d}$ must satisfy the condition $\sum_{i=1}^{n} m_i \mathbf{c_i} = \mathbf{0}$, which can be compactly expressed as $\mathbf{c}[m_1\, m_2 \ldots m_n]^T = \mathbf{0}$. This, in turns, implies the following condition on $\tilde{\mathbf{c}}$: $\tilde{\mathbf{c}}[\sqrt{m_1}\, \sqrt{m_2} \ldots \sqrt{m_n}]^T = \mathbf{0}$. So the problem of finding $\mathbf{c}$ such that $K = \mathbf{c}M\mathbf{c}^T$ reduces to finding three n-dimensional vectors (the rows of $\tilde{\mathbf{c}}$) which are orthogonal to each other and to the vector $[\sqrt{m_1}\, \sqrt{m_2} \ldots \sqrt{m_n}]$ and have norms $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$. One possible solution is to construct such vectors iteratively, by starting with as few non-zero elements as possible and introducing the constraints one at a time (i.e. row 1 has only two non-zero elements and is orthogonal to the vector of masses, row 2 has three non-zero elements and is also orthogonal to row 1, and so on). It is easy to verify algebraically that the following $\tilde{\mathbf{c}}$, constructed in such fashion, satisfies all the requirements:

$$
\begin{bmatrix}
\frac{\sqrt{\lambda_1 m_1}}{\mu_1} & 0 & 0 & -\frac{\sqrt{\lambda_1 m_4}}{\mu_1}\left(\frac{m_1}{m_4}\right) & 0 \cdots 0 \\
\frac{\sqrt{\lambda_2 m_1}}{\mu_2} & -\frac{\sqrt{\lambda_2 m_2}}{\mu_2}\left(\frac{m_1+m_4}{m_2}\right) & 0 & \frac{\sqrt{\lambda_2 m_4}}{\mu_2} & 0 \cdots 0 \\
\frac{\sqrt{\lambda_3 m_1}}{\mu_3} & \frac{\sqrt{\lambda_3 m_2}}{\mu_3} & -\frac{\sqrt{\lambda_3 m_3}}{\mu_3}\left(\frac{m_1+m_2+m_4}{m_3}\right) & \frac{\sqrt{\lambda_3 m_4}}{\mu_3} & 0 \cdots 0
\end{bmatrix}
$$

where μ_1, μ_2, μ_3 are normalizing factors given by:

$$\mu_1 = \sqrt{\frac{m_1(m_1+m_4)}{m_4}}, \quad \mu_2 = \sqrt{\frac{(m_1+m_4)(m_1+m_2)}{m_2}}$$
$$\mu_3 = \sqrt{\frac{(m_1+m_2+m_4)(m_1+m_2+m_3+m_4)}{m_3}}.$$

Finally, by taking $\mathbf{c} = Q^T\tilde{\mathbf{c}}M^{-\frac{1}{2}}$, we have the following configuration $\mathbf{c} \in \mathcal{C}^{3d}$ with the specified symmetric positive definite matrix K as its ensemble inertia tensor:

$$\mathbf{c} = Q^T \begin{bmatrix} \frac{\sqrt{\lambda_1}}{\mu_1} & 0 & 0 & -\frac{\sqrt{\lambda_1}}{\mu_1}\left(\frac{m_1}{m_4}\right) & 0\cdots0 \\ \frac{\sqrt{\lambda_2}}{\mu_2} & -\frac{\sqrt{\lambda_2}}{\mu_2}\left(\frac{m_1+m_4}{m_2}\right) & 0 & \frac{\sqrt{\lambda_2}}{\mu_2} & 0\cdots0 \\ \frac{\sqrt{\lambda_3}}{\mu_3} & \frac{\sqrt{\lambda_3}}{\mu_3} & -\frac{\sqrt{\lambda_3}}{\mu_3}\left(\frac{m_1+m_2+m_4}{m_3}\right) & \frac{\sqrt{\lambda_3}}{\mu_3} & 0\cdots0 \end{bmatrix}.$$

(d) Derivation of (4.11)-(4.12)

Let $K = Q^T\Lambda Q$ be an eigenvalue decomposition of K with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ the diagonal matrix of eigenvalues, and let $\Lambda^{-\frac{1}{2}} = \mathrm{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \lambda_3^{-1/2})$. Then:

$$\pi(\mathbf{c}) = K \Leftrightarrow \pi(Q\mathbf{c}) = \Lambda \Leftrightarrow (\Lambda^{-\frac{1}{2}}Q\mathbf{c}M^{\frac{1}{2}})(\Lambda^{-\frac{1}{2}}Q\mathbf{c}M^{\frac{1}{2}})^T = \mathbb{1}.$$

Hence: $\pi^{-1}(K) = \{\mathbf{c} \in \mathcal{C}^{3d}\ s.t.\ (\Lambda^{-\frac{1}{2}}Q\mathbf{c}M^{\frac{1}{2}})^T \in \mathcal{V}_{n,3}\}$.
Furthermore: $\mathbf{c} \in \mathcal{C}^{3d} \Leftrightarrow \mathbf{c}[m_1 m_2 ... m_n]^T = \mathbf{0} \Leftrightarrow$
$\mathbf{c}M^{\frac{1}{2}}[\sqrt{m_1}\sqrt{m_2}...\sqrt{m_n}]^T = \mathbf{0} \Leftrightarrow \Lambda^{-\frac{1}{2}}Q\mathbf{c}M^{\frac{1}{2}}[\sqrt{m_1}\sqrt{m_2}...\sqrt{m_n}]^T = \mathbf{0}$.
So we can also rewrite $\pi^{-1}(K)$ as:

$$\pi^{-1}(K) = \left\{ \mathbf{c} \in \mathbb{R}^{3\times n}\ s.t.\ \tilde{V}(\mathbf{c}) \triangleq \left[M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} \,\middle|\, \begin{array}{c} \sqrt{m_1/m_{tot}} \\ \vdots \\ \sqrt{m_n/m_{tot}} \end{array} \right] \in \mathcal{V}_{n,4} \right\}.$$

Now observe that, given any fixed $\tilde{Q} \in O(n)$, $\tilde{V}(\mathbf{c}) \in \mathcal{V}_{n,4} \Leftrightarrow \tilde{Q}\tilde{V}(\mathbf{c}) \in \mathcal{V}_{n,4}$ (in fact $(\tilde{Q}\tilde{V})^T(\tilde{Q}\tilde{V}) = \tilde{V}^T\tilde{V} = \mathbb{1}$). In particular choose $\tilde{Q} \in O(n)$ in the form: $\tilde{Q} = \tilde{Q}_W \triangleq \begin{bmatrix} \sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}} \\ W^T \end{bmatrix}$, for any $W \in \mathcal{V}_{n,n-1}$ with columns orthogonal to the vector $[\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]^T$. Then: $\tilde{V}(\mathbf{c}) \in \mathcal{V}_{n,4} \Leftrightarrow \tilde{Q}_W\tilde{V}(\mathbf{c}) \in \mathcal{V}_{n,4} \Leftrightarrow W^T M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = V$ for some $V \in \mathcal{V}_{n-1,3}$ $\Leftrightarrow W^T M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = W^T W V \Leftrightarrow M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = WV$, for some $V \in \mathcal{V}_{n-1,3}$. Note that, in the last two steps, we have used the facts that $W^T W = \mathbb{1}$ and that the columns of $M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}}$ are in the range of W (in fact they are orthogonal to $[\sqrt{m_1} \cdots \sqrt{m_n}]^T$, and thus to the kernel of W^T: $[\sqrt{m_1} \cdots \sqrt{m_n}]M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = [m_1 \cdots m_n]\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = \mathbf{0}$). So we conclude that:

$$\pi^{-1}(K) = \left\{ \mathbf{c} \in \mathbb{R}^{3\times n}\ s.t.\ M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} = WV,\ V \in \mathcal{V}_{n-1,3} \right\}.$$

