Invisibility exposed: physical bounds on passive cloaking: supplementary material

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This document provides supplementary information to “Invisibility exposed: physical bounds on passive cloaking,” <http://dx.doi.org/10.1364/optica.3.000718>. The supplementary document provides the full derivation of the physical bounds on 1D cloaking [Eq. (1) of the main text] and 3D cloaking [Eqs. (2-4) of the main text], as well as further physical insight on the considered scattering problems. Supplementary material also includes the geometrical and material properties of all the considered scatterers and cloaks.

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S1. Physical Bounds on 1D Cloaking – Details and Derivation

The electromagnetic wave interaction with a transversely homogenous slab can be conveniently modeled with transmission-line theory [S1], in which the transverse components of the electric and magnetic fields in the different regions become voltage and current waves in appropriate transmission-line segments. Moreover, if the slab thickness d is much smaller than the operating wavelength, the transmission-line model of the slab reduces to a shunt impedance. For example, for a dielectric slab under normal plane-wave incidence, the impedance is given by [S2]:

$$Z_{slab,0} = -\frac{1}{i\omega\epsilon_0(\epsilon_r - 1)d}, \quad (\text{S1})$$

where ϵ_r is the relative permittivity of the slab, ϵ_0 is the permittivity of free space, and we have assumed a $e^{-i\omega t}$ time-harmonic convention. Therefore, a lossless dielectric slab is formally equivalent to a shunt capacitor of value $C_{slab,0} = \epsilon_0 d(\epsilon_r - 1)$, sandwiched between two semi-infinite transmission lines (with characteristic impedance $\eta_0 = \sqrt{\mu_0/\epsilon_0}$) representing the surrounding medium. This expression for the equivalent capacitance can be easily verified by considering the Taylor expansion, for $k_0 d \rightarrow 0$ (k_0 is the free-space wavenumber), of the reflection coefficient of the slab (or its input impedance), and comparing it with that of a lumped parallel RC circuit, in which the resistance $R = \eta_0$ represents the semi-infinite space behind the slab. With this approach, we have reduced a field problem into a much simpler circuit problem, which correctly

captures the physical response of the structure when $k_0 d \rightarrow 0$. In other words, the RC circuit correctly models the low-pass response (lower reflection at lower frequencies) of an electrically thin slab.

Similarly, we can model the response of thicker slabs, in the neighborhood of any Fabry-Perot resonance (zeros of the reflection coefficient), by using parallel RLC circuits, which capture the local band-pass response of the structure. A dielectric slab periodically resonates at

$$\omega_{FP,m} = \frac{m\pi c_0}{d\sqrt{\epsilon_r}}, \quad (\text{S2})$$

where $m = 1, 2, 3, \dots$ is the order of the Fabry-Perot resonance. In addition, the Taylor expansion of the reflection coefficient of the slab, for $\omega \rightarrow \omega_{FP,m}$, is equal to

$$\Gamma_{slab,FP}(\omega) = \frac{id(\epsilon_r - 1)}{2c_0}(\omega - \omega_{FP,m}) + O(\omega - \omega_{FP,m})^2. \quad (\text{S3})$$

A parallel RLC circuit resonates at $\omega_r = 1/\sqrt{LC}$, and the Taylor expansion, for $\omega \rightarrow \omega_r$, of its reflection coefficient (with respect to the surrounding medium with impedance $R = \eta_0$) is $\Gamma_{RLC}(\omega) = iCR(\omega - \omega_r) + O(\omega - \omega_r)^2$. By comparing these expressions for the RLC circuit with (S2) and (S3), we can readily find the values of the equivalent capacitance and inductance of the slab in the neighborhood of the m -th Fabry-Perot resonance:

$$\begin{aligned} C_{slab,FP} &= \frac{1}{2}\epsilon_0(\epsilon_r - 1)d \\ L_{slab,FP} &= \frac{2d\epsilon_r\mu_0}{m^2\pi^2(\epsilon_r - 1)} \end{aligned} \quad (\text{S4})$$

which are in units of Farad and Henry, respectively. Note that the equivalent capacitance is independent of the resonance order m , since the derivative of the reflection coefficient of the slab is the same at any Fabry-Perot resonance, in agreement with Eq. (S3). As we write in the main text, these values of the equivalent circuit elements are intrinsic properties of the scattering system under consideration. In particular, the reactive elements “capture” the locally resonant behavior of the slab, whereas the resistance represents an inherent property of the background medium (namely, its wave impedance). In other words, the equivalent circuit “extracts” the response of an individual complex pole of the input impedance of the slab $Z_{slab}(\omega)$ (which has infinite complex poles and zeros on the complex frequency plane): the reactive elements determine the real frequency of the pole (frequency of the Fabry-Perot resonance), while both resistive and reactive elements control its imaginary frequency, which represents the oscillation damping of the resonance, and determines its quality factor. It is also important to stress that not any circuit equivalent would work as a local model for the slab. It is not sufficient that the circuit correctly matches the local dispersion of the reflection coefficient (which could be done, for example, also with a series RLC resonator); instead, in order to model the relevant physics of the wave interaction with the slab, it is necessary that the circuit correctly captures the response of the local pole (as done by the parallel RLC resonator).

If we interpret the above circuit as a load to be matched (in order to suppress its reflection coefficient), we can apply the well-established Bode-Fano theory to determine theoretical limitations on the performance of any linear and passive (lossless) matching network. To better understand how this theory works, consider the circuit network shown in Fig. 2C of the main text, with a generic matching network (which we denote as N) connected to the RLC load. In general, a given load can always be represented as a purely reactive two-port network (in this case a parallel LC circuit, which we denote as D) terminated in a pure resistance (this is the so-called “Darlington representation” of the load). H. W. Bode [S3] and R. M. Fano [S4] recognized that the zeros of transmission of the reactive network D are also zeros of transmission of the entire system; therefore, at these points, the performance of the total system is independent of the matching network N . Fano also showed that, depending on the order of the load zeros, a certain number of the power-series coefficients of the reflection coefficient, evaluated at these transmission zeros, are determined only by the load, independent of the applied matching network, as long as it is causal, linear, and purely reactive. These coefficients express constraints, or conditions of physical realizability, for the reflection coefficient, uniquely determined by the load (the matching network instead affects the higher-order coefficients) [S4]. By then applying Cauchy’s integral relations, it is possible to transform these conditions of physical realizability into a set of relations that involve an observable physical quantity, namely, the magnitude of the reflection coefficients over the real frequency axis, suitable to derive theoretical limitations on the impedance-matching problem, as detailed in the original paper by Fano [S4]. In the case of a parallel RLC circuit load, as in Fig. 2C, these relations take the form of the following integral inequalities:

