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Far-field linear optical superresolution via heterodyne detection in a higher-order local oscillator mode: supplementary material

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This document provides supplementary information to "Far-field linear optical superresolution via heterodyne detection in a higher-order local oscillator mode," http://dx.doi.org/10.1364/optica.3.001148. We theoretically calculate the heterodyne detector output with a local oscillator in TEM_{01}. We also show how we use the Fisher information to estimate our experimental uncertainties.

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Theoretical prediction for the signal. Here we calculate the heterodyne detector signal in response to the electromagnetic field generated by an object of a specific shape. We used this method to obtain the theoretical curves in Figs. 2 and 3 of the main text. The calculation is for one-dimensional objects, but is readily extended to two dimensions.

We start by briefly reviewing Abbe's microscope resolution theory.

$$\tilde{E}(k_\perp) = \int_{-\infty}^{+\infty} E(x) e^{ik_\perp x} \mathrm{d}x, \tag{S1}$$

where $k_\perp$ is the orthogonal component of the wavevector and constant normalization factors are neglected throughout the calculation. The objective lens is located in the far field at distance L from the object plane. The position X in the lens plane is then related to $k_\perp$ according to

$$X = L\frac{k_\perp}{k} = \frac{L\lambda k_\perp}{2\pi}. \tag{S2}$$

The lens, in turn, generates the inverse Fourier image in its focal plane:

$$\begin{aligned} E'(x') &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{E}(k_\perp)\tilde{T}(k_\perp) e^{-ik_\perp x'} \mathrm{d}k_\perp \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x)\tilde{T}(k_\perp) e^{ik_\perp(x-x')} \mathrm{d}x\mathrm{d}k_\perp, \end{aligned} \tag{S3}$$

where $\tilde{T}(k_\perp)$ is the transmissivity of the lens as a function of the transverse position in its plane. If this transmissivity is a constant, corresponding to an infinitely wide lens, the image is identical to the object: $E'(x') = E(x)$. If the lens is of finite width, the image is distorted according to

$$E'(x') = \int_{-\infty}^{+\infty} E(x) T(x' - x) \mathrm{d}x, \tag{S4}$$

where $T(x'-x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{T}(k_\perp) e^{ik_\perp(x-x')} \mathrm{d}k_\perp$ is the Fourier image of the lens. In other words, the image is a convolution of the object with $T(\cdot)$.

Heterodyne detection of the image field yields the electronic signal given by Eq.(3) in the main text. If the field $E(x)$ is coherent, that equation is sufficient to calculate the output signal.

If the object is incoherent, we must take the average of the signal over all realizations of $E(x)$ to find the output power of the heterodyne detector photocurrent:

$$\langle P\rangle \propto \left\langle \left| \int_{-\infty}^{+\infty} E'(x')E^*_{LO}(x')\mathrm{d}x' \right|^2 \right\rangle \tag{S5}$$
$$= \iint_{-\infty}^{+\infty} \left\langle E'(x')E'^*(x'')\right\rangle E^*_{LO}(x')E_{LO}(x'')\mathrm{d}x'\mathrm{d}x''$$

Now using Eq. (S4) we find

$$\left\langle E'(x')E'^*(x'')\right\rangle \tag{S6}$$
$$= \iint_{-\infty}^{+\infty} \langle E(x_1)E^*(x_2)\rangle\, T(x'-x_1)T^*(x''-x_2)\mathrm{d}x_1\mathrm{d}x_2.$$

For an incoherent object,

$$\langle E(x_1)E^*(x_2)\rangle \propto I(x_1)\delta(x_1-x_2), \tag{S7}$$

and hence

$$\langle P\rangle \propto \iiint_{-\infty}^{+\infty} I(x)T(x'-x)T^*(x''-x) \tag{S8}$$
$$\times E_{LO}(x')E^*_{LO}(x'')\mathrm{d}x\mathrm{d}x'\mathrm{d}x''.$$

Our goal is to determine the heterodyne detector signal as a function of the object configuration, $E(x)$ in the coherent case and $I(x)$ in the incoherent case. These calculations are simplified if we assume the transmissivity function to be of top-hat shape: $T(k_\perp) = \theta(k_{\perp\max} - |k|)$ with $k_{\perp\max} = 2\pi R/(L\lambda)$ according to Eq. (S2), $R = 0.4 \pm 0.05$ mm is the radius of the diaphragm and $\theta(\cdot)$ is the Heaviside step function. In the Fourier domain this translates into

$$T(x'-x) \propto \frac{\sin(k_{\perp\max}(x-x'))}{k_{\perp\max}(x-x')} \approx e^{-(x-x')^2/4\sigma^2}, \tag{S9}$$

with $\sigma \approx 0.21\lambda L/R = 0.31 \pm 0.03$ mm. In two dimensions, the transmissivity function is given by the first-order Bessel function whose Gaussian approximation is similar.

