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Image retrodiction at low light levels: supplementary material

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This document provides supplementary information to "Image retrodiction at low light levels," http://dx.doi.org/10.1364/optica.2.000950. It includes derivations of some central results of the main work for the more general case of data corrupted by dark counts as well as a short calculation on the retrodicted intensity difference between two pixels.

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1. SINGLE PIXEL RETRODICTION INCLUDING DARK COUNTS

Any real photon detection device will suffer from dark counts, thermal fluctuations in the photon detectors causing erroneous detection signals. These can be modelled as an additional independent Poisson processes such that the number of dark counts d is distributed as $p(d|\delta) = \mathrm{Pois}(d;\delta)$ where δ is the average dark-count rate. If we knew that we had n photons and d dark counts, the probability for m detection events would be

$$p(m|n,d,\delta) = p(m-d|n)\,\mathrm{Pois}(d;\delta), \tag{S1}$$

where $p(m-d|n)$, the probability to get $m-d$ counts given the presence of n photons, is the binomial distribution shown in Eq. (1).

A. Intensity retrodiction

As we know neither the original number of photons n nor the number of dark counts d we find

$$p(m|\lambda,\delta) = \sum_{d=0}^{m} \mathrm{Pois}(d;\delta) \sum_{n=m-d}^{\infty} p(m-d|n)\,\mathrm{Pois}(n;\lambda) = \mathrm{Pois}(m;\delta+\eta\lambda). \tag{S2}$$

This is simply the convenient result that the sum of two independent Poisson distributed variables again follows a Poisson distribution.

If the images show on average $\overline{m} > \delta$ counts per pixel we expect the true intensity λ approximately at $(\overline{m}-\delta)/\eta$ which gives us a prior $p(\lambda|\overline{m},\delta) \simeq \exp(-\eta\lambda/(\overline{m}-\delta))$. To calculate $p(\lambda|m,\overline{m},\delta)$ we need

$$p(m|\lambda,\delta)p(\lambda|\overline{m},\delta) = \sum_{k=0}^{m} \mathrm{Pois}(m-k;\delta)\,\mathrm{Geom}(k;\overline{m}-\delta) \times \mathrm{Gam}\left(\lambda; k+1, [\eta(1+1/(\overline{m}-\delta))]^{-1}\right) \tag{S3}$$

and $p(m|\overline{m},\delta) = \int_0^\infty p(m|\lambda,\delta)p(\lambda|\overline{m},\delta)d\lambda$, that is

$$p(m|\overline{m},\delta) = \sum_{k=0}^{m} \mathrm{Pois}(m-k;\delta)\,\mathrm{Geom}(k;\overline{m}-\delta). \tag{S4}$$

Using the known expectation values of the gamma distribution we get immediately

$$E(\lambda|m,\overline{m},\delta) = \frac{1}{p(m|\overline{m},\delta)} \sum_{k=0}^{m} \mathrm{Pois}(m-k;\delta)\,\mathrm{Geom}(k;\overline{m}-k) \times \frac{k+1}{\eta(1+1/(\overline{m}-\delta))}. \tag{S5}$$

The sums appearing above can be condensed into incomplete gamma functions, but the current form of the expressions is far more intuitive: For instance, the probability to measure m counts in Eq. (S4) is a result of combining $m-k$ Poisson distributed dark-counts with k events from the geometric distribution with mean $\overline{m}-\delta$ corresponding to "true" photodetections, see also Eq. (5).

The effect of increasing values of δ on the probability distribution for λ is shown in figure S1.

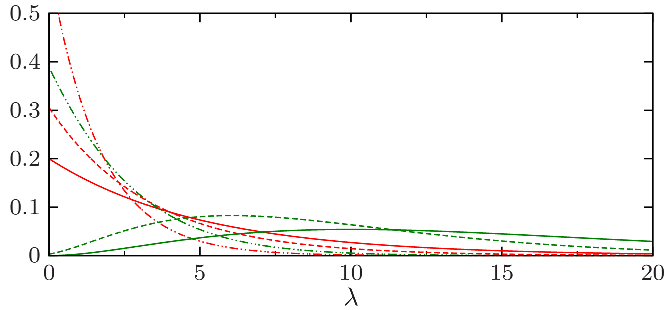

Fig. S1. Illustration of the change in single-pixel intensity distribution $p(\lambda|m,\overline{m},\delta)$ for an increasing dark-count rate δ as given in Eq. (S3) for $m = 0$ (green lines) and $m = 2$ (red lines). With an average count-rate $\overline{m} = 2$ and $\eta = 0.2$ the solid lines show the case without dark-counts, $\delta = 0$, the dashed lines are for $\delta = 0.1$ and the dash-dotted lines are for $\delta = 1.5$.