$$\begin{cases} \int_0^\infty \omega^{-2} \ln \frac{1}{|\Gamma(\omega)|} d\omega \leq \frac{\pi L}{R} \\ \int_0^\infty \ln \frac{1}{|\Gamma(\omega)|} d\omega \leq \frac{\pi}{RC} \end{cases} \quad (\text{S5})$$

We then assume a box-like shape for the reflection coefficient, as in the inset of Fig. 2D in the main text, with bandwidth $\Delta\omega$, central frequency ω_c , and in-band maximum reflection coefficient $|\Gamma_b|$. This shape for the reflection coefficient guarantees the optimal utilization of the area represented by the integrals in (S5). With this assumption the integral inequalities can be simplified as

$$\begin{cases} \ln \frac{1}{|\Gamma_b|} \left(\frac{4\Delta\omega}{4\omega_c^2 - \Delta\omega^2} \right) \leq \frac{\pi L}{R} \\ \ln \frac{1}{|\Gamma_b|} \Delta\omega \leq \frac{\pi}{RC} \end{cases}, \quad (\text{S6})$$

which can then be written in terms of fractional bandwidth, $B = \Delta\omega/\omega_c$:

$$\begin{cases} \ln \frac{1}{|\Gamma_b|} \frac{1}{\omega_c} \left(\frac{B}{1 - B^2/4} \right) \leq \frac{\pi L}{R} \\ \ln \frac{1}{|\Gamma_b|} B \omega_c \leq \frac{\pi}{RC} \end{cases}. \quad (\text{S7})$$

If we further assume that the fractional cloaking bandwidth is small, i.e., $B^2/4 \ll 1$, which is indeed true in the case of cloaking, the inequalities can be rewritten in a more symmetric fashion. Finally, by substituting the values of the circuit elements from Eq. (S4) (and $R = \eta_0$), and after some manipulation, we obtain

$$B \leq -\frac{\pi}{\ln(|\Gamma_b|)} \min \left[\frac{2d\epsilon_r}{m^2\pi^2(\epsilon_r - 1)c_0} \omega_c, \frac{2c_0}{d(\epsilon_r - 1)} \frac{1}{\omega_c} \right], \quad (\text{S8})$$

in which we choose the most stringent bound between an “inductive bound” (the first term under square brackets) and a “capacitive bound”. In particular, the former dominates if ω_c is smaller than the Fabry-Perot resonance under consideration, i.e., $\omega_c < \omega_{FP,m}$, while the latter dominates for $\omega_c > \omega_{FP,m}$. For a given cloaking frequency ω_c , the order m of the closest resonance [which determines the inductive bound in (S8)] can be found as

$$m = \text{round} \left(\frac{\omega_c d \sqrt{\epsilon_r}}{\pi c_0} \right). \quad (\text{S9})$$

It is therefore clear that if $\omega_c = \omega_{FP,m}$ (i.e., if the cloaking frequency corresponds to a Fabry-Perot resonance of the slab), inductive and capacitive bounds become exactly equal, and Eq. (S8) simplifies as

$$B \leq -\frac{\pi}{\ln(|\Gamma_b|)} \frac{2c_0}{d(\epsilon_r - 1)} \frac{1}{\omega_c}, \quad (\text{S10})$$

which corresponds to Eq. (1) of the main text. For any value of ω_c different from $\omega_{FP,m}$, Eq. (S8) yields more stringent bounds than (S10); therefore, Eq. (S10) gives the most optimistic cloaking bound in the neighborhood of the m -th Fabry-Perot resonance.

As noted in the main text, the physical bound expressed by Eq. (1) implies that if a cloak is designed to have large bandwidth, the scattering reduction will be weak; conversely, if the cloak is designed to have strong scattering reduction, the cloaking bandwidth will inevitably be small. In particular, perfect invisibility can be achieved only over an infinitesimal bandwidth. It is also clear from Eq. (S10) that the bound on the fractional bandwidth becomes more stringent if the slab thickness d , the permittivity ϵ_r , or the cloaking frequency ω_c increase.

S2. Physical Bounds on the Scattering from a Sphere – Details and Derivation

Notable similarities and analogies exist between the reflection by a slab and the scattering by a spherically symmetric object [S5]. For instance, as a transversely symmetric (transversely homogeneous) slab conserves the *transverse momentum* of an incident plane wave, a spherically symmetric object conserves the *angular momentum* of an incident spherical wave (a consequence of Noether's theorem [S6]). In addition, the frequency behavior of the planar reflection coefficient Γ (Fig. 2B) and of the spherical scattering coefficients c_n (Fig. 3B) bear a strong resemblance, as they are both characterized by an alternation of peaks and zeros that suggests an interference origin for both phenomena.