Hence, given any fixed matrix $W \in \mathcal{V}_{n,n-1}$ with columns orthogonal to the vector $[\sqrt{m_1} \cdots \sqrt{m_n}]^T$, the following diffeomorphism between $\pi^{-1}(K)$ and the Stiefel manifold $\mathcal{V}_{n-1,3}$ can be established:

$$f_{K,W} : \pi^{-1}(K) \to \mathcal{V}_{n-1,3}$$
$$\mathbf{c} \in \pi^{-1}(K) \mapsto V = f_{K,W}(\mathbf{c}) = W^T M^{\frac{1}{2}}\mathbf{c}^T Q^T \Lambda^{-\frac{1}{2}} \in \mathcal{V}_{n-1,3},$$

with inverse:

$$f_{K,W}^{-1} : \mathcal{V}_{n-1,3} \to \pi^{-1}(K)$$
$$V \in \mathcal{V}_{n-1,3} \mapsto \mathbf{c} = f_{K,W}^{-1}(V) = Q^T \Lambda^{\frac{1}{2}} V^T W^T M^{-\frac{1}{2}} \in \pi^{-1}(K).$$

As a corollary of the above derivation, we obtain the following identity that will be useful later:

$$\tilde{Q}_W^T \tilde{Q}_W = [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]^T [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}] + WW^T = \mathbb{1}$$
$$\Rightarrow WW^T = \mathbb{1} - [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]^T [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]. \tag{2}$$

(e) Derivation of (4.14)

For any given $K \in \mathcal{K}$, the fiber over K is composed of all the configurations having K as their ensemble inertia tensor, given by (using (4.12)):

$$\pi^{-1}(K) = \{Q^T \Lambda^{\frac{1}{2}} V^T W^T M^{-\frac{1}{2}} s.t.\, V \in \mathcal{V}_{n-1,3}\}.$$

Assume we have a reference configuration: $\tilde{\mathbf{c}} = Q^T \Lambda^{\frac{1}{2}} \tilde{V}^T W^T M^{-\frac{1}{2}} \in \pi^{-1}(K)$, for some $\tilde{V} \in \mathcal{V}_{n-1,3}$. Since $\mathcal{V}_{n-1,3} \equiv O(n-1)/O(n-4)$ [22], $\tilde{V}$ can be mapped to any other element of the Stiefel manifold via premultiplication by orthogonal matrices in $O(n-1)$: $\tilde{V} \mapsto Q_{\tilde{V}} \tilde{V}, \forall Q_{\tilde{V}} \in O(n-1)$. This allows to map $\tilde{\mathbf{c}}$ into any other $\mathbf{c} \in \pi^{-1}(K)$ as follows: $\tilde{\mathbf{c}} \mapsto Q^T \Lambda^{\frac{1}{2}} \tilde{V}^T Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}} = \tilde{\mathbf{c}}(M^{\frac{1}{2}} W Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}}), \forall Q_{\tilde{V}} \in O(n-1)$. Hence, given any $\tilde{\mathbf{c}} \in \pi^{-1}(K)$, we have the following alternative expression for $\pi^{-1}(K)$:

$$\pi^{-1}(K) = \{\tilde{\mathbf{c}} M^{\frac{1}{2}} W Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}} \; s.t.\, Q_{\tilde{V}} \in O(n-1)\}.$$

If we define the *democracy* group as:

$$\mathcal{D} = \{D = M^{\frac{1}{2}} W Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}} \; s.t.\, Q_{\tilde{V}} \in O(n-1)\},$$

then the fiber over K can be compactly expressed as:

$$\pi^{-1}(K) = \{\tilde{\mathbf{c}} D \; s.t.\, D \in \mathcal{D}\},$$

i.e. the orbit of the democracy group action starting from $\tilde{\mathbf{c}}$.

If all the masses are equal, the orbits of the democracy group include the discrete family of individual relabelings (or position exchanges between individuals). To show this, observe that a configuration $\tilde{\mathbf{c}}'$ is obtained from $\tilde{\mathbf{c}}$ via individual relabeling if and only if $\tilde{\mathbf{c}}' = \tilde{\mathbf{c}} P$ for some permutation matrix $P \in P(n) \subset O(n)$. Now $P \in \mathcal{D}$ if and only if $M^{\frac{1}{2}} W Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}} = P$ for some $Q_{\tilde{V}} \in O(n-1)$, i.e. if and only if $W^T M^{-\frac{1}{2}} P M^{\frac{1}{2}} W \in O(n-1)$. If $M = m\mathbb{1}$, i.e. the masses are all equal, then $W^T M^{-\frac{1}{2}} P M^{\frac{1}{2}} W = W^T P W$ certainly satisfies the orthogonality property:
$(W^T P W)^T (W^T P W) = W^T P^T W W^T P W =$
$= \mathbb{1} - W^T P^T [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]^T [\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}] P W = \mathbb{1}$,
where we have used (2), the facts that $W^T W = \mathbb{1}$ and $P^T P = \mathbb{1}$, and the fact that a permutation does not affect the range of a matrix (hence the columns of PW are, like those of W, orthogonal to the vector $[\sqrt{m_1/m_{tot}} \cdots \sqrt{m_n/m_{tot}}]^T$).

(f) Invariance of Riemannian metric (4.2) to the democracy group action

Let $D \in \mathcal{D}$ be an arbitrary matrix belonging to the democracy group, and $\mathbf{v_c}, \mathbf{w_c} \in T_{\mathbf{c}}\mathcal{C}^{3d}$ be arbitrary tangent vectors to $\mathcal{C}^{3d}$ at configuration $\mathbf{c}$. Then:

$$\begin{aligned}
\langle \mathbf{v_c} D, \mathbf{w_c} D \rangle_{\mathbf{c}D} =& \mathrm{tr}(\mathbf{v_c} D M D^T \mathbf{w_c}^T) = \\
=& \mathrm{tr}(\mathbf{v_c} M^{\frac{1}{2}} W Q_{\tilde{V}}^T W^T M^{-\frac{1}{2}} M M^{-\frac{1}{2}} W Q_{\tilde{V}} W^T M^{\frac{1}{2}} \mathbf{w_c}^T) = \\
=& \mathrm{tr}(\mathbf{v_c} M^{\frac{1}{2}} W W^T M^{\frac{1}{2}} \mathbf{w_c}^T) = \\
=& \mathrm{tr}(\mathbf{v_c} M \mathbf{w_c}^T) = \\
=& \langle \mathbf{v_c}, \mathbf{w_c} \rangle_{\mathbf{c}},
\end{aligned}$$

where we have used (2) and the fact that $\mathbf{v_c}[m_1\cdots m_n]^T = [m_1\cdots m_n]\mathbf{w_c}^T = \mathbf{0}$. Hence the Riemannian metric (4.2) is invariant to the action of any $D \in \mathcal{D}$.

(g) Orthogonality property of the mechanical connection (5.1)

To prove that the mechanical connection (5.1) is the one giving orthogonal splitting of tangent vectors, observe that, $\forall \xi \in \mathfrak{g}$:

$$\begin{aligned}(J(v_p - (I_p^{-1}J(v_p))_P(p)))(\xi) &= \langle v_p - (I_p^{-1}J(v_p))_P(p), \xi_P(p)\rangle_p = \\ &= \langle v_p, \xi_P(p)\rangle_p - \langle (I_p^{-1}J(v_p))_P(p), \xi_P(p)\rangle_p = \\ &= (J(v_p))(\xi) - (I_p(I_p^{-1}J(v_p)))(\xi) = 0.\end{aligned}$$

Hence, $\forall v_p \in T_pP$:

$$\langle v_p - (I_p^{-1}J(v_p))_P(p), (I_p^{-1}J(v_p))_P(p)\rangle_p = (J(v_p - (I_p^{-1}J(v_p))_P))(I_p^{-1}J(v_p)) = 0.$$

(h) Proof of Lemma 5.1

The Lie algebra of the Lie group $SO(3)$ is $so(3) \triangleq \mathbb{R}^{3\times3}_{skew}$, the space of 3×3 skew-symmetric matrices. In turns, $\mathbb{R}^{3\times3}_{skew} \cong \mathbb{R}^3$ using the following mapping:

$$\xi \in \mathbb{R}^{3\times3}_{skew} \quad\leftrightarrow\quad \boldsymbol{\xi} \in \mathbb{R}^3 \; s.t. \; \xi\mathbf{v} = \boldsymbol{\xi}\times\mathbf{v},\; \forall \mathbf{v}\in\mathbb{R}^3. \tag{3}$$