B. Transmission retrodiction

Just as above we can derive the probabilities for the transmission coefficients τ given that we measured m and that the incoming photons follow a Poisson distribution of mean value ν such that $p(n|\tau,\nu) = \text{Pois}(n;\tau\nu)$. Then the equivalent to Eq. (S2) from intensity retrodiction is $p(m|\tau,\nu,\delta) = \text{Pois}(m;\tau\eta\nu+\delta)$ such that the probability distribution for the transmission parameter τ reads

$$p(\tau|m,\nu,\delta) = \frac{\eta\nu(\delta+\eta\tau\nu)^m e^{-(\delta+\eta\tau\nu)}}{\Gamma(m+1,\delta+\eta\nu)-\Gamma(m+1,\delta)} \tag{S6}$$

with a mean value

$$E(\tau|m,\nu,\delta) = \frac{1}{\eta\nu}\frac{\Gamma(m+2,\delta+\eta\nu)-\Gamma(m+2,\delta)}{\Gamma(m+1,\delta+\eta\nu)-\Gamma(m+1,\delta)} - \frac{\delta}{\eta\nu}. \tag{S7}$$

2. MIXED RETRODICTION INCLUDING DARK-COUNTS

In section 3 we discussed the probability distribution for a situation where we a priori assume that there is a chance W that the intensity λ was responsible for a measurement m, cf. Eq. (14). Adding an average dark-count rate δ gives

$$p(m|\lambda,\lambda',W,\delta) = \text{Pois}(m;\eta W\lambda+\eta(1-W)\lambda'+\delta). \tag{S8}$$

Using the exponential prior $p(\lambda'|\overline{m},\delta) \sim \exp\big(-\eta\lambda'/(\overline{m}-\delta)\big)$ we find

$$p(m|\lambda,W,\overline{m},\delta) = \\ = \sum_{k=0}^{m} \text{Geom}\left(m-k;(\overline{m}-\delta)(1-W)\right)\text{Pois}(k;\delta+\eta W\lambda), \tag{S9}$$

which is the equivalent to Eq. (15). Similarly we have $p(\lambda|m,W,\overline{m},\delta) \sim p(m|\lambda,W,\overline{m},\delta)p(\lambda|\overline{m},\delta)$,

$$p(\lambda|m,W,\overline{m},\delta) \sim \sum_{k=0}^{m}\text{Geom}(m-k;(\overline{m}-\delta)(1-W)) \\ \times \sum_{l=0}^{m}\text{Pois}(k-l;\delta)\,\text{Geom}(l;W(\overline{m}-\delta)) \\ \times \text{Gam}\left(\lambda;l+1,[\eta(W+1/(\overline{m}-\delta))]^{-1}\right). \tag{S10}$$

As described in Eq. (20), a Bayesian update of the probability distribution for λ_i using measurements $m_1,\ldots,m_N$, the corresponding weights $W_{i1},\ldots,W_{iN}$ and a dark-count rate δ is then calculated as

$$p(\lambda|\{m_j,W_{ij}\}_{j=1,\ldots,N},\overline{m},\delta) \sim \\ \sim \prod_{j\neq i} p(m_j|\lambda,W_{ij},\overline{m},\delta)p(\lambda|m_i,W_{ii},\overline{m},\delta). \tag{S11}$$

3. EXPECTED DISTANCE BETWEEN TWO PIXELS

In section 3D we introduce the weights for the retrodiction inspired by the non-local means averaging algorithm and use the expected distance between two retrodicted intensities as a measure for the similarity between two pixels. The expression given there in Eq. (22) is derived below.

Having measurements m_1 and m_2 we may also calculate the probability distribution of the difference $\Delta = \lambda_1 - \lambda_2$. If $\Delta \geq 0$, then $\lambda_1 = \lambda_2 + \Delta$ and for $\Delta < 0$ we have $\lambda_2 = \lambda_1 + |\Delta|$. Therefore the probability to measure a certain distance is

$$\Pr(\{\Delta = \lambda_1 - \lambda_2\}) = \\ = \begin{cases} \int_0^\infty \Pr(\{\lambda_1 = \lambda_2+\Delta\})\Pr(\{\lambda_2\})\,d\lambda_2 & \text{for } \Delta \geq 0 \\ \int_0^\infty \Pr(\{\lambda_2 = \lambda_1+|\Delta|\})\Pr(\{\lambda_1\})\,d\lambda_1 & \text{for } \Delta < 0 \end{cases}$$

such that using $p(\lambda_i|m_i,\overline{m}) = \text{Gam}(\lambda_i;m_i+1,1/\overline{\eta})$ from Eq. (7) we get

$$p(\Delta|m_1,m_2,\overline{m}) = \\ = \begin{cases} \sum_{k=0}^{m_2}\binom{m_1+m_2-k}{m_1}\frac{\text{Gam}(\Delta;k+1,1/\overline{\eta})}{2^{m_1+m_2-k+1}} & \text{for } \Delta \geq 0 \\ \sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\frac{\text{Gam}(|\Delta|;k+1,1/\overline{\eta})}{2^{m_1+m_2-k+1}} & \text{for } \Delta < 0 \end{cases} \tag{S12}$$

with a mean value

$$\text{E}(\Delta|m_1,m_2,\overline{m}) = \sum_{k=0}^{m_2}\binom{m_1+m_2-k}{m_1}\frac{k+1}{2^{m_1+m_2-k+1}\overline{\eta}} \\ - \sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\frac{k+1}{2^{m_1+m_2-k+1}\overline{\eta}}. \tag{S13}$$