Nevertheless, it is important to also recognize several subtle differences between the two quantities. While Γ represents the reflection coefficient in a planar transmission-line model of the structure, excited by a propagating wave, a given scattering coefficient c_n does not represent the conventional spherical reflection coefficient in the corresponding spherical transmission line. In fact, due to the well-known expansion in spherical harmonics (here written in scalar form):

$$e^{ikr \cos \theta} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(kr) P_n(\cos \theta), \quad (\text{S11})$$

where $j_n(\cdot)$ indicates the spherical Bessel function of order n , and $P_n(\cdot)$ the Legendre polynomial, a propagating plane wave that illuminates the spherical scatterer under consideration can be represented as a discrete sum of *standing* spherical waves, determined by two counter-propagating spherical waves [i.e., $j_n(\cdot) = (h_n^{(1)}(\cdot) + h_n^{(2)}(\cdot))/2$, where $h_n^{(1)}(\cdot)$ and $h_n^{(2)}(\cdot)$ are spherical Hankel functions of first and second kind, representing incoming and out-going spherical waves, respectively]. Therefore, the scattering coefficient c_n represents the “reaction” of the scatterer to such standing-wave “Bessel-function” excitation. Instead, a different quantity, denoted here with s_n , corresponds to the conventional spherical reflection by the scatterer, illuminated by a propagating spherical wave $h_n^{(1)}(\cdot)$, and its amplitude is always unitary if the scatterer is lossless. In other words, considering a spherical transmission-line model of the system for the n -th spherical harmonic (see, e.g., Ref. [S7]), the incident energy completely “bounces back” from the origin of the spherical coordinate system, and the effect of any linear and lossless scatterer is simply to change the phase (not the amplitude) of the out-going spherical wave. However, it can easily be shown that the coefficients s_n and c_n are not independent, as intuitively expected, and their relation is

$$c_n = (s_n e^{-2ik_0 a} - 1)/2, \quad (\text{S12})$$

where a is the radius of the spherical scatterer under consideration. This relation has a clear physical interpretation: for a given scatterer, the n -th scattering coefficient corresponds to the n -th spherical reflection coefficient after applying a phase shift $\phi = 2k_0 a$ (which corresponds to changing the reference plane of the reflection coefficient from the surface of the sphere to its center), and making it interfere with an outgoing spherical wave of unitary amplitude [i.e., the outgoing portion of the standing-wave excitation $j_n(\cdot) = (h_n^{(1)}(\cdot) + h_n^{(2)}(\cdot))/2$]. It is exactly this interference effect between the reflected spherical wave and the

outgoing portion of the “incident” standing wave that determines the alternation of scattering zeros and peaks as a function of frequency, as seen in Fig. 3B.

It is worth discussing the fact that, although the conventional definition of scattering coefficients is physically meaningful (it correctly predicts the scattered energy, which is a measurable quantity), it surprisingly leads to important non-causality issues, as extensively discussed in the literature [S8-10]. In fact, since the interaction of the scatterer with the excitation starts when the incident wave hits the surface, an outgoing scattering signal can appear before the input signal reaches the origin (corresponding to the time instant $t = 0$), where the scattering coefficients are evaluated, consistent with the shift in reference plane in Eq. (S12). Although the total scattering (proportional to the forward scattering) from a material body obviously follows a causal response, the “partial” scattering for a given spherical harmonic is non-causal, due to the phase shift in (S12). Surprisingly, however, if this phase shift was not included, or artificially removed by phase-delaying c_n , the modified scattering coefficients would not correctly predict the scattered energy (any trivial attempt to artificially delay the scattering response would actually lead to breaking the passivity condition, as discussed in [S9], and in the following). Despite these causality issues and the fact that the scattering coefficients do not directly correspond to spherical reflection, we still propose to locally represent c_n as the causal reflection coefficient of a passive circuit, in order to apply Bode-Fano theory to the scattering problem. In particular, we look for a way to capture the interference effect that leads to the alternation of scattering zeros and peaks in Fig. 2B. We shall see in the following that this can indeed be achieved in a physically meaningful way.

First, we notice that the causality issues that affect the conventional scattering coefficients can be well modeled by isolated poles in the complex plane [S9], and are not expected to largely affect the causal resonances that determine the dynamic response of the system after the excitation. This implies that, for the purpose of our work, in which we focus on a typically narrow frequency bandwidth around a central frequency of interest, the response is expected to be well approximated by a fully causal circuit. In addition, while the response of individual scattering harmonics is noncausal, the sum of the responses from all harmonics is fully causal, as we discuss in the following.

The scattering coefficients can be obtained by first calculating the spherical reflection s_n , based on a spherical transmission-line model of the system under considerations [S7], and then applying Eq. (S12). Alternatively, they can be directly obtained by solving a system of equations derived from imposing boundary conditions on the concentric spherical interfaces [S11]. As mentioned in the main text, the scattering coefficients can then be conveniently written as [S12]-[S13]

$$c_n = -\frac{1}{1 + i \frac{V_n}{U_n}}, \quad (\text{S13})$$

where the quantities U_n and V_n (real in the lossless case) are expressed as a combination of spherical Bessel and Neumann functions (see, e.g., [S13] for details). If we assume that c_n is the reflection coefficient of a certain circuit, with respect to an arbitrary reference impedance $Z_0 = 1 \Omega$, the normalized input impedance of the circuit is given by

$$Z_{in,n} = \frac{1 + c_n}{1 - c_n}, \quad (\text{S14})$$

which, after substituting Eq. (S13), can be written as

$$Z_{in,n} = \left[1 + \left(\frac{i V_n}{2 U_n} \right)^{-1} \right]^{-1}. \quad (\text{S15})$$

Eq. (S15) can readily be interpreted as the input impedance of a circuit composed of a unitary resistor $R = 1$, connected in parallel with a purely reactive element $X_n = V_n/2U_n$ (see Fig. S1c), which we named “scattering reactance” in the main text. The reflection coefficient of this circuit is exactly equal to the scattering coefficient c_n . Interestingly, it is easy to verify that, if the resistor is removed from the circuit in Fig. S1c, and the scattering reactance is doubled, the reflection coefficient of this new circuit (Fig. S1b) becomes equal to the phase-shifted spherical reflection coefficient $s_n e^{-2ik_0 a} = -(U_n - iV_n)/(U_n + iV_n)$, which has always unitary amplitude. Therefore, we see that the resistor R in Fig. S1b allows modeling the interference effect that leads to the frequency behavior of the scattering coefficient. As a result, although in the original spherical transmission-line model of a scatterer (Fig. S1a) the reflection coefficient s_n includes both the scattered and non-scattered energy, whereas a transmission coefficient is not defined, we can obtain a model that discriminates between scattered and non-scattered waves by simply: (a) shifting the reference plane to the origin (multiplying by $e^{-2ik_0 a}$, which leads to the circuit in Fig. S1b), and (b) closing the circuit on a resistor $R = 1$ (circuit in Fig. S1c). In the latter circuit, in fact, the reflected energy corresponds to scattering, while the transmitted energy that is dissipated in the resistor represents the portion of input energy that is not scattered.