The LO field in the TEM_{00} mode is optimized to match the mode $E(x')$ in the coherent case for $E(x) \propto \delta(x)$, so $E_{LO}(x') \propto e^{-x'^2/4\sigma^2}$. Subsequently it is switched to the TEM_{01} mode with $E_{LO}(x') \propto x'e^{-x'^2/4\sigma^2}$. Substituting this mode into the expression Eq.(3) in the main text for the coherent case, we find

$$P \propto J^2 \propto \left(\int_{-\infty}^{+\infty} xE(x)e^{-x^2/8\sigma^2} \right)^2 \mathrm{d}x. \tag{S10}$$

For the incoherent case, the mean signal power (S8) equals

$$\langle P\rangle \propto \int_{-\infty}^{+\infty} x^2 I(x)e^{-x^2/4\sigma^2}\mathrm{d}x. \tag{S11}$$

While the above results are valid for coherent and incoherent objects of any shape, a simple analysis leading to Eq.(5) in the main text is sufficient for the purposes of our experiment, as evidenced by Fig. S1. The only visible difference between the two models is that the curve calculated for the incoherent case using the complete model does not reach zero at the slit position $x_p = 0$. This is because an incoherent slit of a finite width, which can be seen as a combination of multiple mutually incoherent point sources positioned around $x_p = 0$, makes a nonzero contribution to TEM_{01}.

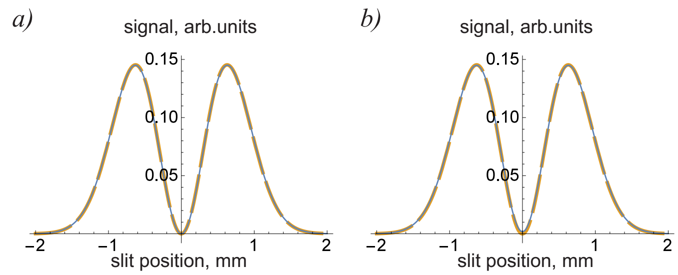

Fig. S1. Comparison of the theoretical predictions for the signal power taking into account the finite width of the slit (blue thin solid line) and assuming an infinitely narrow slit (yellow thick dashed line). a) Coherent case, b) incoherent case.

Error analysis. Our experimental setup yields the electronic signal power $P(x_p)$ with root mean square (rms) uncertainty $\Delta(P(x_p))$ as a function of the slit position x_p. Suppose the task is to estimate x_p from the observed power. Below, we present a method for determining and minimizing the error of this estimation.

We can treat the observed power as a random variable whose probability distribution

$$f(P, x_p) = \frac{1}{\sqrt{2\pi}\Delta} e^{-[P-P(x_p)]^2/2\Delta^2} \tag{S12}$$

is Gaussian with rms width Δ centered on the mean power $P(x_p)$. The problem then reduces to that of estimating parameter x_p from this random variable. The uncertainty of this estimation can be found using the Cramér-Rao error bound, $\delta x_p \geq \sqrt{1/F}$, where

$$F = \int_{-\infty}^{+\infty} \left[\left(\frac{\partial f(P,x_p)}{\partial x_p} \right)^2 \bigg/ f(P,x_p) \right] \mathrm{d}P \tag{S13}$$

is the Fisher information. In the neighborhood of $x_p = 0$, the power $P(x_p)$ can be approximated as ax_p^2, while the uncertainty $\Delta^2 = c^2 + [gP(x_p)]^2$ as discussed in the main text, with constants a, c and g evaluated from the experimental data. Accordingly, we find for the Fisher information in the limit $g \ll 1$

$$F = \frac{4a^2x_p^2}{a^2g^2x_p^4 + c^2}. \tag{S14}$$

Next, we determine the value of x_p where the Fisher information is maximized so the measurement is the most sensitive. Taking the derivative of F, we find that the maximum value $F_{\max} = 2a/gc$ is reached at $x_p = \sqrt{c/ag}$. It is these values that we use to evaluate the estimation uncertainties of x_p in the main text.

Derivation of Eq. (10) in the main text. We use Eq. (S7) to write the output power of the heterodyne detector as the average

$$\langle P_{0n}\rangle \propto \left\langle J_{0n}^2 \right\rangle = \iint_{-\infty}^{+\infty} \langle E(x_1)E(x_2)\rangle\, J(x_1)J(x_2)\mathrm{d}x_1\mathrm{d}x_2$$
$$= \int_{-\infty}^{+\infty} I(x)J^2(x)\mathrm{d}x. \tag{S15}$$

Substituting (8) from the main text into the above result, we obtain (10).