We will interchangeably use either side of (3) to describe elements of $so(3)$. Given $\xi \in so(3)$, the infinitesimal generator of the group action Φ_Q on $\mathcal{C}^{2^+d}$ induced by ξ is defined as:

$$\xi_{\mathcal{C}^{2^+d}}(\mathbf{c}) = \frac{d}{dt}\bigg|_{t=0} \Phi_{exp(t\xi)}(\mathbf{c}) = \frac{d}{dt}\bigg|_{t=0} (exp(t\xi))\mathbf{c} = \xi\mathbf{c}, \tag{4}$$

or alternatively:

$$\xi_{\mathcal{C}^{2^+d}}(\mathbf{c}) = [\boldsymbol{\xi}\times\mathbf{c_1}\; \boldsymbol{\xi}\times\mathbf{c_2} \ldots \boldsymbol{\xi}\times\mathbf{c_n}]. \tag{5}$$

Then the momentum map (5.2) can be computed as follows, $\forall \mathbf{c} \in \mathcal{C}^{2^+d}$:

$$\begin{aligned}(\mathbf{J}(\mathbf{c},\mathbf{v_c}))(\xi) &\triangleq \langle \mathbf{v_c}, \xi_{\mathcal{C}^{2^+d}}(\mathbf{c})\rangle = \sum_{i=1}^n m_i((\boldsymbol{\xi}\times\mathbf{c_i})^T\mathbf{v_{ci}}) = \sum_{i=1}^n m_i\boldsymbol{\xi}^T(\mathbf{c_i}\times\mathbf{v_{ci}}) \\ &= \boldsymbol{\xi}^T\sum_{i=1}^n m_i(\mathbf{c_i}\times\mathbf{v_{ci}}) \quad \forall\xi\in so(3),\end{aligned} \tag{6}$$

which is simply the $\boldsymbol{\xi}$-component of the angular momentum with respect to the center of mass (5.4). A similar computation specializing (5.3) yields the locked inertia tensor, $\forall \mathbf{c} \in \mathcal{C}^{2^+d}$:

$$\begin{aligned}(I_\mathbf{c}(\eta))(\xi) &\triangleq \langle \eta_{\mathcal{C}^{2^+d}}(\mathbf{c}), \xi_{\mathcal{C}^{2^+d}}(\mathbf{c})\rangle = \sum_{i=1}^n m_i((\boldsymbol{\xi}\times\mathbf{c_i})^T(\boldsymbol{\eta}\times\mathbf{c_i})) \\ &= \sum_{i=1}^n m_i\boldsymbol{\xi}^T(\mathbf{c_i}\times(\boldsymbol{\eta}\times\mathbf{c_i})) = \boldsymbol{\xi}^T\sum_{i=1}^n m_i(|\mathbf{c_i}|^2\mathbb{1} - \mathbf{c_i}\mathbf{c_i}^T)\boldsymbol{\eta} \quad \forall\eta,\xi\in so(3),\end{aligned} \tag{7}$$

where we have used the triple product formulae for vectors in $\mathbb{R}^3$: $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$. Hence the locked inertia tensor is the moment of inertia tensor with respect to the center of mass (5.5). □

(i) Curvature of the ensemble connection

The *curvature* of an Ehresmann connection A on $\mathcal{C}^{3d}$ is defined as the vertical-valued two-form:

$$B_{\mathbf{c}}(X(\mathbf{c}), Y(\mathbf{c})) = -A_{\mathbf{c}}([\mathrm{hor}(X), \mathrm{hor}(Y)](\mathbf{c}))\ \forall \mathbf{c} \in \mathcal{C}^{3d}, \tag{8}$$

where $X, Y \in \mathcal{X}(\mathcal{C}^{3d})$ are any two tangent vector fields, $\mathrm{hor}(X)$, $\mathrm{hor}(Y)$ are their horizontal projections and $[\mathrm{hor}(X), \mathrm{hor}(Y)](\mathbf{c})$ is their Jacobi-Lie bracket (also a vector field on $\mathcal{C}^{3d}$) at $\mathbf{c}$ (see for example [25]). For most Ehresmann connections, even a trajectory with tangent vectors that are horizontal everywhere produces some vertical displacement along the fibers. The curvature form determines the infinitesimal vertical motion obtained from an infinitesimal loop constructed using any two horizontal vector fields $\mathrm{hor}(X)(\mathbf{c})$ and $\mathrm{hor}(Y)(\mathbf{c})$ (which is how the Jacobi-Lie bracket $[\mathrm{hor}(X), \mathrm{hor}(Y)](\mathbf{c})$ can be interpreted).

Theorem 7.1 The curvature form of the ensemble connection (5.10) is:

$$\begin{aligned} B_{\mathbf{c}}(X(\mathbf{c}), Y(\mathbf{c})) &= A_{\mathbf{c}}((S_X(\mathbf{c})S_Y(\mathbf{c}) - S_Y(\mathbf{c})S_X(\mathbf{c}))\,\mathbf{c}) \\ &= Q^T(\mathbf{c}) \begin{bmatrix} 0 & \frac{2\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & \frac{2\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} \\ \frac{-2\lambda_2}{\lambda_1+\lambda_2}\tilde{a}_{12} & 0 & \frac{2\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} \\ \frac{-2\lambda_3}{\lambda_1+\lambda_3}\tilde{a}_{13} & \frac{-2\lambda_3}{\lambda_2+\lambda_3}\tilde{a}_{23} & 0 \end{bmatrix} Q(\mathbf{c})\mathbf{c}, \end{aligned} \tag{9}$$

where

$$\tilde{A}(\mathbf{c}, \mathbf{v_c}) = \begin{bmatrix} 0 & \tilde{a}_{12} & \tilde{a}_{13} \\ -\tilde{a}_{12} & 0 & \tilde{a}_{23} \\ -\tilde{a}_{13} & -\tilde{a}_{23} & 0 \end{bmatrix} = Q(\mathbf{c})(S_X(\mathbf{c})S_Y(\mathbf{c}) - S_Y(\mathbf{c})S_X(\mathbf{c})Q^T(\mathbf{c}), \tag{10}$$

$K(\mathbf{c}) = Q^T(\mathbf{c})\Lambda(\mathbf{c})Q(\mathbf{c})$, $\Lambda(\mathbf{c}) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, form an eigendecomposition of $K(\mathbf{c})$, and $S_X(\mathbf{c}), S_Y(\mathbf{c})$ are solutions to Lyapunov equations $S_X(\mathbf{c})K(\mathbf{c}) + K(\mathbf{c})S_X(\mathbf{c}) = 2\,\mathrm{sym}(\mathbf{c}MX(\mathbf{c})^T)$ and $S_Y(\mathbf{c})K(\mathbf{c}) + K(\mathbf{c})S_Y(\mathbf{c}) = 2\,\mathrm{sym}(\mathbf{c}MY(\mathbf{c})^T)$ respectively.

Proof: Let $X, Y \in \mathcal{X}(\mathcal{C}^{3d})$ be two arbitrary vector fields. Their Jacobi-Lie bracket can be computed, at each $\mathbf{c} \in \mathcal{C}^{3d}$, in the ambient space $\mathbb{R}^{3\times n}$ (of which $\mathcal{C}^{3d}$ is a subset):

$$\begin{aligned} [X, Y](\mathbf{c}) =& (DY)_{\mathbf{c}} X(\mathbf{c}) - (DX)_{\mathbf{c}} Y(\mathbf{c}) = \\ &= \left.\frac{d}{dt}\right|_{t=0} Y(\mathrm{Fl}_t^X \mathbf{c}) - \left.\frac{d}{dt}\right|_{t=0} X(\mathrm{Fl}_t^Y \mathbf{c}) \\ &= \left.\frac{d}{dt}\right|_{t=0} Y(\mathbf{c} + tX(\mathbf{c})) - \left.\frac{d}{dt}\right|_{t=0} X(\mathbf{c} + tY(\mathbf{c})). \end{aligned} \tag{11}$$

Here we have denoted with Fl_t^X and Fl_t^Y the flows of the vector fields. Now consider the horizontal projections of X, Y with respect to connection (5.10). These are given by, $\forall \mathbf{c} \in \mathcal{C}^{3d}$, $\mathrm{hor}(X)(\mathbf{c}) = S_X(\mathbf{c})\mathbf{c}$ and $\mathrm{hor}(Y)(\mathbf{c}) = S_Y(\mathbf{c})\mathbf{c}$, with $S_X(\mathbf{c})$ the solution to Lyapunov equation $S_X(\mathbf{c})K(\mathbf{c}) + K(\mathbf{c})S_X(\mathbf{c}) = 2\,\mathrm{sym}(\mathbf{c}MX(\mathbf{c})^T)$ and $S_Y(\mathbf{c})$ the solution to Lyapunov equation $S_Y(\mathbf{c})K(\mathbf{c}) + K(\mathbf{c})S_Y(\mathbf{c}) = 2\,\mathrm{sym}(\mathbf{c}MY(\mathbf{c})^T)$.

