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Opto-mechanical inter-core cross-talk in multi-core fibers: supplementary material

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This document provides supplementary information to "Opto-mechanical inter-core cross-talk in multi-core fibers," https://doi.org/10.1364/optica.4.000289. The following provides a mathematical analysis of the inter-core, cross-phase modulation that is induced by guided acoustic waves Brillouin scattering in a multi-core fiber. First, expressions are derived for phase modulation of probe waves propagating in different cores, due to radial acoustic modes that are driven by pump light at the inner central core. The modulation spectra consist of a series of narrowband resonances. Calculations are carried out for the geometry of a commercially-available, seven-core fiber. The calculated, normalized spectra are in excellent agreement with measurements of the same fiber, reported in the Main Text. The analysis is then extended to pump light which propagates in an outer, off-axis core. Due to the removal of radial symmetry, high-order torsional-radial modes of the cylindrical cross-section, which cannot be addressed in standard single-mode fibers, are stimulated as well. Pump light in an outer core stimulates hundreds of individual modes, and gives rise to broad, quasi-continuous inter-core phase modulation spectra up to 1 GHz frequency. Corresponding measurements of Brillouin scattering due to high-order guided torsional-radial acoustic modes are reported in the Main Text. The strength of opto-mechanical cross-talk is quantified in terms of equivalent nonlinear coefficients, which depend on the choices of acoustic modes, pump and probe cores, the probe state of polarization, and frequency. The nonlinear coefficients may be as high as 1.9 [W×km]$^{-1}$, and their magnitudes are comparable to that of intra-core Kerr nonlinearity in the same fiber. The magnitude of the effect is also supported by experiment.

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1. Inter-core opto-mechanical cross-phase modulation driven by pump waves at the central core

Consider a pump pulse of instantaneous power $P(t)$, where t stands for time, that is propagating in the central core of a multi-core fiber. The Fourier transform of $P(t)$ is denoted by $\tilde{P}(\Omega)$, where Ω is a radio-frequency variable. (Throughout this analysis, the overhanging ~ sign represents frequency-domain variables.) The normalized transverse profile of the optical mode of the central core of the fiber is $E_T^{(1)}(r)$, with r the radial coordinate.

Acoustic modes are driven by an electro-strictive force per unit volume that is induced by the pump pulse. In general, the driving force vector consists of radial and azimuthal components that depend on both r and the azimuthal coordinate φ [1]. However, in our experiment we deliberately scramble the polarization of pump pulses, so that φ-dependent terms average out. The remaining radial force term is given by [1]:

$$\tilde{F}_r(r,\Omega) = -\frac{1}{4nc}(a_1+4a_2)E_T^{(1)}(r)\frac{\partial E_T^{(1)}(r)}{\partial r}\tilde{P}(\Omega). \tag{S1}$$

Here n is the refractive index of silica, c is the speed of light in vacuum, and the parameters $a_{1,2}$ are drawn from elements of the photo-elastic tensor $\mathbf{P}$ of silica [1]: $a_1 = -n^4(P_{11}-P_{12})$ and $a_2 = -n^4 P_{12}$, where P_{11} = 0.121 and P_{12} = 0.27.

The electro-strictive force stimulates radial acoustic modes of the fiber, denoted by R_{0m} where m is an integer. The local Fourier component of the material displacement vector in mode R_{0m} and frequency Ω may be expressed as $\tilde{\vec{U}}^{(m)}(r,\Omega)=\tilde{A}_r^{(m)}(\Omega)u_r^{(m)}(r)\hat{r}$, where $\hat{r}$ is the radial unit vector, $\tilde{A}_r^{(m)}(\Omega)$ denotes the Fourier component of modal displacement magnitude (in units of m²/Hz), and $u_r^{(m)}(r)$ is the normalized transverse profile:

$$u_r^{(m)}=J_1\left[\left(\Omega_m r\right)/v_d\right]\Big/\sqrt{2\pi\int_0^a\left\{J_1\left[\left(\Omega_m r\right)/v_d\right]\right\}^2 r\mathrm{d}r}\,. \quad \text{(S2)}$$