We should stress that the equivalent circuit in Fig. S1c is not just an abstraction aimed at modeling the frequency response of the scattering coefficients, but it has a clear relation to the underlying physical mechanism. In particular, in order to clarify this claim, it is important to establish that the scattering reactive energy in the near-field of a scatterer, associated to each harmonic (e.g., the fringe capacitance), can be truly represented by a reactive circuit. A first relevant observation comes from examining the current flowing in the equivalent circuit. For simplicity, let us consider the first TM spherical harmonic (corresponding to the electric dipole term in a multipole expansion of the electromagnetic field [S14]), with scattering reactance $X_1^{TM} = V_1^{TM}/2U_1^{TM}$. We drive the circuit with the Norton equivalent of a generator with shunt resistance $R = 1$, as shown in Fig. S1d, with

$$I_0 = \frac{6\pi\omega\epsilon_0}{k_0^3} |\mathbf{E}| \delta(\mathbf{r}), \quad (\text{S16})$$

where $\mathbf{E}$ is the electric field and $\delta(\mathbf{r})$ indicates a delta function centered in the origin. The current flowing through the scattering reactance, for this excitation, can be easily obtained:

$$I_s = I_0 \frac{1}{1 + i \frac{V_1^{TM}}{U_1^{TM}}} = \frac{6\pi\omega\epsilon_0}{k_0^3} c_1^{TM} |\mathbf{E}| \delta(\mathbf{r}). \quad (\text{S17})$$

Since the electric dipole polarizability is given by $\alpha_e = -6\pi i \epsilon_0 c_1^{TM}/k_0^3$ [S13], Eq. (S17) can also be written as

$$I_s = -i\omega\alpha_e |\mathbf{E}| \delta(\mathbf{r}), \quad (\text{S18})$$

which corresponds exactly to the amplitude of the polarization current (actually its electric dipolar contribution) induced over a scatterer with polarizability α_e . This, therefore, suggests that the equivalent circuit in Fig. S1c, correctly captures the underlying physical mechanism that determines the scattering, namely, the excitation of currents on the scatterer by the incident field. In addition, it is important to recognize that the scattering reactance is proportional to the real part of the inverse polarizability

$$\text{Re}[\alpha_e^{-1}] = \frac{k_0^3}{6\pi\epsilon_0} \frac{V_1^{TM}}{U_1^{TM}}, \text{ which indeed controls the electric dipolar}$$

scattering response of the object [S9], whereas the resistance R in the equivalent circuit is associated with the imaginary part of the inverse polarizability $\text{Im}[\alpha_e^{-1}] = -\frac{k_0^3}{6\pi\epsilon_0}$, which is completely

independent of the properties of the scatterer and has the role of fulfilling power conservation (it represents the local photonic density of states of the surrounding environment [S9], [S15]). Similar considerations also apply to c_1^{TE} , which corresponds to the magnetic dipole polarizability, and higher-order multipoles.

Fig. S1. Different models of spherical reflection/scattering. (a) Conventional spherical transmission-line model for the n -th spherical harmonic, in the case of a sphere of radius a . The transmission-line segment modeling the sphere terminates on the origin of the spherical coordinate system (here represented by an open circuit), and is characterized by an impedance ζ_n (see [S7], Section 2.7, for details). The spherical-wave impedance of the surrounding medium is not purely resistive as in the planar case, but it has also a reactive component that becomes important at low frequencies (the spherical-wave impedance can be modeled as a cascade of reactive elements closed on a resistance, as shown by Chu [S17]). (b) Equivalent circuit model for the phase-shifted spherical reflection coefficient $s_n e^{-2ik_0 a}$. The reactance $2X_n$ models the scatterer and its reactive near-field (i.e., the reactive portion of the spherical-wave impedance). As an example, for a sphere in quasi-static regime, the reactance is given by $2X_n = -(3/2k_0^3 a)(\epsilon_r + 2)/(\epsilon_r - 1)$. (c) Equivalent circuit model for the scattering coefficient c_n , which, after suitably enforcing causality, we use as a “scattering load” for the application of Bode-Fano theory. (d) Circuit in (c) driven by the Norton equivalent generator [see Eq. (S16)]. The current flowing through the scattering reactance is equal to

the relevant multipole projection of the polarization current induced on the scatterer.

The above discussion shows that the scattering reactance represents an impedance that relates the local electric field to the induced polarization current (notice that such field-polarization relation is causal). However, in order to apply the analytical results of circuit and network theory (such as Bode-Fano theory), we need to determine whether $X_n = V_n/2U_n$ can formally be considered a reactance function and, if not, suitably modify it in a meaningful way. To give an example, we plot in Fig. S2 the scattering reactance X_1^{TM} for a dielectric sphere, as a function of frequency. It is clear that, generally, the reactance is monotonously increasing with frequency, with an alternation of zeros and poles along the real axis. Therefore, the function $X_1^{TM}(\omega)$ seems to fulfill the fundamental Foster's reactance theorem dictated by causality [S16], which ensures the realizability of a given function in terms of reactive circuit elements (for example a cascade of series or parallel LC resonators). However, superimposed to the general zero-pole monotonous behavior, we also observe a series of "ripples" that, in some cases, may even determine a locally negative slope of the function, which corresponds to a locally non-passive response, even though the scatterer is perfectly passive and lossless (the stored energy is proportional to the frequency derivative of the reactance function). It can be shown that this locally non-passive behavior of $X_n(\omega)$ is a consequence of the inherent non-causal response of the scattering coefficients (or, equivalently, of the polarizabilities) that we have discussed above. In particular, the non-causality of c_n determines the presence of non-causal poles of $X_n(\omega)$ on the complex frequency plane, which may result in a negative slope of the function along the real frequency axis. It is clear that this problem cannot be solved by simply phase-delaying c_n (i.e., multiplying it by e^{2ik_0a}), because this would lead to a completely distorted scattering reactance, drastically violating Foster's theorem. Therefore, trying to solve the passivity issues of $X_n(\omega)$ by phase delaying c_n , determines even worse passivity issues.