If we specialize the Jacobi-Lie bracket computation (11) to the case $X(\mathbf{c}) = S_X(\mathbf{c})\mathbf{c}$, $Y(\mathbf{c}) = S_Y(\mathbf{c})\mathbf{c}$, we obtain:

$$\begin{aligned}
[S_X(\mathbf{c})\mathbf{c}, S_Y(\mathbf{c})\mathbf{c}](\mathbf{c}) &= \left.\frac{d}{dt}\right|_{t=0} S_Y(\mathbf{c}+tS_X(\mathbf{c})\mathbf{c})(\mathbf{c}+tS_X(\mathbf{c})\mathbf{c}) \\
&\quad - \left.\frac{d}{dt}\right|_{t=0} S_X(\mathbf{c}+tS_Y(\mathbf{c})\mathbf{c})(\mathbf{c}+tS_Y(\mathbf{c})\mathbf{c}) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} S_Y(\mathbf{c}+tS_X(\mathbf{c})\mathbf{c})\right)\mathbf{c} + S_Y(\mathbf{c})S_X(\mathbf{c})\,\mathbf{c} \\
&\quad - \left(\left.\frac{d}{dt}\right|_{t=0} S_X(\mathbf{c}+tS_Y(\mathbf{c})\mathbf{c})\right)\mathbf{c} - S_X(\mathbf{c})S_Y(\mathbf{c})\,\mathbf{c}.
\end{aligned} \tag{12}$$

Since $S_X(\mathbf{c}+tS_Y(\mathbf{c})\mathbf{c})$, $S_Y(\mathbf{c}+tS_X(\mathbf{c})\mathbf{c}) \in \mathbb{R}^{3\times 3}_{sym}\ \forall t$, their derivatives with respect to t are also symmetric; on the other hand, the matrix commutation of two symmetric matrices, and in particular of $S_X(\mathbf{c}), S_Y(\mathbf{c})$, is always skew-symmetric. Hence we can rewrite (12) as follows:

$$[S_X(\mathbf{c})\mathbf{c}, S_Y(\mathbf{c})\mathbf{c}](\mathbf{c}) = J_{XY,sym}(\mathbf{c})\mathbf{c} + J_{XY,skew}(\mathbf{c})\mathbf{c}, \tag{13}$$

where

$$J_{XY,sym}(\mathbf{c}) \triangleq \left.\frac{d}{dt}\right|_{t=0} S_Y(\mathbf{c}+tS_X(\mathbf{c})\mathbf{c}) - \left.\frac{d}{dt}\right|_{t=0} S_X(\mathbf{c}+tS_Y(\mathbf{c})\mathbf{c}) \tag{14}$$

$$J_{XY,skew}(\mathbf{c}) \triangleq S_Y(\mathbf{c})S_X(\mathbf{c}) - S_X(\mathbf{c})S_Y(\mathbf{c}). \tag{15}$$

Now, by the definition of curvature (8):

$$\begin{aligned}
B_{\mathbf{c}}(X(\mathbf{c}), Y(\mathbf{c})) &= -A_{\mathbf{c}}([S_X(\mathbf{c})\mathbf{c}, S_Y(\mathbf{c})\mathbf{c}](\mathbf{c})) = -A_{\mathbf{c}}(J_{XY,sym}(\mathbf{c})\mathbf{c} + J_{XY,skew}(\mathbf{c})\mathbf{c}) \\
&= A_{\mathbf{c}}(-J_{XY,skew}(\mathbf{c})\mathbf{c}),
\end{aligned} \tag{16}$$

since the component $J_{XY,sym}(\mathbf{c})\mathbf{c}$ is horizontal (recall lemma 5.3) and thus its connection vanishes. The formulae (9)-(10) then directly follow from the results of section (l) below. □

(j) Proof of Lemma 5.4

We start by deriving the differential map $d\pi_{SO(3)}$ for the projection map (5.21), $\forall S_K \in T_K\mathcal{K}^*$:

$$d\pi_{SO(3)}(S_K) = \begin{bmatrix} d[\mathrm{tr}(K)](S_K) \\ \frac{1}{2}d[(\mathrm{tr}K)^2 - \mathrm{tr}(K^2)](S_K) \\ d[\det(K)](S_K) \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(S_K) \\ \mathrm{tr}(K)\mathrm{tr}(S_K) - \mathrm{tr}(KS_K) \\ \det(K)\mathrm{tr}(K^{-1}S_K) \end{bmatrix}, \tag{17}$$

where we have used the linearity of the trace operator and Jacobi's formula for the derivative of the determinant of a matrix ($d[\det(K)] = \mathrm{tr}(\mathrm{adj}(K)dK)$, where the *adjunct* of matrix K is equal to $\det(K)K^{-1}$ since K is invertible. From (17), it trivially follows that the vertical space $V_K = \ker d\pi_{SO(3)}(K)$ corresponds exactly to (5.22). Now consider the unique eigendecomposition of K specified by (5.20): $K = Q^T \Lambda Q$, $Q \in SO(3)$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$, and define the matrix $\tilde{S}_K \triangleq QS_KQ^T$. By the invariance of the trace operator to similarity transformations: $\mathrm{tr}(S_K) = 0 \Leftrightarrow \mathrm{tr}(QS_KQ^T) = \mathrm{tr}(\tilde{S}_K) = 0$, $\mathrm{tr}(KS_K) = 0 \Leftrightarrow \mathrm{tr}(QKS_KQ^T) = \mathrm{tr}(\Lambda\tilde{S}_K) = 0$, and $\mathrm{tr}(K^{-1}S_K) = 0 \Leftrightarrow \mathrm{tr}(QK^{-1}S_KQ^T) = \mathrm{tr}(\Lambda^{-1}\tilde{S}_K) = 0$. In terms of the elements $\tilde{S}_{Kij}$ of $\tilde{S}_K$, we have the following set of conditions equivalent to (5.22):

$$\begin{aligned}
\mathrm{tr}(\tilde{S}_K) &= \tilde{S}_{K11} + \tilde{S}_{K22} + \tilde{S}_{K33} = 0 \\
\mathrm{tr}(\Lambda\tilde{S}_K) &= \lambda_1\tilde{S}_{K11} + \lambda_2\tilde{S}_{K22} + \lambda_3\tilde{S}_{K33} = 0 \\
\mathrm{tr}(\Lambda^{-1}\tilde{S}_K) &= \frac{1}{\lambda_1}\tilde{S}_{K11} + \frac{1}{\lambda_2}\tilde{S}_{K22} + \frac{1}{\lambda_3}\tilde{S}_{K33} = 0,
\end{aligned}$$

which we can also rewrite in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{bmatrix} \begin{bmatrix} \tilde{S}_{K11} \\ \tilde{S}_{K22} \\ \tilde{S}_{K33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

Finally, we can show that the matrix on the left side in (18) is always full rank when $\lambda_1 > \lambda_2 > \lambda_3 > 0$ by putting it in upper triangular form via basic row operations:

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{bmatrix} \xrightarrow[R_3' = R_3 - \frac{1}{\lambda_1} R_1]{R_2' = R_2 - \lambda_1 R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 \\ 0 & \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} & \frac{\lambda_1 - \lambda_3}{\lambda_1 \lambda_3} \end{bmatrix}$$

$$\xrightarrow{R_3'' = R_3' + \frac{1}{\lambda_1 \lambda_2} R_2'} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 \\ 0 & 0 & \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}{\lambda_1 \lambda_2 \lambda_3} \end{bmatrix}. \tag{19}$$

Since $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$, (19) can never be singular, and therefore the only possible solution to (18) is $\tilde{S}_{K11} = \tilde{S}_{K22} = \tilde{S}_{K33} = 0$. Therefore we conclude that (5.23) is equivalent to (5.22) for characterizing the vertical space $V_K = \ker d\pi_{SO(3)}(K)$.