Here a is the cladding radius, and v_d = 5,996 m/s is the velocity of longitudinal sound waves in silica. The modal cut-off frequency is given by $\Omega_m=\left(v_d/a\right)\xi_m$, where ξ_m is the m^{th}-order solution to the equation [2]:

$$\left(1-\frac{v_s^2}{v_d^2}\right)J_0(\xi)=\frac{v_s^2}{v_d^2}J_2(\xi)\,. \quad \text{(S3)}$$

In Eq. (S3), v_s = 3740 m/s is the shear velocity of sound waves in silica. Each mode R_{0m} is also characterized by a linewidth Γ_m, which is determined by the acoustic dissipation in silica, the mechanical impedance matching between silica and the polymer coating or other surrounding medium, and radius inhomogeneity and ellipticity of the cladding [3,4]. Throughout this entire analysis, we assume that the transverse profiles and dispersion relations of the acoustic modes are unaffected by the presence of the fiber cores.

The frequency-domain, modal displacement magnitude may be found by solving the elastic wave equation, subject to the driving force of Eq. (S1) [1]. The solution is of the form:

$$\begin{aligned}\tilde{A}_r^{(m)}(\Omega)&=\frac{1}{4nc\rho_0}\left(a_1+4a_2\right)\times\\&\times\left[2\pi\int_0^a E_T^{(1)}(r)\frac{\mathrm{d}E_T^{(1)}(r)}{\mathrm{d}r}u_r^{(m)}(r)r\mathrm{d}r\right]\times\\&\times\frac{2}{j\Gamma_m\Omega}\left[\frac{\tilde{P}(\Omega)}{1+2j\left(\Omega-\Omega_m\right)/\Gamma_m}\right]\\&\approx\frac{1}{2nc\rho_0\Gamma_m\Omega_m}Q_{ES}^{(m)}\left[\frac{\tilde{P}(\Omega)}{j-2\left(\Omega-\Omega_m\right)/\Gamma_m}\right].\end{aligned} \quad \text{(S4)}$$

Here ρ_0 is the density of silica, and we approximate $\Omega\approx\Omega_m$ in the term outside the squared brackets in the final expression. The electro-strictive transverse overlap integral, which affects the stimulation of R_{0m}, is defined as follows:

$$Q_{ES}^{(m)}\equiv\left(a_1+4a_2\right)\cdot 2\pi\int_0^a E_T^{(1)}(r)\frac{\mathrm{d}E_T^{(1)}(r)}{\mathrm{d}r}u_r^{(m)}(r)r\mathrm{d}r. \quad \text{(S5)}$$

The stimulation of the acoustic wave is resonant around the modal cut-off frequency Ω_m. The Fourier component of the acoustic displacement $\tilde{A}_r^{(m)}(\Omega)$ is related to $\tilde{P}(\Omega)$ through multiplication by a narrowband, electro-strictive transfer function that is of Lorentzian line-shape. In cases of practical interest $\left|\Omega_{m\pm1}-\Omega_m\right|>>\Gamma_m$ for all m. Therefore, only a single radial mode (at most) may be stimulated at each Ω.

The acoustic disturbances induce perturbations of magnitude $\delta\tilde{\boldsymbol{\varepsilon}}^{(m)}(r,\Omega)$ to the local dielectric tensor. Only transverse components of optical fields are considered in this analysis, hence $\delta\tilde{\boldsymbol{\varepsilon}}^{(m)}(r,\Omega)$ is regarded as a 2×2 tensor. Changes are given by the product of the photo-elastic tensor $\mathbf{P}$ and the tensor of local acoustic strain magnitude $\tilde{\mathbf{S}}^{(m)}(r,\Omega)$. For radial acoustic modes, the normal strain components in the radial and azimuthal directions are given by the following two expressions, respectively:

$$\tilde{S}_{rr}^{(m)}(r,\Omega)=\tilde{A}_r^{(m)}(\Omega)\frac{\mathrm{d}u_r^{(m)}(r)}{\mathrm{d}r}, \quad \text{(S6)}$$

$$\text{and } \tilde{S}_{\varphi\varphi}^{(m)}(r,\Omega)=\tilde{A}_r^{(m)}(\Omega)\frac{u_r^{(m)}(r)}{r}, \quad \text{(S7)}$$

and there is no shear strain [1]. The photo-elastic corrections to the radial and azimuthal elements of the local dielectric tensor equal:

$$\delta\tilde{\varepsilon}_{rr}^{(m)}(r,\Omega)=\left(a_1+a_2\right)\tilde{S}_{rr}^{(m)}(r,\Omega)+a_2\tilde{S}_{\varphi\varphi}^{(m)}(r,\Omega), \quad \text{(S8)}$$

$$\delta\tilde{\varepsilon}_{\varphi\varphi}^{(m)}(r,\Omega)=\left(a_1+a_2\right)\tilde{S}_{\varphi\varphi}^{(m)}(r,\Omega)+a_2\tilde{S}_{rr}^{(m)}(r,\Omega). \quad \text{(S9)}$$

The acoustic modification to the effective dielectric constant seen by the probe wave as a whole, $\overline{\delta\tilde{\boldsymbol{\varepsilon}}}^{(m)}(\Omega)$, is obtained by the transverse overlap integral between the local perturbation tensor $\delta\tilde{\boldsymbol{\varepsilon}}^{(m)}(r,\Omega)$ and the profile of the probe optical mode. (The overhanging bar sign denotes transverse spatial averaging, weighted by the modal profile of the probe wave.) The results depend on the choice of core in which the probe wave is propagating, and possibly on its state of polarization as well. When the probe wave is propagating at the inner core of the multi-core fiber, alongside the pump pulse, the tensor $\overline{\delta\tilde{\boldsymbol{\varepsilon}}}^{(m)}(\Omega)$ becomes diagonal for any choice of Cartesian axes, with diagonal values:

$$\begin{aligned}\overline{\delta\tilde{\varepsilon}}^{(m)}(\Omega)&=\tilde{A}_r^{(m)}(\Omega)\left(\frac{a_1}{2}+a_2\right)\times\\&\times 2\pi\int_0^a\left[\frac{\mathrm{d}u_r^{(m)}(r)}{\mathrm{d}r}+\frac{u_r^{(m)}(r)}{r}\right]\left|E_T^{(1)}(r)\right|^2 r\mathrm{d}r\\&=\tilde{A}_r^{(m)}(\Omega)Q_{PE}^{(m)}.\end{aligned} \quad \text{(S10)}$$

Here we defined the photo-elastic overlap integral [1]:

$$\begin{aligned}Q_{PE}^{(m)}&\equiv\left(\frac{a_1}{2}+a_2\right)\times\\&\times 2\pi\int_0^a\left[\frac{\mathrm{d}u_r^{(m)}(r)}{\mathrm{d}r}+\frac{u_r^{(m)}(r)}{r}\right]\left|E_T^{(1)}(r)\right|^2 r\mathrm{d}r.\end{aligned} \quad \text{(S11)}$$

Since $\overline{\delta\tilde{\varepsilon}}^{(m)}(\Omega)<<n^2$, we may well approximate the modification to the effective index of the probe wave according to: $\overline{\delta\tilde{n}}^{(m)}(\Omega)\approx\overline{\delta\tilde{\varepsilon}}^{(m)}(\Omega)\big/(2n)$, leading to:

$$\overline{\delta\tilde{n}}^{(m)}(\Omega)=\frac{1}{4n^2c\rho_0\Gamma_m\Omega_m}Q_{ES}^{(m)}Q_{PE}^{(m)}\times \\ \times\frac{\tilde{P}(\Omega)}{j-2(\Omega-\Omega_m)/\Gamma_m}. \tag{S12}$$

When the probe wave is propagating in an outer core, and the separation between cores is much larger than the mode field diameter (MFD) in the outer core, we may distinguish between two orthogonal, linear states of polarization of the probe wave. Let us denote the normalized transverse profile of the probe wave as $E_T^{(2)}(r,\varphi)$. The radial unit vector $\hat{r}$, throughout the extent of $E_T^{(2)}(r,\varphi)$, is closely aligned with the unit vector $\vec{e}_1$ that connects between the centers of the inner and outer cores. The azimuthal unit $\hat{\varphi}$ vector is closely aligned with the orthogonal unit vector $\vec{e}_2$. Hence, at the basis $\vec{e}_{1,2}$ we may identify: $\tilde{S}_{rr}^{(m)}(r,\Omega)\approx\tilde{S}_{11}^{(m)}(r,\Omega)$ and $\tilde{S}_{\varphi\varphi}^{(m)}(r,\Omega)\approx\tilde{S}_{22}^{(m)}(r,\Omega)$. These considerations suggest two principal axes of birefringence for opto-mechanical index perturbations in the outer core. The two corresponding expressions of the dielectric constant perturbation, $\overline{\delta\tilde{\varepsilon}}_{11}^{(m)}(\Omega)$ and $\overline{\delta\tilde{\varepsilon}}_{22}^{(m)}(\Omega)$, are of the same form of Eq. (S10). However, the photo-elastic overlap integral of Eq. (S11) is replaced by either of the following two expressions:

$$Q_{PE}^{(m,1)}=\int_0^{2\pi}\int_0^a\left[(a_1+a_2)\frac{\mathrm{d}u_r^{(m)}(r)}{\mathrm{d}r}+a_2\frac{u_r^{(m)}(r)}{r}\right]\times \\ \times\left|E_T^{(2)}(r,\varphi)\right|^2 r\mathrm{d}r\mathrm{d}\varphi \tag{S13}$$

for the $\vec{e}_1$ axis, and:

$$Q_{PE}^{(m,2)}=\int_0^{2\pi}\int_0^a\left[a_2\frac{\mathrm{d}u_r^{(m)}(r)}{\mathrm{d}r}+(a_1+a_2)\frac{u_r^{(m)}(r)}{r}\right]\times \\ \times\left|E_T^{(2)}(r,\varphi)\right|^2 r\mathrm{d}r\mathrm{d}\varphi \tag{S14}$$

for the $\vec{e}_2$ axis. Lastly, the Fourier components of cross-phase modulation (XPM) to the probe wave are given by the product of the index perturbation $\overline{\delta\tilde{n}}^{(m)}(\Omega)$, the vacuum wavenumber of the probe wave k_0 and the length of the fiber L:

$$\delta\tilde{\phi}_{OM}^{(m,i)}(\Omega)=\frac{k_0}{4n^2c\rho_0}\frac{Q_{ES}^{(m)}Q_{PE}^{(m,i)}}{\Gamma_m\Omega_m}L\frac{\tilde{P}(\Omega)}{j-2(\Omega-\Omega_m)/\Gamma_m} \\ =\gamma_{OM}^{(m,i)}L\frac{\tilde{P}(\Omega)}{j-2(\Omega-\Omega_m)/\Gamma_m}. \tag{S15}$$

In Eq. (S15) $i=1,2$. This expression also appears as Eq. (1) in the Main Text. Here we have defined the equivalent opto-mechanical nonlinear coefficient, in units of [W×km]$^{-1}$, following [5]:

$$\gamma_{OM}^{(m,i)}\equiv\frac{k_0}{4n^2c\rho_0}\frac{Q_{ES}^{(m)}Q_{PE}^{(m,i)}}{\Gamma_m\Omega_m}. \tag{S16}$$

Note that the opto-mechanical nonlinear coefficient may be of positive or negative sign. As seen in Eq. (S15), $\gamma_{OM}^{(m,i)}$ holds a similar role to that of the Kerr effect nonlinear coefficient γ_{Kerr}, and the two may be compared. The coefficient $\gamma_{OM}^{(m,i)}$ is mode-specific and core-specific, depends on the geometry of the fiber, and may also vary with the state of polarization of the probe wave as discussed above. For $m\geq 2$, $\left|\gamma_{OM}^{(m,2)}\right|>\left|\gamma_{OM}^{(m,1)}\right|$. The instantaneous phase perturbation of the probe wave, $\delta\phi_{OM}^{(i)}(t)$, may be obtained by the inverse Fourier transform of Eq. (S15) and summation over m.