Nevertheless, we observe that, for subwavelength scatterers and large scatterers with simple spherical geometry (for example dielectric spheres, or impenetrable spheres), the local non-passivity of $X_n(\omega)$ is typically not strong, whereas the alternation of zeros and poles is the most important feature of the function. Therefore, we are commonly able to define a slightly modified scattering reactance that matches such alternation, while being passive everywhere. In particular, a true reactance function should be purely real and have only real poles (namely, it should be expressed as a sum of causal Lorentzian functions with poles at real frequencies). Therefore, a general recipe to obtain a causal scattering reactance is to build a modified function $X_{n,c}(\omega)$ composed of only the causal real poles of $X_n(\omega)$, following a procedure similar to that developed in [S9] to enforce the causality of the susceptibility of an homogenized metamaterial (notice the analogy between the susceptibility of a lossless material and the reactance of a circuit, as they both follow Kramers-Kronig-like dispersions).

For a given scattering reactance, a fitting algorithm can be used to extract the causal poles and build the modified reactance function. It should be stressed, however, that this procedure can be used only if the resulting causal reactance is not too different from

the original reactance function on the real axis. While this is typically true for several cases of interest for cloaking, for more complex objects the quantity $X_n(\omega)$ may locally deviate from the behavior expected from Foster's theorem around the frequency of interest for some of the scattering harmonics, preventing any attempt to enforce causality. Assuming this is not the case, the fitting algorithm is expected to give good results. In addition, for some classes of scatterers, a simple analytical expression for the approximate causal scattering reactance can be obtained. For example, for moderately large homogenous dielectric and impenetrable spheres, we found that the alternation of causal zeros and poles of $X_n(\omega)$ can be well approximated by the function (for TM spherical harmonics):

$$X_{n,c}(\omega) = -\frac{x j_n(x)}{d[x j_n(x)]/dx}, \quad (\text{S19})$$

where $x = k_0 a_m$, and a_m is a modified radius, with value $a_m = a(\sqrt{\epsilon_r} - 1)$ for dielectric spheres with relative permittivity ϵ_r , and $a_m = a$ for impenetrable spheres. Eq. (S19) can be interpreted as the input impedance of a spherical transmission-line segment (made of air and terminating on the origin), as defined in [S7], assuming that the spherical-wave impedance of the surrounding environment is above cutoff (i.e., it is approximately equal to η_0). This model essentially captures the phase difference that the spherical wave undergoes due to the presence of the scatterer, with respect to the case without it (spherical wave just 'bouncing back' from the origin), and therefore it correctly captures the alternation of zeros and poles of $X_n(\omega)$, as shown in the example in Fig. S2. Incidentally, we also note that the "speed" of the zero-pole alternation for a dielectric sphere with permittivity ϵ_r becomes faster than for a corresponding PEC sphere when $(\sqrt{\epsilon_r} - 1) > 1$, that is, $\epsilon_r > 4$. This observation is related to the fact that the static polarizability of a dielectric sphere with $\epsilon(0) > 4$ is larger than that of a PEC sphere of same size, and therefore, as discussed in [S18] in the context of global cloaking bounds, the dielectric sphere scatters more over the entire electromagnetic spectrum.

Consistent with the physical interpretation above, the reactance $X_{n,c}(\omega)$ in Eq. (S19), which approximates $X_n(\omega)$ for real frequencies, is passive and causal everywhere [the input impedance of the circuit in Fig. S1c, with modified reactance $X_{n,c}(\omega)$, is a *positive-real function* [S19]]. Therefore, fundamental theorems guarantee that such impedance is always realizable as a passive circuit, and all the results of circuit and network theory can be applied to it. In particular, we can exploit Bode-Fano theory to derive bandwidth limitations on the suppression of the reflection coefficient of the n -th passive "scattering load" represented by the circuit in Fig. S1c, which corresponds to the n -th scattering coefficient. At this point, however, one may ask whether these physical bounds, derived based on causal circuit equivalents, really apply to the original scattering coefficients, which are intrinsically non-causal, or they just apply to the reflection coefficients of the equivalent circuits. It turns out that, although the derived bounds may not strictly apply *individually* to the corresponding scattering coefficients, the sum of all the bounds apply to the sum of the scattering coefficients, which is a strictly causal function. We have verified in several examples that, as expected, when a scattering coefficient exhibits a non-causal response, all the other non-

negligible coefficients compensate it, such that their sum remains a causal function. This is clarified in the following paragraphs.

Consider a generic causal function of frequency $A(f)$, which can be written as the sum of two (or more) different “partial” functions $A(f) = B(f) + C(f)$, which, individually, are not causal. Imagine now to define an approximate causal version of these partial functions, which we denote as B_c and C_c , and to derive physical bounds on them in virtue of their causality (e.g., based on Kramers-Kronig relations, or Bode-Fano integrals). For example, we could obtain $\text{Re}[B_c] > b_0$ and $\text{Re}[C_c] > c_0$ over some real frequency interval. These bounds do not necessarily apply to the original partial functions B and C , as they are non-causal (B and B_c are not strictly required to have the same constraints on dispersion). However, if $B_c + C_c \approx B + C$ over the complex frequency plane (in particular, if they have a similar constellation of poles), which is now possible since both sums are causal by definition, then the sum of the bounds becomes an approximate bound on A , namely, $\text{Re}[A] = \text{Re}[B + C] \approx \text{Re}[B_c + C_c] > b_0 + c_0$ (notice that, due to the triangle inequality, this would not generally hold if the bounds were defined in terms of amplitude instead of real part).