Next we characterize the horizontal space orthogonal to the vertical space. A vector $S_K \in T_K \mathcal{K}^*$ is defined as horizontal if and only if $\langle S_K, T_K \rangle_K = 0 \, \forall \, T_K \in V_K$. From (5.23), vertical vectors $T_K \in V_K$ have the property that all diagonal elements of $\tilde{T}_K = Q T_K Q^T$ are equal to zero. Therefore, using (5.17),

$$\begin{aligned} \langle S_K, T_K \rangle_K &= \mathrm{tr}(S_{K\Lambda} \Lambda T_{K\Lambda}) \\ &= \mathrm{tr}\left(\begin{bmatrix} \frac{\tilde{S}_{K11}}{2\lambda_1} & \frac{\tilde{S}_{K12}}{\lambda_1+\lambda_2} & \frac{\tilde{S}_{K13}}{\lambda_1+\lambda_3} \\ \frac{\tilde{S}_{K12}}{\lambda_1+\lambda_2} & \frac{\tilde{S}_{K22}}{2\lambda_2} & \frac{\tilde{S}_{K23}}{\lambda_2+\lambda_3} \\ \frac{\tilde{S}_{K13}}{\lambda_1+\lambda_3} & \frac{\tilde{S}_{K23}}{\lambda_2+\lambda_3} & \frac{\tilde{S}_{K33}}{2\lambda_3} \end{bmatrix} \begin{bmatrix} 0 & \frac{\lambda_1 \tilde{T}_{K12}}{\lambda_1+\lambda_2} & \frac{\lambda_1 \tilde{T}_{K13}}{\lambda_1+\lambda_3} \\ \frac{\lambda_2 \tilde{T}_{K12}}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_2 \tilde{T}_{K23}}{\lambda_2+\lambda_3} \\ \frac{\lambda_3 \tilde{T}_{K13}}{\lambda_1+\lambda_3} & \frac{\lambda_3 \tilde{T}_{K23}}{\lambda_2+\lambda_3} & 0 \end{bmatrix} \right) \\ &= \frac{1}{\lambda_1+\lambda_2} \tilde{S}_{K12} \tilde{T}_{K12} + \frac{1}{\lambda_1+\lambda_3} \tilde{S}_{K13} \tilde{T}_{K13} + \frac{1}{\lambda_2+\lambda_3} \tilde{S}_{K23} \tilde{T}_{K23}, \end{aligned} \tag{20}$$

$\forall T_K \in V_K$. It is clear that $\langle S_K, T_K \rangle_K = 0 \, \forall \, T_K \in V_K$ (i.e. for each possible $\tilde{T}_{K12}, \tilde{T}_{K13}, \tilde{T}_{K23} \in \mathbb{R}$ in (20)) if and only if $\tilde{S}_{K12} = \tilde{S}_{K13} = \tilde{S}_{K23} = 0$. Equivalently, $S_K \in T_K \mathcal{K}^*$ is horizontal if and only if $\tilde{S}_K = Q S_K Q^T$ is a diagonal matrix, i.e. (5.24) holds. □

(k) Proof of (5.27)

Any given inertia tensor transformation $S(\mathbf{c}, \mathbf{v_c})\mathbf{c}$, starting at configuration $\mathbf{c}$ with $K(\mathbf{c}) \in \mathcal{K}^*$, can be projected down to $T_K \mathcal{K}^*$ to find the corresponding tangent vector in base space, denoted as S_K:

$$d\pi_{\mathbf{c}}(S(\mathbf{c}, \mathbf{v_c})\mathbf{c}) = K S(\mathbf{c}, \mathbf{v_c}) + S(\mathbf{c}, \mathbf{v_c}) K \triangleq S_K. \tag{21}$$

Replacing K with its eigendecomposition $K = Q^T \Lambda Q$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, we obtain:

$$Q^T \Lambda Q S(\mathbf{c}, \mathbf{v_c}) + S(\mathbf{c}, \mathbf{v_c}) Q^T \Lambda Q = S_K.$$

Pre-multiplying by Q and post-multiplying by Q^T both sides, and using the fact that $Q Q^T = \mathbb{1}$:

$$\Lambda \tilde{S}(\mathbf{c}, \mathbf{v_c}) + \tilde{S}(\mathbf{c}, \mathbf{v_c}) \Lambda = \tilde{S}_K, \tag{22}$$

where we have introduced $\tilde{S}(\mathbf{c}, \mathbf{v_c}) \triangleq Q S(\mathbf{c}, \mathbf{v_c}) Q^T$ and $\tilde{S}_K \triangleq Q S_K Q^T$.

Considering only the diagonal elements of the matrices, and using the facts that $(\Lambda\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag} = \Lambda(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}$, $(\tilde{S}(\mathbf{c},\mathbf{v_c})\Lambda)^{on}_{diag} = (\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}\Lambda$:

$$\Lambda(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag} + (\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}\Lambda = (\tilde{S}_K)^{on}_{diag}.$$

Pre-multiplying by Q^T and post-multiplying by Q both sides, and using again the fact that $QQ^T = \mathbb{1}$:

$$\begin{aligned} Q^T\Lambda QQ^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}Q + Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}QQ^T\Lambda Q = Q^T(\tilde{S}_K)^{on}_{diag}Q \Rightarrow \\ KQ^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}Q + Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}QK = \mathrm{hor}(S_K). \end{aligned} \tag{23}$$

Comparing (23) with (21), it's evident that $d\pi_{\mathbf{c}}(Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}Q\mathbf{c}) = \mathrm{hor}(S_K)$, and conversely $\mathrm{lift}_{\mathbf{c}}\mathrm{hor}(S_K) = Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{on}_{diag}Q\mathbf{c}$.

Considering instead the off-diagonal elements of the matrices in (22), and using the facts that $(\Lambda\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag} = \Lambda(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}$, $(\tilde{S}(\mathbf{c},\mathbf{v_c})\Lambda)^{off}_{diag} = (\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}\Lambda$, we obtain:

$$KQ^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}Q + Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}QK = Q^T(\tilde{S}_K)^{off}_{diag}Q = A_K(S_K). \tag{24}$$

Thus $d\pi_{\mathbf{c}}(Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}Q\mathbf{c}) = A_K(S_K)$ and $\mathrm{lift}_{\mathbf{c}}A_K(S_K) = Q^T(\tilde{S}(\mathbf{c},\mathbf{v_c}))^{off}_{diag}Q\mathbf{c}$. □

(I) Ensemble connection applied to $\mathbf{v_c} = A(\mathbf{c},\mathbf{v_c})\mathbf{c} \in T_{\mathbf{c}}\mathcal{C}^{3d}$, $\forall A(\mathbf{c},\mathbf{v_c}) \in \mathbb{R}^{3\times3}_{skew}$

From the ensemble connection definition (theorem 5.2):

$$\begin{aligned} F(\mathbf{c},\mathbf{v_c}) &= 2\,\mathrm{sym}(\mathbf{c}M\mathbf{v_c}^T) = 2\,\mathrm{sym}(\mathbf{c}M\mathbf{c}^TA(\mathbf{c},\mathbf{v_c})^T) = 2\,\mathrm{sym}(K(\mathbf{c})A(\mathbf{c},\mathbf{v_c})^T) \\ &= A(\mathbf{c},\mathbf{v_c})K(\mathbf{c}) - K(\mathbf{c})A(\mathbf{c},\mathbf{v_c}), \end{aligned} \tag{25}$$

thus the horizontal component of $\mathbf{v_c}$ is:

$$\mathrm{hor}(A(\mathbf{c},\mathbf{v_c})\mathbf{c}) = S_A(\mathbf{c},\mathbf{v_c})\mathbf{c}, \tag{26}$$

where $S_A(\mathbf{c},\mathbf{v_c})$ is the solution to the Lyapunov equation:

$$S_A(\mathbf{c},\mathbf{v_c})K(\mathbf{c}) + K(\mathbf{c})S_A(\mathbf{c},\mathbf{v_c}) = A(\mathbf{c},\mathbf{v_c})K(\mathbf{c}) - K(\mathbf{c})A(\mathbf{c},\mathbf{v_c}). \tag{27}$$