The power spectral density $\left|\delta\tilde{\phi}_{OM}^{(m,i)}(\Omega)\right|^2$ of inter-core XPM due to guided acoustic waves Brillouin scattering in a commercial seven-core fiber was calculated numerically, using the above expressions. The obtained normalized spectra are in excellent agreement with corresponding experimental measurements (see Fig. 3(d) of Main Text). The largest nonlinear coefficient for a probe wave in an outer core was obtained for mode R_{08}, at the resonance frequency Ω_8 = 2π·369.2 MHz. Calculations suggest $\left|\gamma_{OM}^{(8,1)}\right|$ = 0.9 ± 0.1 [W×km]$^{-1}$ and $\left|\gamma_{OM}^{(8,2)}\right|$ = 1.9 ± 0.1 [W×km]$^{-1}$. Uncertainty is due to tolerance in MFD specifications. These values are comparable with γ_{Kerr} = 4 [W×km]$^{-1}$ in the same fiber.

2. Modelling of guided acoustic wave Brillouin scattering within a Sagnac loop

Opto-mechanical, inter-core XPM was experimentally measured in a seven-core fiber that was placed in a Sagnac interferometer loop (see Main Text, [4,6]). The probe wave propagated along the loop in both directions, whereas the pump pulses propagated in the clockwise (CW) direction only. Due to the wave-vector matching characteristics of guided acoustic wave Brillouin scattering, the CW-propagating probe wave was subjected to linear birefringence in the fiber as well as opto-mechanical XPM, whereas the counter-clockwise (CCW) probe wave was affected by birefringence only [4,6].

The experimental procedure is modelled by dividing the fiber to N segments of equal lengths $dz=L/N$. Calculations are formulated at the orthogonal basis $\vec{e}_{1,2}$. Let us denote the input Jones vector of the probe wave as $\vec{A}_{in}$. The Jones matrix describing the propagation of the probe in the CW direction along segment $j=1\ldots N$, in the absence of pump, is noted by $\mathbf{T}_j$. Each such matrix represents linear birefringence with randomly-drawn magnitude and principal axes. The statistics of the local birefringence magnitude are defined by a beat length L_b [7], and the local axes of birefringence evolve according to a coupling length L_c [7]. The polarization transformation of a polarization controller, located at the input end of the loop in the CW direction, is represented by Jones matrix $\mathbf{T}_{PC}$. The output Jones vector of the CW-propagating probe wave, in the absence of pump, is therefore given by:

$$\vec{A}_{CW}^{ref}=\tfrac{1}{2}\prod_{j=1}^{N}\mathbf{T}_j\mathbf{T}_{PC}\vec{A}_{in}. \tag{S17}$$

The factor of $1/2$ represents the two paths through the loop input/output coupler.

We consider a pump that is modulated by a sine wave at the frequency of an acoustic resonance, so that: $\tilde{P}(\Omega)=P_p\cdot\delta(\Omega-\Omega_m)$ with P_p the modulation magnitude in

[W]. The pump wave introduces modulation of the CW probe wave, at frequency Ω_m. In order to determine the magnitude of probe modulation, we calculate the following, modified output Jones vector for the CW direction of propagation:

$$\vec{A}_{CW}^{(m)} = \tfrac{1}{2}\prod_{j=1}^{N}\left(\mathbf{T}_{OM}^{(m)}\mathbf{T}_j\right)\times\mathbf{T}_{PC}\vec{A}_{in}, \tag{S18}$$

where:

$$\mathbf{T}_{OM}^{(m)} \equiv \begin{bmatrix} \exp\left(j\gamma_{OM}^{(m,1)}P_p dz\right) & 0 \\ 0 & \exp\left(j\gamma_{OM}^{(m,2)}P_p dz\right) \end{bmatrix}. \tag{S19}$$

The pump power is chosen so that $\gamma_{OM}^{(m,i)}P_p L << 1,\ \ i = 1,2$. The CCW-propagating probe wave is unaffected by the pump. It passes through the same set of polarization transformations in reverse order, and in the opposite direction:

$$\vec{A}_{CCW}^{ref} = -\tfrac{1}{2}\mathbf{T}_{PC}^{T}\prod_{j=0}^{N-1}\mathbf{T}_{N-j}^{T}\vec{A}_{in}. \tag{S20}$$

Here the superscript T denotes the transpose operation.