In our case, the causal function A is the total forward scattering $S(0^\circ)$, which can be written as the sum of the scattering coefficients [S11], corresponding to the non-causal partial functions B and C above. According to the discussion in the previous paragraph, if we define a set of modified causal scattering coefficients, the sum of the individual bounds on their real part becomes an approximate bound on the original function, namely, a bound on $\text{Re}[S(0^\circ)]$, provided that the sum of the original scattering coefficients is approximately equal to the sum of the modified causal scattering coefficients, which is typically true if the latter correctly capture the causal poles of the former. Notice also that, for a lossless scatterer, $\text{Re}[S(0^\circ)]$ is proportional to the total SCS according to the optical theorem [S5]. Besides, power conservation implies that the real part of the scattering coefficients must be equal to their amplitude square, i.e., $\text{Re}[c_n] = |c_n|^2$, in order for the extinction cross section to be equal to the scattering cross section [S5]. Therefore, an overall bound on the total SCS can be obtained from the sum of the bounds on the amplitude of the modified causal scattering coefficients, which can be conveniently derived via Bode-Fano theory as discussed above and in the main text (instead, it is clear that the individual bounds may not strictly apply to the corresponding individual non-causal scattering coefficients). As it can be seen in the results reported in the main text, this approach indeed leads to meaningful bounds on the total SCS, which agree very well with other heuristic cloaking limitations derived from pure causality considerations (as also discussed at the end of this section).

Fig. S2. Example of scattering reactance, for the first TM spherical harmonic, $X_1^{TM}(\omega)$, for a dielectric sphere with permittivity $\epsilon_r = 3$ and radius $a = \lambda_c$ at the frequency f_c . The red curve indicates the original non-causal scattering reactance, which exhibits a clear alternation of zeros and poles along the real frequency axis, accompanied by “ripples” associated with non-causal complex poles. The blue curve indicates the modified causal scattering reactance, obtained with Eq. (S19), which correctly captures the alternation of zeros and poles of the original reactance, particularly at high frequencies (the model is not accurate for spheres smaller than the wavelength).

Having established in the previous paragraphs how the physical bounds on the scattering coefficients should be treated, we can now proceed in our derivation of such bounds based on causal circuit equivalents. Since the analytical Bode-Fano theory can easily handle only impedances having a single pole, we need to locally model the dispersion of the modified scattering reactance with a simple circuit. Generalization of Bode-Fano theory exists that can handle more complicated impedance functions [S20], but they heavily rely on numerical optimization procedures, which makes them less appealing for our goal of deriving cloaking bounds in closed form. Therefore, here we locally approximate the passive function $X_{n,c}(\omega)$ with a lumped-element parallel LC resonator, which captures its locally resonant response, analogous to what we have done for the dielectric slab in Section S1. In particular, by requiring that the LC circuit matches the value of the function $X_{n,c}(\omega)$ and its first derivative at the frequency of interest ω_c , we obtain the following expressions for the equivalent inductance and capacitance:

$$\begin{aligned} L_n &= \frac{2}{\omega_c^2} \left[\frac{dY_n(\omega)}{d\omega} - \frac{Y_n(\omega)}{\omega} \right]^{-1} \Bigg|_{\omega=\omega_c}, \\ C_n &= \frac{1}{2} \left[\frac{dY_n(\omega)}{d\omega} + \frac{Y_n(\omega)}{\omega} \right] \Bigg|_{\omega=\omega_c}, \end{aligned} \quad (\text{S20})$$

where $Y_n(\omega) = 1/X_{n,c}(\omega)$. This approach to derive physical bounds, based on transforming a field problem into a local circuit problem for each spherical harmonic, is analogous to the approach used to derive the well-known Chu limit on the bandwidth of small antennas [S17]. It should also be stressed that this equivalent circuit must be derived from the modified causal reactance $X_{n,c}(\omega)$. If we used, instead, the original reactance $X_n(\omega)$, we

would obtain inaccurate, or even unphysical, results, due to the ripples and the potentially non-passive response of $X_n(\omega)$.

Finally, by applying Bode-Fano theory to the “scattering load” represented by this parallel RLC circuit (the resistance $R=1$ always closes the circuit, as in Fig. S1c), we obtain physical bounds on the *local* suppression of the n -th modified (causal) scattering coefficient c_n , analogous to the physical bounds on the reflection coefficient studied in Section S1. By substituting Eqs. (S20) into Eq. (S7), and after a few manipulations (assuming again that $B^2/4 \ll 1$), we obtain

$$\Delta f \ln \frac{1}{|c_{n,b}|} \leq \frac{1}{\max[C_{n,+}, C_{n,-}]}, \quad (\text{S21})$$

where Δf is the absolute cloaking bandwidth around the central frequency f_c , $|c_{n,b}|$ is the in-band maximum scattering coefficient, and we choose the most stringent bound according to the quantity

$$C_{n,\pm} = \left[\frac{dY_n(\omega)}{d\omega} \pm \frac{Y_n(\omega)}{\omega} \right]_{\omega=\omega_c}. \quad (\text{S22})$$

Eq. (S21) expresses, in closed form, the physical bound on the suppression of the n -th TE or TM modified scattering coefficient, in terms of cloaking bandwidth and scattering reduction.

Recalling that the total SCS of an object is given by a weighted sum of the amplitude square of all scattering coefficients:

$$SCS = \frac{\lambda_c^2}{2\pi} \sum_{n=1}^N (2n+1) \left(|c_n^{TM}|^2 + |c_n^{TE}|^2 \right), \quad (\text{S23})$$

we can easily obtain an overall cloaking bound on the total SCS by properly combining the individual bounds on each $|c_{n,b}|$, yielding Eq. (2) in the main text. Note that this is possible only because all the operations in (S23) (essentially squaring and summing positive numbers) preserve the direction of the inequality in (S21). The number N of contributions in Eq. (2) of the main text should be equal to the number of non-negligible scattering coefficients, which is typically of the same order as $\text{round}[k_0 a]$ [S11]. It should

also be noted that the cloaking bounds derived here assume an optimistic scenario in which each scattering coefficient can be suppressed independently of the others. In practice, a cloak may be able to suppress a specific scattering order at the expense of enhancing other scattering harmonics, leading to cloaking performance far from the predicted optimal bound. In general, the cloaking bound expressed by Eq. (2) can be considered the most optimistic bound that only assumes causality and passivity, without any further assumptions on the cloaking technique.