We now specialize (5.13) to the Lyapunov equation (27). First of all, we can express $\tilde{F}$ as follows:

$$\begin{aligned} \tilde{F} &= Q(\mathbf{c})[A(\mathbf{c},\mathbf{v_c})Q^T(\mathbf{c})\Lambda(\mathbf{c})Q(\mathbf{c}) - Q^T(\mathbf{c})\Lambda(\mathbf{c})Q(\mathbf{c})A(\mathbf{c},\mathbf{v_c})]Q^T(\mathbf{c}) \\ &= \tilde{A}(\mathbf{c},\mathbf{v_c})\Lambda(\mathbf{c}) - \Lambda(\mathbf{c})\tilde{A}(\mathbf{c},\mathbf{v_c}), \end{aligned} \tag{28}$$

where $\tilde{A}(\mathbf{c},\mathbf{v_c}) \triangleq Q(\mathbf{c})A(\mathbf{c},\mathbf{v_c})Q^T(\mathbf{c}) \in \mathbb{R}^{3\times3}_{skew}$.
Let $\tilde{a}_{12}, \tilde{a}_{13}, \tilde{a}_{23}$ be the components of $\tilde{A}(\mathbf{c},\mathbf{v_c})$:

$$\tilde{A}(\mathbf{c},\mathbf{v_c}) = \begin{bmatrix} 0 & \tilde{a}_{12} & \tilde{a}_{13} \\ -\tilde{a}_{12} & 0 & \tilde{a}_{23} \\ -\tilde{a}_{13} & -\tilde{a}_{23} & 0 \end{bmatrix}. \tag{29}$$

Then:

$$\begin{aligned} \tilde{F} &= \begin{bmatrix} 0 & \lambda_2\tilde{a}_{12} & \lambda_3\tilde{a}_{13} \\ -\lambda_1\tilde{a}_{12} & 0 & \lambda_3\tilde{a}_{23} \\ -\lambda_1\tilde{a}_{13} & -\lambda_2\tilde{a}_{23} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda_1\tilde{a}_{12} & \lambda_1\tilde{a}_{13} \\ -\lambda_2\tilde{a}_{12} & 0 & \lambda_2\tilde{a}_{23} \\ -\lambda_3\tilde{a}_{13} & -\lambda_3\tilde{a}_{23} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (\lambda_2-\lambda_1)\tilde{a}_{12} & (\lambda_3-\lambda_1)\tilde{a}_{13} \\ (\lambda_2-\lambda_1)\tilde{a}_{12} & 0 & (\lambda_3-\lambda_2)\tilde{a}_{23} \\ (\lambda_3-\lambda_1)\tilde{a}_{13} & (\lambda_3-\lambda_2)\tilde{a}_{23} & 0 \end{bmatrix}. \end{aligned} \tag{30}$$

From (5.13), we thus obtain:

$$S_A(\mathbf{c},\mathbf{v_c}) = Q^T(\mathbf{c}) \begin{bmatrix} 0 & \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} \\ \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & 0 & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} \\ \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} & 0 \end{bmatrix} Q(\mathbf{c}), \tag{31}$$

and hence:

$$\mathrm{hor}(A(\mathbf{c},\mathbf{v_c})\mathbf{c}) = S_A(\mathbf{c},\mathbf{v_c})\mathbf{c} = Q^T(\mathbf{c}) \begin{bmatrix} 0 & \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} \\ \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & 0 & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} \\ \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} & 0 \end{bmatrix} Q(\mathbf{c})\mathbf{c}. \tag{32}$$

Finally, the ensemble connection $A_\mathbf{c}$ (9) applied to $\mathbf{v_c} = A(\mathbf{c},\mathbf{v_c})\mathbf{c}$ can be indirectly obtained as the residual component $\mathbf{v_c} - \mathrm{hor}(A(\mathbf{c},\mathbf{v_c})\mathbf{c})$:

$$\begin{aligned} A_\mathbf{c}(A(\mathbf{c},\mathbf{v_c})\mathbf{c}) &= A(\mathbf{c},\mathbf{v_c})\mathbf{c} - S_A(\mathbf{c},\mathbf{v_c})\mathbf{c} \\ &= Q^T(\mathbf{c}) \left(\tilde{A}(\mathbf{c},\mathbf{v_c}) - \begin{bmatrix} 0 & \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} \\ \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & 0 & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} \\ \frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} & \frac{\lambda_3-\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} & 0 \end{bmatrix} \right) Q(\mathbf{c})\mathbf{c} \\ &= Q^T(\mathbf{c}) \begin{bmatrix} 0 & \frac{2\lambda_1}{\lambda_1+\lambda_2}\tilde{a}_{12} & \frac{2\lambda_1}{\lambda_1+\lambda_3}\tilde{a}_{13} \\ \frac{-2\lambda_2}{\lambda_1+\lambda_2}\tilde{a}_{12} & 0 & \frac{2\lambda_2}{\lambda_2+\lambda_3}\tilde{a}_{23} \\ \frac{-2\lambda_3}{\lambda_1+\lambda_3}\tilde{a}_{13} & \frac{-2\lambda_3}{\lambda_2+\lambda_3}\tilde{a}_{23} & 0 \end{bmatrix} Q(\mathbf{c})\mathbf{c}, \end{aligned} \tag{33}$$

with $\tilde{A}(\mathbf{c},\mathbf{v_c})$ as in (29).

(m) Verification of the orthogonality between the terms in (5.30)

From theorem 5.4, the rigid rotation $(\mathbf{v_c})_{rot}$ and the inertia tensor deformation $(\mathbf{v_c})_{ens.def}$ are certainly orthogonal. Moreover, $(\mathbf{v_c})_{rot}$ is orthogonal to any other shape transformation, including the residual shape transformation $(\mathbf{v_c})_{shp.res}$ not accounted for by the inertia tensor deformation. Hence we only need to verify that the two shape transformation terms $(\mathbf{v_c})_{ens.def}$ and $(\mathbf{v_c})_{shp.res}$ are orthogonal to each other. Combining (5.29) and (5.30), we can rewrite $(\mathbf{v_c})_{shp.res}$ as:

$$\begin{aligned} (\mathbf{v_c})_{shp.res} &= \mathbf{v_c} \;-\; (\mathbf{v_c})_{ens.def} \;-\; (\mathbf{v_c})_{rot} \\ &= (\mathbf{v_c})_{ens.rot} \;+\; (\mathbf{v_c})_{dem} \;-\; (\mathbf{v_c})_{rot}, \end{aligned}$$

i.e. as the sum of an inertia tensor rotation, a democratic motion, and a rigid rotation. All three motions are orthogonal to any inertia tensor deformation, and in particular to $(\mathbf{v_c})_{ens.def}$. Hence $(\mathbf{v_c})_{shp.res}$ and $(\mathbf{v_c})_{ens.def}$ are orthogonal to each other.

(n) Proof of theorem 5.5

Any arbitrary inertia tensor deformation is in the form $Q^T\mathrm{diag}(\tilde{S}_{11},\tilde{S}_{22},\tilde{S}_{33})Q\mathbf{c}$, for some $\tilde{S}_{11},\tilde{S}_{22},\tilde{S}_{33} \in \mathbb{R}$. Given any $\alpha \in \mathbb{R}$, we can rewrite it as follows:

$$Q^T\mathrm{diag}(\tilde{S}_{11},\tilde{S}_{22},\tilde{S}_{33})Q\mathbf{c} = \alpha\,\mathbf{c} \;+\; Q^T\mathrm{diag}(\tilde{S}_{11}-\alpha,\tilde{S}_{22}-\alpha,\tilde{S}_{33}-\alpha)Q\mathbf{c}, \tag{34}$$

i.e. as the sum of a volume-changing motion ($\alpha\,\mathbf{c}$) and a residual inertia tensor transformation. However, we are interested in the only choice of $\alpha \in \mathbb{R}$ that makes the two components orthogonal to each other:

$$
\begin{aligned}
&\langle \alpha \mathbf{c}, Q^T \operatorname{diag}(\tilde{S}_{11}-\alpha, \tilde{S}_{22}-\alpha, \tilde{S}_{33}-\alpha) Q \mathbf{c} \rangle = \\
&\operatorname{tr}(\alpha \mathbf{c} M \mathbf{c}^T Q^T \operatorname{diag}(\tilde{S}_{11}-\alpha, \tilde{S}_{22}-\alpha, \tilde{S}_{33}-\alpha) Q) = \\
&\operatorname{tr}(\alpha Q^T \Lambda \operatorname{diag}(\tilde{S}_{11}-\alpha, \tilde{S}_{22}-\alpha, \tilde{S}_{33}-\alpha) Q) = \\
&\alpha \operatorname{tr}(\operatorname{diag}(\lambda_1(\tilde{S}_{11}-\alpha), \lambda_2(\tilde{S}_{22}-\alpha), \lambda_3(\tilde{S}_{33}-\alpha))) = \\
&\alpha(\lambda_1 \tilde{S}_{11} + \lambda_2 \tilde{S}_{22} + \lambda_3 \tilde{S}_{33} - \alpha(\lambda_1+\lambda_2+\lambda_3)) = 0,
\end{aligned} \tag{35}
$$

where we have first used the fact that $\mathbf{c} M \mathbf{c}^T = K = Q^T \Lambda Q$ and then the invariance of the trace to similarity transformations. Clearly the choice that satisfies (35) is:

$$
\alpha = (\lambda_1 \tilde{S}_{11} + \lambda_2 \tilde{S}_{22} + \lambda_3 \tilde{S}_{33})/(\lambda_1+\lambda_2+\lambda_3) = \operatorname{tr}(\Lambda \tilde{S})/\operatorname{tr}(K). \tag{36}
$$