The probe power at the loop output is oscillating at Ω_m about a bias value $P^{ref} = 2nc\varepsilon_0 A_{eff}\left|\vec{A}_{CCW}^{ref} + \vec{A}_{CW}^{ref}\right|^2$, with a magnitude that is given by $\delta P^{(m)} = 2nc\varepsilon_0 A_{eff}\left(\left|\vec{A}_{CCW}^{ref} + \vec{A}_{CW}^{(m)}\right|^2 - \left|\vec{A}_{CCW}^{ref} + \vec{A}_{CW}^{ref}\right|^2\right)$. Here A_{eff} is the effective area of the probe optical mode, and ε_0 is the vacuum permittivity. For each realization $\{\mathbf{T}_j\}$, both $\mathbf{T}_{PC}$ and $\vec{A}_{in}$ are varied over hundreds of arbitrary states in attempt to maximize $\left|\delta P^{(m)}\right|$. The strongest oscillation of probe output power is achieved when the states of polarization of $\vec{A}_{CCW}^{ref}$ and $\vec{A}_{CW}^{ref}$ are parallel, and the difference between the phase delays acquired in the CW and CCW directions without pump is $\pi/2$. In these conditions $P^{ref} = \tfrac{1}{2}\cdot 2nc\varepsilon_0 A_{eff}\left|\vec{A}_{in}\right|^2 \equiv \tfrac{1}{2}P_{in}$. The adjustments of the input state of polarization and of the controller inside the loop replicate the experimental procedure. An estimate of the inter-core, opto-mechanical nonlinear coefficient for the particular fiber realization can be obtained according to:

$$\left|\left\langle \gamma_{OM}^{(m)}\right\rangle_z\right| = \frac{\max\left\{\left|\delta P^{(m)}\right|\right\}}{\tfrac{1}{2}P_{in}P_p L}. \tag{S21}$$

Since the CW probe wave accumulates XPM over the entire length L, and the probe wave polarization varies at random as a function of position z, the estimated nonlinear coefficient above represents an averaged value, denoted here as $\langle\ \rangle_z$. Equation (S21) is equivalent to Eq. (5) of the Main Text, which is used in the interpretation of experimental data. An intermediate value of $\left|\left\langle \gamma_{OM}^{(m)}\right\rangle_z\right|$ between $\left|\gamma_{OM}^{(m,1)}\right|$ and $\left|\gamma_{OM}^{(m,2)}\right|$ can be expected.

Calculations were carried out for pump wave modulation at frequency Ω_8, over hundreds of arbitrary fiber realizations. The obtained nonlinear coefficients were $\left|\left\langle \gamma_{OM}^{(8)}\right\rangle_z\right|$ = 1.4 ± 0.2 [W×km]$^{-1}$, in agreement with expectations. The uncertainty represents the standard deviation among the results obtained for many fiber realizations. The results showed little sensitivity to the specific choices of beat length L_b or coupling length L_c, provided that both are shorter than the 30 meters-long fiber. Note that the coiling of a multi-core fiber over tens-of-cm radii induces strain in off-axis cores, giving rise to comparatively large birefringence and an estimated beat length shorter than 1 meter. The corresponding experimentally measured value of the nonlinear coefficient is $\left|\left\langle \gamma_{OM}^{(8)}\right\rangle_z\right|$ = 1.3 ± 0.2 [W×km]$^{-1}$ (see Main Text). Hence the analysis supports not only the spectral shape of inter-core, opto-mechanical XPM by radial acoustic modes, but also its absolute magnitude.

3. Inter-core opto-mechanical cross-phase modulation driven by pump waves at outer cores

Optical pump waves propagating at a central radially-symmetric core may only stimulate two classes of guided acoustic modes: the radial modes R_{0m} discussed above, and torsional-radial modes TR_{2m} that exhibit two-fold azimuthal symmetry [2]. Brillouin scattering due to the latter category averages out when the polarization of the pump wave is scrambled. When both pump and probe waves propagate in outer, off-axis cores, guided acoustic waves Brillouin scattering may take place through more general torsional-radial modes TR_{pm} as well, where $m \geq 1$ and $p \geq 0$ are integers. Stimulated scattering involving these modes takes place even when the state of polarization of the pump wave in an outer core is scrambled. In this section, the previous analysis of guided acoustic wave Brillouin scattering in multi-core fibers is extended to include TR_{pm} modes. For simplicity, we continue to assume hereunder the scrambling of pump polarization, so that the electro-strictive driving force retains its shape of Eq. (S1), with the proper transverse offset:

$$\begin{aligned}\vec{\tilde{F}}(\vec{r},\Omega) &= -\frac{1}{4nc}(a_1 + 4a_2)\times \\ &\times E_T^{(1)}\left(|\vec{r}-\vec{r}_0|\right)\frac{\partial E_T^{(1)}\left(|\vec{r}-\vec{r}_0|\right)}{\partial r}\hat{f}\cdot\tilde{P}(\Omega) \\ &\equiv \frac{1}{4nc}\vec{G}(\vec{r})\tilde{P}(\Omega).\end{aligned} \tag{S22}$$