As the Chu's limit on the antenna bandwidth [S17] was derived independently from the specific structure of the radiating system, modeled as a generic coupling network for spherical harmonics, in a dual fashion our physical bounds on the cloaking bandwidth are independent of the scheme employed to suppress the coupling to the scattering channels. It is, however, important to mention that, although the cloaking bounds are independent of the structure of the cloak, there is a relevant exception, namely, when the cloak includes an impenetrable layer (e.g., a PEC shell) that completely isolates the space inside it from external fields. In this case, cloaking bounds should be derived for the impenetrable object, and not for the original scatterer. Since an impenetrable sphere may scatter less than a dielectric sphere, it may happen that the new cloaking bounds for the impenetrable scatterer are less

stringent than for the original one. This is not surprising, because also in Bode-Fano theory for circuit networks, if a shunt short circuit (together with perhaps other circuit elements) is introduced in front of the load, then the original load does not matter any longer for the physical bounds, since the voltages and currents never really reach it. Therefore, whenever Bode-Fano theory is applied to a scatterer, it should always be verified whether an equivalent impenetrable sphere would lead to more relaxed cloaking bounds.

Eq. (2) allows quantitatively predicting how the maximum cloaking bandwidth shrinks as the electrical size of an object $x = k_0 a = 2\pi a/\lambda_c$, is scaled up. However, for extremely large objects, e.g., larger than hundreds of wavelengths, it may not be convenient to sum all the contributions in Eq. (2). Nevertheless, for electrically large scatterers the physical bounds derived above can be largely simplified by exploiting the asymptotic form of the spherical Bessel functions for large arguments [S21]:

$$\begin{aligned} j_n(z) &\approx \frac{1}{z} \sin\left(z - \frac{1}{2}n\pi\right) \\ y_n(z) &\approx -\frac{1}{z} \cos\left(z - \frac{1}{2}n\pi\right). \end{aligned} \quad (\text{S24})$$

For example, for a large impenetrable PEC sphere, the (causal) scattering reactances for the n -th TE and TM scattering order become

$$\begin{aligned} X_{n,c}^{TM} &\approx \frac{\sin(k_0 a - n\pi/2)}{2 \cos(k_0 a - n\pi/2)} = \tan(k_0 a - n\pi/2)/2 \\ X_{n,c}^{TE} &\approx -\frac{\cos(k_0 a - n\pi/2)}{2 \sin(k_0 a - n\pi/2)} = -\cot(k_0 a - n\pi/2)/2 \end{aligned}, \quad (\text{S25})$$

which are both positive-real functions and are therefore always realizable as passive circuits. The corresponding approximated scattering coefficients are

$$\begin{aligned} c_n^{TM} &\approx \left[(-1)^{n+1} e^{2ik_0 a} - 1 \right] / 2 \\ c_n^{TE} &\approx \left[(-1)^n e^{2ik_0 a} - 1 \right] / 2. \end{aligned} \quad (\text{S26})$$

[Incidentally, we note that similar asymptotic approximations are often used in the definition of non-relativistic quantum scattering from spherically symmetric potentials [S6]]. As before, we locally model the scattering coefficients (S26) as the reflection of a parallel RLC circuit, in the neighborhood of a zero of c_n (a pole of $X_{n,c}$), as we have done for the slab reflection in section S1. Since we have approximated the spherical Bessel functions with trigonometric functions, the zeros of the scattering coefficients can easily be found, as they are periodic along the real frequency axis. The frequency of the m -th zero of the n -th scattering coefficient is given by

$$\begin{aligned} \omega_{0,mm}^{TM} &= \frac{c_0}{2a} \pi \left(2m + \frac{(-1)^{n+1} - 1}{2} \right) \\ \omega_{0,mm}^{TE} &= \frac{c_0}{2a} \pi \left(2m + \frac{(-1)^n - 1}{2} \right), \end{aligned} \quad (\text{S27})$$

with $m = 1, 2, 3, \dots$. In addition, it can readily be shown that the first derivative of the scattering coefficients at any zero is always equal to

$$\left. \frac{dc_n(\omega)}{d\omega} \right|_{\omega=\omega_{0,nn}} = ia/c_0, \quad (\text{S28})$$

independent of the polarization and of the indices n and m . Finally, by recalling that a parallel RLC circuit resonates at $\omega_r = 1/\sqrt{LC}$, and the first derivative of its reflection coefficient at resonance is equal to iCR , we can easily find the values of the equivalent capacitance and inductance (assuming, as usual, $R=1$):

$$C = \frac{a}{c_0}, \quad L = \frac{a}{c_0} \frac{4}{\pi^2 M_{nm}^{TE/TM}}, \quad (\text{S29})$$

where the quantity $M_{nm}^{TE/TM}$ is defined as

$$M_{nm}^{TE/TM} = \begin{cases} \left(2m + \frac{(-1)^{n+1} - 1}{2} \right)^2 & \text{TM} \\ \left(2m + \frac{(-1)^{n+1} - 1}{2} \right)^2 & \text{TE} \end{cases}. \quad (\text{S30})$$

[Incidentally, we note that these values of the equivalent circuit elements have some similarities with the equivalent circuits for spherical-wave impedances derived by Chu [S17]]. Then, by following exactly the same procedure as in section S1, we obtain theoretical limitations on the suppression of the n -th modified scattering coefficient at a given frequency ω_c . In addition, to further simplify the physical bound, we assume that the cloaking frequency is always very close to a zero of the scattering coefficients, i.e., $\omega_c \approx \omega_{0,nn}$ (for every n), which is clearly the most optimistic assumption. In this case, the inductive and capacitive bounds become approximately equal, as discussed in Section S1. After some manipulation, the physical bound, in terms of fractional cloaking bandwidth B and in-band scattering coefficient, can be written as

$$B \ln \frac{1}{|c_{n,b}|} \leq \pi \frac{c_0}{a} \frac{1}{\omega_c}, \quad (\text{S31})$$

