Coherent random walks in free space: supplementary material

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This document provides supplementary information to "Coherent random walks in free space," <http://dx.doi.org/10.1364/optica.1.000268>, and comprises two sections. The first section contains a rigorous mathematical derivation of how the lattice-like evolution arises in free space. It also provides a region of validity within which the free space evolution is indeed equal to the evolution within a discrete lattice. The second section contains an example which corresponds to a two site excitation of a discrete lattice.

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ANALYTICAL CONNECTION BETWEEN LATTICE AND FREE SPACE

Purpose

Our aim here is to demonstrate that the amplitude distribution described by the equation

$$\psi(x, z) = \left(1 - \frac{iz}{k\sigma^2}\right)^{-1/2} \exp\left[\frac{-2kx^2 + ia^2\sigma^2 z}{4(k\sigma^2 - iz)}\right] \sum_{l=-\infty}^{\infty} (-i)^l J_{2l}(A(x, z)) J_l(B(z)) , \quad (\text{S1})$$

with $A(x, z)$ and $B(z)$ given in the paper, is related, within a certain longitudinal region and at certain transverse distances labeled as x_m , to the probability amplitude observed along the m -th channel of a periodic lattice when excited by a single quantum walker [1, 2]

$$\psi_m(\zeta) = (-i)^m J_m(\kappa\zeta) , \quad (\text{S2})$$

where κ represents the coupling coefficients between adjacent lattice elements.

As already mentioned in the paper, we start by assuming a very broad Gaussian apodizing the initial transverse profile

$J_0(\alpha x)$ ($k\sigma^2 \gg z; \sigma \gg x$), such that Eq.(S1) reduces to

$$\psi(x, z) = \sum_{l=-\infty}^{\infty} (-i)^l J_{2l}(\alpha x) J_l(a^2 z / 4k) . \quad (\text{S3})$$

Subsequently, it will be shown that Eq. (S3) is indeed formally equivalent to Eq. (S2) within some region of validity.

Mathematical preliminaries

Before going into the details of the derivation, we start by presenting the most important mathematical tools which we will use. In the following, we will apply the two asymptotic approximations of the Bessel functions of the first kind, namely [3,

9.1.7]

$$J_m(x) \approx \frac{1}{m!} \left(\frac{x}{2}\right)^m, \quad (\text{S4})$$

valid for small arguments

$$x < \sqrt{m+1}, \quad (\text{S5})$$

and [4, 9.2.1]

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m}{2}\pi - \frac{\pi}{4}\right) \quad (\text{S6})$$

for large arguments

$$x > m^2 - \frac{1}{4}. \quad (\text{S7})$$

Moreover, we will also apply the Jacobi-Anger-identity [5, 9.1.42-45]

$$e^{ix \cos \phi} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\phi},$$

which can be used to show that

$$J_0 + 2 \sum_{n=1}^{\infty} J_{4n}(x) = \frac{1}{2} [1 + \cos x] \quad (\text{S9})$$

as well as

$$2 \sum_{n=0}^{\infty} J_{4n+2}(x) = \frac{1}{2} [1 - \cos x]. \quad (\text{S10})$$

To obtain Eq.(S13) we have used the identity $J_{-m}(x) = (-1)^m J_m(x)$ [7, 9.1.5], as well as Eq. (S9) and Eq. (S10). From Eq.(S13) it can be inferred that for all $x_m = m\pi/\alpha$, m being an integer, either the first $[1 + \cos(\alpha x_{2m+1})]$ or the second

$$\psi(x_a, z) \approx \sqrt{\frac{8k}{\pi\alpha^2 z}} \left\{ \cos\left(\frac{\alpha^2 z}{4k} - \frac{\pi}{4}\right) \frac{1}{2} [1 + \cos(\alpha x_a)] - i \sin\left(\frac{\alpha^2 z}{4k} - \frac{\pi}{4}\right) \frac{1}{2} [1 - \cos(\alpha x_a)] \right\}. \quad (\text{S13})$$

Equations (S9) and (S10) can be easily obtained from the Jacobi-Anger-identity by adding (or subtracting) the two special cases of $\phi = 0$ and $\phi = \pi/2$.

Derivation

The first step in deriving the equivalence between Eq. (S3) and Eq. (S2) is to show that for a given $x = x_a$ the sum in Eq. (S3) can be terminated at a corresponding maximum order m_a , i.e.

$$\psi(x_a, z) \approx \sum_{l=-m_a}^{m_a} (-i)^l J_{2l}(\alpha x_a) J_l(\alpha^2 z/4k). \quad (\text{S11})$$

The validity of Eq. (S11) can be conducted from Eq. (S4) and Stirling's formula [6, 6.1.37], from which follows that for every x_a there is an m_a , such that Cond. Eq. (S5) holds and for all $m > m_a$ one has $J_m(x_a) \approx 0$.