Here $\vec{r}$ is the transverse position vector, r denotes again the radial coordinate, and $E_T^{(1)}(r)$ stands for the transverse profile of an optical mode, defined with its center at the origin. Also in Eq. (S22), $\vec{r}_0$ is the location of the center of an outer core, $\hat{f}$ is a unit vector in the direction of $\vec{r}-\vec{r}_0$, and:

$$\begin{aligned}\vec{G}(\vec{r}) &\equiv -(a_1 + 4a_2)\times \\ &\times E_T^{(1)}(\vec{r}-\vec{r}_0)\frac{\partial E_T^{(1)}\left(|\vec{r}-\vec{r}_0|\right)}{\partial r}\hat{f}.\end{aligned} \tag{S23}$$

In order to calculate the cutoff frequencies and transverse profiles of modes TR_{pm}, let us define the quantities $\Psi \equiv (\Omega a)/v_s$ and $\Phi \equiv (\Omega a)/v_d$, the operator $\Theta_p(\xi) \equiv \xi J_{p-1}(\xi)/J_p(\xi)$, and the matrix $\mathbf{M}_p(\Omega)$ [8,9]:

$$\mathbf{M}_p(\Omega) \equiv \begin{pmatrix} p^2 - 1 - \tfrac{1}{2}\Psi^2 & 2\left(p^2-1\right)\left[\Theta_p(\Psi) - p\right] - \Psi^2 \\ \Theta_p(\Phi) - p - 1 & 2p^2 - 2\left[\Theta_p(\Psi) - p\right] - \Psi^2 \end{pmatrix}. \tag{S24}$$

The cutoff frequency of mode TR_{pm}, noted as Ω_{pm}, is given by the m^{th}-order solution to the equation [8,9]:

$$\det \mathbf{M}_p(\Omega) = 0 \tag{S25}$$

Two orthogonal solutions exist for the normalized transverse displacement vectors $\vec{u}^{(pm)}(\vec{r})$ of mode TR_{pm}. These are determined by a pair of coefficients: D_{pm}, C_{pm}. The calculation of these coefficients is provided in detail in [10]. The radial and azimuthal components of the first solution for $\vec{u}^{(pm)}(\vec{r})$ are given by [10,11]:

$$\begin{aligned} u_r^{(pm)}(\vec{r}) = B_{pm}\Bigg[& D_{pm}\frac{\Omega_{pm}}{v_d}J_p{}'\left(\frac{\Omega_{pm}}{v_d}r\right)+ \\ & +\frac{p}{r}C_{pm}J_p\left(\frac{\Omega_{pm}}{v_s}r\right)\Bigg]\cos(p\varphi) \end{aligned} \tag{S26}$$

and:

$$\begin{aligned} u_\varphi^{(pm)}(\vec{r}) = B_{pm}\Bigg[& \frac{p}{r}D_{pm}J_p\left(\frac{\Omega_{pm}}{v_d}r\right)+ \\ & +C_{pm}\frac{\Omega_{pm}}{v_s}J_p{}'\left(\frac{\Omega_{pm}}{v_s}r\right)\Bigg]\sin(-p\varphi), \end{aligned} \tag{S27}$$

respectively. In Eq. (S26) and Eq. (S27), B_{pm} is a normalization constant, chosen so that: $\iint \left|\vec{u}^{(pm)}(\vec{r})\right|^2 ds = 1$. Integration is carried out over the cladding cross-section, and ds is an area element. In the second orthogonal solution for $\vec{u}^{(pm)}(\vec{r})$, the azimuthal dependence of $u_r^{(pm)}(\vec{r})$ in Eq. (S26) is replaced with $\sin(p\varphi)$, and that of $u_\varphi^{(pm)}(\vec{r})$ in Eq. (S27) is changed to $\cos(p\varphi)$. The following analysis, from this point onwards, must be repeated twice for each TR_{pm}, once for each azimuthal orientation. The two contributions to the overall process should be added together.