which is independent of the polarization (TE or TM), the scattering order n , and the zero order m . This can be expected since, in the asymptotic approximation used here, the derivative of the scattering coefficients at a zero is always constant, as seen in Eq. (S28), which makes the capacitive bound independent of n, m , and the polarization. Finally, the overall bound on the total scattering cross section is obtained by substituting the inequality (S31) in the expression for the SCS, Eq. (S23), as done before. Here, however, this formula can be greatly simplified since the bound on the individual scattering coefficients does not depend on n , and

$$\frac{\lambda_c^2}{2\pi} \sum_{n=1}^{2\pi a/\lambda_c} (2n+1) \rightarrow 2\pi a^2 \quad \text{for } \lambda_c \rightarrow 0, \quad (\text{S32})$$

namely, the sum converges to twice the geometrical cross section of the sphere for very small wavelengths, consistent with the well-known “extinction paradox” [S11]. Eq. (4) of the main text can then be readily obtained. Interestingly, this cloaking bound for electrically large impenetrable objects, which directly follows from Bode-Fano theory applied to the scattering problem under consideration, is consistent with heuristic cloaking bounds derived earlier in the literature, based on completely different arguments (see, e.g., Refs. [8,13,15] in the main text).

S3. Geometrical and material properties of all the considered scatterers and cloaks

- **Impenetrable sphere in Fig. 1C:** perfect electric conducting (PEC) sphere with radius $a = \lambda_c/6$ at the cloaking frequency λ_c .
- **Plasmonic cloak in Fig. 1C:** homogenous and isotropic shell with radius $a_c = 1.42a$, relative permittivity following lossless Drude dispersion $\epsilon_r = 1 - f_p^2/f^2$ with plasma frequency $f_p = 0.85f_c$ (f_c denotes the cloaking frequency) and dispersionless relative permeability $\mu_r = 2.14$. Parameters obtained through numerical optimization to maximize bandwidth and scattering suppression.
- **Transformation-optics cloak in Fig. 1C:** shell with radius $a_c = 3a$ with ideal anisotropy and inhomogeneity as in Ref. [3] of the main text. We assumed no frequency dispersion for any component of the constitutive tensors larger than unity, while a lossless Lorentz dispersion was used for the permeability components less than unity, $\mu_r = 1 - F/(1 - f_0^2/f^2)$ with $F = 0.55$, and a lossless Drude dispersion for the permittivity components less than unity, $\epsilon_r = 1 - f_p^2/f^2$, consistent with Ref. [S22] (f_0 and f_p continuously change along the radial direction to ensure the required permeability and permittivity, respectively, at each radial position). Scattering cross section numerically computed with COMSOL Multiphysics®.
- **Impenetrable sphere in Fig. 3:** perfect electric conducting (PEC) sphere with radius $a = \lambda_c$ at the cloaking frequency λ_c .
- **Plasmonic cloak in Figs. 3D-E:** homogenous and isotropic shell with radius $a_c = 1.066a$, relative permittivity following lossless Drude dispersion $\epsilon_r = 1 - f_p^2/f^2$ with plasma frequency $f_p = 0.925f_c$ (f_c denotes the cloaking frequency) and dispersionless relative permeability $\mu_r = 4.69$. Parameters obtained through numerical optimization to maximize bandwidth and scattering suppression.
- **Transformation-optics cloak in Figs. 3D-E:** exactly as in Fig. 1C but with outer radius $a_c = 2a$ and $F = 0.7$. The values of a_c and F slightly modify the bandwidth, but never exceeding the cloaking bounds in Fig. 3D. Scattering cross sections numerically computed with COMSOL Multiphysics®.
- **Dielectric slab in Fig. 2:** relative permittivity $\epsilon_r = 10$ and thickness $d = \lambda_c/10$ at the cloaking wavelength λ_c .
- **Planar mantle cloaks in Fig. 2E.** Each metasurface is composed of an infinite array of perfectly conducting square patches having period $P = \lambda_c/4$ and negligible thickness at the cloaking wavelength λ_c . With reference to Fig. S3 all the other dimensions are as follows:
 - 1-layer mantle cloak: $s_1 \approx 0.040\lambda_c$, $d_1 \approx P - 0.017\lambda_c$.
 - 2-layer mantle cloak: $s_1 \approx 0$, $d_1 \approx P - 0.043\lambda_c$, $s_2 \approx 0.058\lambda_c$, $d_2 \approx P - 0.012\lambda_c$.
 - 3-layer mantle cloak: $s_1 \approx 0$, $d_1 \approx P - 0.040\lambda_c$, $s_2 \approx 0.058\lambda_c$, $d_2 \approx P - 0.012\lambda_c$, $s_3 \approx 0.700\lambda_c$, $d_3 \approx P - 0.087\lambda_c$.
- All the parameters have been obtained through numerical optimization to maximize bandwidth and scattering

suppression. Reflection coefficients numerically computed with CST Microwave Studio®.

- **Dielectric sphere in Fig. 4A:** diameter $2a = \lambda_c/2$ at the cloaking wavelength λ_c , and relative permittivity $\epsilon_r = 5$.
- **Chebyshev-like mantle cloak in Fig. 4A.** Two concentric metasurfaces with radii $a_{s1} = a$ (conformal to the dielectric sphere) and $a_{s2} = 1.3a$. The separation between layers is filled with air. The two metasurfaces have been modeled as ideal impedance sheets with zero thickness. A capacitive-like frequency dispersion has been assumed for both impedances: $Z_{s1} = 1/(i\omega C_1)$ and $Z_{s2} = 1/(i\omega C_2)$, where ω is the angular frequency and the values of C_1 and C_2 have been chosen to have reactances $X_{s1} = -64 \Omega$ and $X_{s2} = -212 \Omega$ at the cloaking frequency. All the parameters have been obtained through numerical optimization to maximize bandwidth and scattering suppression

Fig. S3. Planar mantle cloaks based on stacks of metasurfaces. (a) Front view of a single metasurface composed of a periodic array of perfectly conducting (PEC) square patches (gray), in front of a dielectric slab (yellow). Relevant dimensions are the period P and the patch size d_i (the index corresponds to the layer number). (b) Top view of a stack of three metasurfaces, with geometry as in (a), in front of a dielectric slab. Interlayer distances are denoted by s_1, s_2, s_3 .

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