Therefore, once we fix m_a , it is possible to find $z > z_a$, such that Cond. Eq. (S7) holds for every term contained in the sum of Eq.(S11). Using these facts, Eq.(S11) becomes

$$\psi(x_a, z > z_a) \approx \sum_{l=-m_a}^{m_a} \sqrt{\frac{8k}{\pi\alpha^2 z}} (-i)^l J_{2l}(\alpha x_a) \cos\left(\frac{\alpha^2 z}{4k} - \frac{l\pi}{2} - \frac{\pi}{4}\right). \quad (\text{S12})$$

After some algebra, this sum can be cast as follows

$[1 - \cos(\alpha x_{2m})]$ term vanishes depending on whether m is odd or even. Finally, using Eq. (S6) in Eq. (S13) and considering $x_m = m\pi/\alpha$ leads to

$$\begin{aligned} \psi(x_{2m}, z > z_m) &\approx \sqrt{\frac{8k}{\pi\alpha^2 z}} \cos\left(\alpha^2 z/4k - \frac{\pi}{4}\right) = (-i)^m \sqrt{\frac{8k}{\pi\alpha^2 z}} \cos\left(\alpha^2 z/4k - \frac{2m}{2}\pi - \frac{\pi}{4}\right) \\ &\approx (-i)^{2m} J_{2m}(\alpha^2 z/4k), \end{aligned}$$

in the even case, whereas for the odd case we have

$$\begin{aligned} \psi(x_{2m+1}, z > z_m) &\approx -i \sqrt{\frac{8k}{\pi\alpha^2 z}} \sin\left(\alpha^2 z/4k - \frac{\pi}{4}\right) = (-i)^{2m+1} \sqrt{\frac{8k}{\pi\alpha^2 z}} \cos\left(\alpha^2 z/4k - \frac{2m+1}{2}\pi - \frac{\pi}{4}\right) \\ &\approx (-i)^{2m+1} J_{2m+1}(\alpha^2 z/4k). \end{aligned}$$

Evidently, these two expressions can be condensed into one equation

$$\psi(x_m, z > z_m) \approx (-i)^m J_m(\alpha^2 z/4k), \quad (\text{S14})$$

which is identical to the evolution of the probability amplitude arising in periodic lattices given in Eq.(S2). Comparing the amplitude evolution in an array, Eq.(S2), and the free space evolution, Eq.(S14), we find that they are equal provided $\alpha^2 z/4k = \kappa\zeta$. Consequently, to obtain the same longitudinal dimensions, $z =$

ζ , we just have to make the substitution $\kappa = \alpha^2/4k$.

Region of validity

In order to determine the range of validity for our approach we consider the transverse positions $x_a(n) = n\pi/\alpha$, introduced above. For these positions $x_a(n)$ one can specify $m_a = 4n + 4$, such that Cond. Eq. (S5) and Eq. (S11) are satisfied. Inserting m_a into Cond. Eq. (S7) the parabolic region of validity can be

given as

$$z_a(n) \approx 16k/a^2 (n^2 + 2n + 1). \quad (\text{S15})$$

Note that additionally $ko^2 \gg z$ as well as $\sigma \gg x$ still has to hold.

Fig. S1. Theoretical field distributions for a two site excitation. (a) shows the probability distribution $|\psi_{\text{lattice}}(n, z)|^2$ of a random walker traversing a uniform lattice, where the lattice sites are labeled by n . In (b) the free space intensity $|\psi_{\text{fs}}(x_n, z)|^2$ is plotted for the different transverse positions $x_n = n\pi/a$. Plot (c) depicts the difference $|\psi_{\text{fs}}(x_n, z) - \psi_{\text{lattice}}(n, z)|^2$.

TWO SITE EXCITATION

For the sake of completeness, in this section we quantify the difference between the probability amplitude exhibited by a single quantum walker, prepared in a coherent superposition of states, propagating in a periodic lattice, and the amplitude dynamics of its counterpart free-space state.

consider the coherent superposition In this section order to complete the discussion on the two site excitation presented in

the main paper, Fig. S1 shows the comparison between the lattice propagation and the free space scenario.

Comparing Fig. S1 (a) and (b), one can qualitatively see the similarities between both systems. From Fig. S1 (c) also the quantitative agreement becomes visible. From the superposition principle of linear systems it follows in a straight forward fashion, that the parabolic region of validity for the two excitation szenario is given by the overlap of the two transversely displaced regions corresponding to single excitations. As a result for the two site excitation the region of validity is narrower compared to the case of a single excitation.

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