We finally need to specialize this result to the inertia tensor deformation component of an arbitrary collective motion $\mathbf{v_c} \in T_{\mathbf{c}}\mathcal{C}^{3d}$, which is given by (5.28). Hence we need to replace $\tilde{S}_{11} = \tilde{F}_{11}/(2\lambda_1)$, $\tilde{S}_{22} = \tilde{F}_{22}/(2\lambda_2)$ and $\tilde{S}_{33} = \tilde{F}_{33}/(2\lambda_3)$ in (36), where $\tilde{F}_{ij}$ are the entries of matrix $\tilde{F} = Q(2\operatorname{sym}(\mathbf{c} M \mathbf{v_c}^T))Q^T$. In particular:

$$
\begin{aligned}
\operatorname{tr}(\Lambda \tilde{S}) &= \lambda_1 \tilde{S}_{11} + \lambda_2 \tilde{S}_{22} + \lambda_3 \tilde{S}_{33} = \frac{1}{2}(\tilde{F}_{11} + \tilde{F}_{22} + \tilde{F}_{33}) = \frac{1}{2}\operatorname{tr}(\tilde{F}) \\
&= \frac{1}{2}\operatorname{tr}(Q(2\operatorname{sym}(\mathbf{c} M \mathbf{v_c}^T))Q^T) = \frac{1}{2}\operatorname{tr}(2\operatorname{sym}(\mathbf{c} M \mathbf{v_c}^T)) = \operatorname{tr}(\mathbf{c} M \mathbf{v_c}^T),
\end{aligned} \tag{37}
$$

where we have used the property that $\operatorname{tr}(\operatorname{sym}(M)) = \operatorname{tr}(\frac{M+M^T}{2}) = \operatorname{tr}(M)$ for any matrix $M \in \mathbb{R}^{3\times 3}$. Replacing (37) in (36) completes the proof. □

(o) Derivation of energy contributions in (5.32)

$$
\begin{aligned}
E_{ens.rot} &\triangleq \frac{1}{2}\langle(\mathbf{v_c})_{ens.rot},(\mathbf{v_c})_{ens.rot}\rangle_{\mathbf{c}}\\
&= \frac{1}{2}\mathrm{tr}(Q^T\begin{bmatrix} 0 & \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3}\\ \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & 0 & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3}\\ \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3} & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3} & 0\end{bmatrix} Q\mathbf{c}M\mathbf{c}^T Q^T \begin{bmatrix} 0 & \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3}\\ \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & 0 & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3}\\ \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3} & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3} & 0\end{bmatrix} Q)\\
&= \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} 0 & \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3}\\ \frac{\tilde{F}_{12}}{\lambda_1+\lambda_2} & 0 & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3}\\ \frac{\tilde{F}_{13}}{\lambda_1+\lambda_3} & \frac{\tilde{F}_{23}}{\lambda_2+\lambda_3} & 0\end{bmatrix}\begin{bmatrix} 0 & \frac{\lambda_1\tilde{F}_{12}}{\lambda_1+\lambda_2} & \frac{\lambda_1\tilde{F}_{13}}{\lambda_1+\lambda_3}\\ \frac{\lambda_2\tilde{F}_{12}}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_2\tilde{F}_{23}}{\lambda_2+\lambda_3}\\ \frac{\lambda_3\tilde{F}_{13}}{\lambda_1+\lambda_3} & \frac{\lambda_3\tilde{F}_{23}}{\lambda_2+\lambda_3} & 0\end{bmatrix}\right)\\
&= \frac{1}{2}\left(\frac{\tilde{F}_{12}^2}{\lambda_1+\lambda_2}+\frac{\tilde{F}_{13}^2}{\lambda_1+\lambda_3}+\frac{\tilde{F}_{23}^2}{\lambda_2+\lambda_3}\right),
\end{aligned}
$$

$$
\begin{aligned}
E_{ens.def} &\triangleq \frac{1}{2}\langle(\mathbf{v_c})_{ens.def},(\mathbf{v_c})_{ens.def}\rangle_{\mathbf{c}}\\
&= \frac{1}{2}\mathrm{tr}(Q^T\begin{bmatrix} \frac{\tilde{F}_{11}}{2\lambda_1} & 0 & 0\\ 0 & \frac{\tilde{F}_{22}}{2\lambda_2} & 0\\ 0 & 0 & \frac{\tilde{F}_{33}}{2\lambda_3}\end{bmatrix} Q\mathbf{c}M\mathbf{c}^T Q^T \begin{bmatrix} \frac{\tilde{F}_{11}}{2\lambda_1} & 0 & 0\\ 0 & \frac{\tilde{F}_{22}}{2\lambda_2} & 0\\ 0 & 0 & \frac{\tilde{F}_{33}}{2\lambda_3}\end{bmatrix} Q)\\
&= \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} \frac{\tilde{F}_{11}}{2\lambda_1} & 0 & 0\\ 0 & \frac{\tilde{F}_{22}}{2\lambda_2} & 0\\ 0 & 0 & \frac{\tilde{F}_{33}}{2\lambda_3}\end{bmatrix}\begin{bmatrix} \frac{\tilde{F}_{11}}{2} & 0 & 0\\ 0 & \frac{\tilde{F}_{22}}{2} & 0\\ 0 & 0 & \frac{\tilde{F}_{33}}{2}\end{bmatrix}\right)\\
&= \frac{1}{8}\left(\frac{\tilde{F}_{11}^2}{\lambda_1}+\frac{\tilde{F}_{22}^2}{\lambda_2}+\frac{\tilde{F}_{33}^2}{\lambda_3}\right),
\end{aligned}
$$

$$
\begin{aligned}
E_{rot} &\triangleq \frac{1}{2}\langle(\mathbf{v_c})_{rot},(\mathbf{v_c})_{rot}\rangle_{\mathbf{c}}\\
&= \frac{1}{2}\sum_{i=1}^{n} m_i(I_{\mathbf{c}}^{-1}\mathbf{J}(\mathbf{c},\mathbf{v_c})\times\mathbf{c_i})^T(I_{\mathbf{c}}^{-1}\mathbf{J}(\mathbf{c},\mathbf{v_c})\times\mathbf{c_i})\\
&= \frac{1}{2}\sum_{i=1}^{n} m_i(\mathbf{J}(\mathbf{c},\mathbf{v_c})^T I_{\mathbf{c}}^{-1}(|\mathbf{c_i}|^2\mathbb{1}-\mathbf{c_i}\mathbf{c_i}^T)I_{\mathbf{c}}^{-1}\mathbf{J}(\mathbf{c},\mathbf{v_c}))\\
&= \frac{1}{2}\mathbf{J}(\mathbf{c},\mathbf{v_c})^T I_{\mathbf{c}}^{-1}\mathbf{J}(\mathbf{c},\mathbf{v_c}),
\end{aligned}
$$

$$
\begin{aligned}
E_{vol} &\triangleq \frac{1}{2}\langle(\mathbf{v_c})_{vol},(\mathbf{v_c})_{vol}\rangle_{\mathbf{c}}\\
&= \mathrm{tr}\left(\frac{\mathrm{tr}(\mathbf{c}M\mathbf{v_c}^T)}{\mathrm{tr}(K)}\,\mathbf{c}M\mathbf{c}^T\,\frac{\mathrm{tr}(\mathbf{c}M\mathbf{v_c}^T)}{\mathrm{tr}(K)}\right) = \frac{\mathrm{tr}^2(\mathbf{c}M\mathbf{v_c}^T)}{\mathrm{tr}^2(K)}\mathrm{tr}(K)\\
&= \frac{\mathrm{tr}^2(\mathbf{c}M\mathbf{v_c}^T)}{\mathrm{tr}(K)}.
\end{aligned}
$$

(p) Analysis and visualization of the pigeon flocking data (figure 8)

The pigeon flocking data that we analysed is part of the dataset studied in [14], which can be downloaded from the webpage *http://hal.elte.hu/pigeonflocks*. The dataset includes 11 free flights and 4 homing flights of a small pigeon flock (7-10 individuals), and consists of pre-processed positions and velocities acquired at 5Hz via miniature GPS devices mounted on the pigeons.

