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Topological protection of photonic path entanglement: supplementary material

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We derive an expression for the photon correlation map Γ given the initial two-photon wavefunction, namely:

$$\Gamma_{rs} = \left| \left(U \left(B_0 + B_0^T \right) U^T \right)_{rs} \right|^2, \qquad \text{(S1)}$$

where Γ_{rs} is 2-photon correlation map (at waveguides *r* and *s*), *U* is the single-photon propagator from time *0* to *t*, and B_0 a matrix representing wavefunction at time $t = 0$.
We assume two photons injected into the system, and since photon number is conserved by our Hamiltonian, the wavefunction can always be written as:

$$\left| \psi(t) \right\rangle = \sum_{ij} B_{ij}(t) a_i^\dagger a_j^\dagger \left| 0 \right\rangle, \qquad \text{(S2)}$$

where $a_n^\dagger$ is the creation operator for a photon in waveguide *n*. We denote $B_{ij}(t=0) = B_{0,ij}$. It should be noted that because of the bosonic commutation relations, $B_{ij}(t)$ can always be chosen to be a symmetric matrix.
The hopping coefficient between two waveguides *n* and *m* is c_{nm}, and therefore our Hamiltonian can be written as:

$$H = \sum_{nm} c_{nm} a_n^\dagger a_m . \qquad \text{(S3)}$$

We now aim to write the Schrödinger equation $i\partial_t |\psi\rangle = H|\psi\rangle$ as an equation for the coefficients $B_{ij}(t)$. We evaluate $H|\psi\rangle$:

$$\begin{aligned} &H|\psi\rangle = \sum_{nm} c_{nm} a_n^\dagger a_m \sum_{ij} B_{ij}(t) a_i^\dagger a_j^\dagger |0\rangle = \\ &\sum_{nmij} c_{nm} B_{ij}(t) a_n^\dagger a_m a_i^\dagger a_j^\dagger |0\rangle = 2\sum_{nij} c_{in} B_{nj}(t) a_i^\dagger a_j^\dagger |0\rangle \end{aligned} . \qquad \text{(S4)}$$

The last equality can be easily proven using the well-known commutation relations of the creation and annihilation operators, and remembering that B is a symmetric matrix. Using Eq. (S4), the Schrödinger equation is now written as:

$$i\partial_t \sum_{ij} B_{ij}(t) a_i^\dagger a_j^\dagger |0\rangle = 2\sum_{nij} c_{in} B_{nj}(t) a_i^\dagger a_j^\dagger |0\rangle . \qquad \text{(S5)}$$

Using the following identity, again easily proven using commutation relations:

$$\langle 0| a_p a_q a_i^\dagger a_j^\dagger |0\rangle = \delta_{iq}\delta_{jp} + \delta_{ip}\delta_{jq} , \qquad \text{(S6)}$$

we operate with $\langle 0| a_p a_q$ from the left on our Schrödinger equation (S5), and get for the left hand side of (S5):

$$i\partial_t \sum_{ij} B_{ij}(t) \langle 0| a_p a_q a_i^\dagger a_j^\dagger |0\rangle = 2i\partial_t B_{qp} \qquad \text{(S7)}$$

For the right hand side of (S5) we get:

$$\sum_{nij} 2c_{in} B_{nj}(t) \langle 0| a_p a_q a_i^\dagger a_j^\dagger |0\rangle = 2\sum_n \left(c_{qn} B_{np}(t) + c_{pn} B_{nq}(t) \right) \qquad \text{(S8)}$$

Our Schrödinger equation now takes the form:

$$i\partial_t B_{qp} = \sum_n \left(c_{qn} B_{np} + c_{pn} B_{nq} \right) \qquad \text{(S9)}$$

Eq (S9) can also be written in matrix form:

$$i\partial_t B = CB + (CB)^T \qquad \text{(S10)}$$

With $C_{mn} = c_{mn}$.

The solution to equation (S10) can easily be shown to be

$$B\left(t\right)=e^{-iCt}B\left(0\right)e^{-iC^{T}t}. \qquad \text{(S11)}$$

We write $U\left(t\right)=e^{-iCt}$, and recognize that the matrix $U\left(t\right)$ is the propagator of the Schrödinger equation for a single particle.

It is now possible to write the state at time t in the following form:

$$\left|\psi\left(t\right)\right\rangle=\sum_{ij}B_{0,ij}\left(U^{T}a^{\dagger}\right)_{i}\left(U^{T}a^{\dagger}\right)_{j}\left|0\right\rangle \qquad \text{(S12)}$$

We now wish to evaluate the 2-photon correlation map of the state after propagating time *t*. That is, we wish to evaluate:

$$\Gamma_{rs}=\left\langle\psi\left(t\right)\right|a_{r}^{+}a_{s}^{+}a_{s}a_{r}\left|\psi\left(t\right)\right\rangle \qquad \text{(S13)}$$

We start by writing the bra and ket form of Eq (S13) in a more explicit manner:

$$\left|\psi\left(t\right)\right\rangle=\sum_{ijpq}B_{0,ij}\left(U_{ip}^{T}a_{p}^{+}\right)\left(U_{jq}^{T}a_{q}^{+}\right)\left|0\right\rangle \;,\; \left\langle\psi\right|=\left\langle 0\right|\sum_{i'j'mn}B_{0,i'j'}^{*}\left(U_{i'm}^{\dagger}a_{m}\right)\left(U_{j'n}^{\dagger}a_{n}\right) \qquad \text{(S14)}$$

We now write the correlation at time t:

$$\Gamma_{rs}=\left\langle\psi\left(t\right)\right|a_{r}^{+}a_{s}^{+}a_{s}a_{r}\left|\psi\left(t\right)\right\rangle=\sum_{\substack{ijpq\\ i'j'mn}}B_{0,i'j'}^{*}U_{i'm}^{\dagger}U_{j'n}^{\dagger}B_{0,ij}U_{ip}^{T}U_{jq}^{T}\left\langle 0\right|a_{m}a_{n}a_{r}^{+}a_{s}^{+}a_{s}a_{r}a_{p}^{+}a_{q}^{+}\left|0\right\rangle$$

We use the identity (again, easily proven using commutation relations):

$$\left\langle 0\right|a_{m}a_{n}a_{r}^{+}a_{s}^{+}a_{s}a_{r}a_{p}^{+}a_{q}^{+}\left|0\right\rangle=\delta_{rm}\delta_{ns}\delta_{ps}\delta_{rq}+\delta_{rm}\delta_{ns}\delta_{pr}\delta_{sq}+\delta_{rn}\delta_{ms}\delta_{ps}\delta_{rq}+\delta_{rn}\delta_{ms}\delta_{pr}\delta_{sq} \qquad \text{(S15)}$$

By using the fact that B_{0} is a symmetric matrix we get that the correlation is:

$$\Gamma_{rs}=4\left|\left(UB_{0}U^{T}\right)_{rs}\right|^{2} \qquad \text{(S16)}$$