In our analysis, we excluded time samples at which the data from one or more pigeons was missing, or at which some pigeons were still stationary or taking off. To exclude the latter, we started considering the data only once all the birds had reached a speed of at least 2 m/s. We considered 240 seconds for each trial, or up to the sample when either the speed of the fastest pigeon dropped below 2 m/s, or that of the slowest pigeon dropped below 1 m/s. The last two criteria were introduced to exclude time samples in which the whole flock or some of its members may have started to land. Five of the free flights were excluded from analysis, since no time samples were available with all the pigeons flying simultaneously. Each of the remaining six free flights provided more than 80 seconds of data (400 time-samples). Two of the free flights and all of the homing flights provided 240 seconds of data. From this set, we chose one representative homing flight and one representative free flight for figure 8(a)-8(f). The energy ratios were computed as described in the main text, with unit masses for all pigeons. For the free flight, which had a more interesting distribution of energy, we also computed the histogram relevant to the energy ratios (6.2) (figure 8(g), bins width 0.05). We chose these energy ratios over the alternative ones (6.1) because we observed that typically $E_{rot} > E_{ens.rot}$ (cf. fig.8(c),8(f)). However, the alternative histogram would have been almost identical to the one we displayed.

To verify that the energy distribution in figure 8(g) was representative of a typical free flight, we computed a similar histogram for the other five free flights and then computed the mean and standard deviation of the normalized bin counts across all six flights. Normalization consisted in dividing the bin counts in each trial by the total number of time samples. The resulting mean distribution of the energy ratios was plotted in figure 8(h), along with error bars denoting the standard deviation across six flights. The results were consistent with figure 8(g), confirming that the most relevant energy contributions were E_{com}, E_{rot} and E_{vol}, in this order. We then assessed how much of the total kinetic energy could be accounted for by the first energy term alone (E_{com}), the first two terms alone ($E_{com} + E_{rot}$) and the first three terms alone ($E_{com} + E_{rot} + E_{vol}$). To do that, we computed histograms of these quantities for each trial, normalized, and then derived the mean and standard deviation across trials. The results, plotted in figure 8(i), showed that on average the translational term E_{com} accounted for more than 90% of the energy (last two bins) roughly 80% of the time, with the percentage growing to 91% and then to 97% when E_{rot} and then E_{vol} were included. This highlighted that the energy contributions of the other terms $E_{shp.res}$ and $E_{ens.res}$, especially the latter, were almost always negligible. This fact was visualized in figure 8(i) by also plotting the distribution of $E_{com} + E_{rot} + E_{vol} + E_{ens.res}$, almost identical to $E_{com} + E_{rot} + E_{vol}$.

Finally, we verified that the energy distributions for the free flights (fig.8(g)-8(i)) could not be explained by chance. We produced artificial datasets from the true datasets by independently drawing, at each time step, the individual pigeon velocities from a multivariate normal distribution with mean equal to the true center of mass velocity and covariance equal to the sample covariance of the pigeon velocities (given by $\frac{1}{n-1}\mathbf{v_c}\mathbf{v_c}^T$). At each time step, the artificial pigeon trajectories were updated forward in time with Euler method. From each free flight, we produced ten artificial flocking events in this fashion, computed the histograms of the energy ratios of interest for each of them, and then averaged across the ten events. This yielded a representative artificial distribution of energy for each of the six free flights. We then computed the mean and standard deviation of these distributions across the six flights and compared with those obtained from the true datasets. Figure 9 shows the results of the comparison. Since the distribution of E_{com} in true and artificial trials was analogous by construction, we plotted the fraction of the relative kinetic energy ($E_{rel} = E - E_{com}$) accounted for by each elementary motion, rather than the fraction of the total energy E.

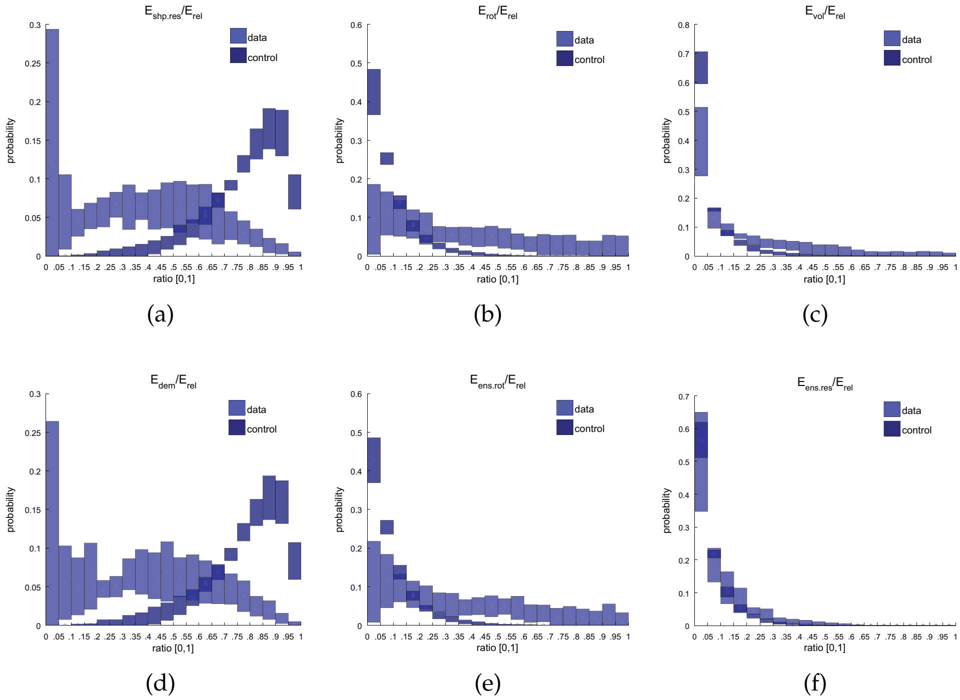

Figure 9. Comparison of mean probability distributions obtained from actual pigeon free flights (*data*, $n = 6$) with those obtained in comparable artificial flocking events generated with random pigeon velocities relative to the center of mass (*control*). All energy ratios are relative to $E_{rel} = E - E_{com}$: (a) $E_{shp.res}/E_{rel}$, (b) E_{rot}/E_{rel}, (c) E_{vol}/E_{rel}, (d) E_{dem}/E_{rel}, (e) $E_{ens.rot}/E_{rel}$, (f) $E_{ens.res}/E_{rel}$. The height of the bars reflects the standard deviation of the distributions ($\pm 1 std$).

The results clearly show that the shape transformation energy $E_{shp.res}$ would have been much larger, and the rotation and volume-changing energies E_{rot}, E_{vol} smaller, if the pigeon velocities were random fluctuations relative to a common velocity vector (fig.9(a)-9(c)). This is not surprising, since shape transformations are horizontal tangent vectors relative to the shape fibering, and the dimensionality of the base space is much higher than that of the fiber space in this fibering (fig.4(a),5(a)). Hence a tangent vector picked at random is likely to have a larger projection onto the horizontal subspace than the vertical one. Similarly, democratic motions are vertical tangent vectors for the ensemble fibering, for which the fiber space has higher dimension than the base space (fig.4(b),5(b)), so they would account for a large fraction of energy in the case of random pigeon velocities. The data, instead, show a smaller contribution of E_{dem} and a larger contribution of $E_{ens.rot}$ (fig.9(d)-9(e)) than what would be obtained by chance. Only the distribution of the energy term $E_{ens.res}$, which is almost negligible in both the real and the artificial datasets, appears to be explainable by chance alone (fig.9(f)).