The material displacement vector is given by $\tilde{\vec{U}}^{(pm)}(\vec{r},\Omega) = \tilde{A}_r^{(pm)}(\Omega)\vec{u}^{pm}(\vec{r})$, with the following modal displacement magnitude:

$$\begin{aligned} \tilde{A}^{(pm)}(\Omega) = & \frac{1}{2nc\rho_0\Gamma_{pm}\Omega_{pm}}\iint \vec{G}(\vec{r})\cdot\vec{u}^{(pm)}(\vec{r})\,ds\times \\ & \times\frac{\tilde{P}(\Omega)}{j-2(\Omega-\Omega_{pm})/\Gamma_{pm}}. \end{aligned} \tag{S28}$$

Here Γ_{pm} is the linewidth of mode TR_{pm}. For brevity, we define next the electro-strictive overlap integral:

$$Q_{ES}^{(pm)} \equiv \iint \vec{G}(\vec{r})\cdot\vec{u}^{(pm)}(\vec{r})\,ds, \tag{S29}$$

and the modal frequency response:

$$H_{pm}(\Omega) \equiv \frac{1}{j-2(\Omega-\Omega_{pm})/\Gamma_{pm}}, \tag{S30}$$

so that the material displacement vector may be written as:

$$\tilde{\vec{U}}^{(pm)}(\vec{r},\Omega) = \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Gamma_{pm}\Omega_{pm}}\vec{u}^{(pm)}(\vec{r}). \tag{S31}$$

This expression is a generalization of Eq. (S4). The elements of the symmetric strain tensor associated with mode TR_{pm}, in Cartesian transverse coordinates x, y, are given by:

$$\begin{aligned} \tilde{S}_{kl}^{(pm)}(\vec{r},\Omega) &= \frac{1}{2}\left[\frac{\partial\tilde{U}_k^{(pm)}(\vec{r},\Omega)}{\partial l}+\frac{\partial\tilde{U}_l^{(pm)}(\vec{r},\Omega)}{\partial k}\right] \\ &= \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}\frac{1}{2}\left[\frac{\partial u_k^{(pm)}(\vec{r})}{\partial l}+\frac{\partial u_l^{(pm)}(\vec{r})}{\partial k}\right] \\ &= \frac{Q_{ES}^{(pm)}}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}s_{kl}^{(pm)}(\vec{r})H_{pm}(\Omega)\tilde{P}(\Omega), \end{aligned} \tag{S32}$$

where $k, l = x, y$, and:

$$s_{kl}^{(pm)}(\vec{r}) \equiv \frac{1}{2}\left[\frac{\partial u_k^{(pm)}(\vec{r})}{\partial l}+\frac{\partial u_l^{(pm)}(\vec{r})}{\partial k}\right]. \tag{S33}$$

In the absence of linear birefringence in the propagation of the probe wave [12,13], the photo-elastic perturbations to the elements of the local dielectric tensor due to mode TR_{pm} may be expressed as:

$$\begin{aligned} \begin{bmatrix}\delta\tilde{\varepsilon}_{xx}^{(pm)}(\vec{r},\Omega)\\ \delta\tilde{\varepsilon}_{yy}^{(pm)}(\vec{r},\Omega)\\ \delta\tilde{\varepsilon}_{xy}^{(pm)}(\vec{r},\Omega)\end{bmatrix} &= -n^4\begin{pmatrix}P_{11} & P_{12} & 0\\ P_{12} & P_{11} & 0\\ 0 & 0 & P_{44}\end{pmatrix}\begin{bmatrix}\tilde{S}_{xx}^{(pm)}(\vec{r},\Omega)\\ \tilde{S}_{yy}^{(pm)}(\vec{r},\Omega)\\ 2\tilde{S}_{xy}^{(pm)}(\vec{r},\Omega)\end{bmatrix} \\ &= \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}\cdot -n^4\begin{pmatrix}P_{11} & P_{12} & 0\\ P_{12} & P_{11} & 0\\ 0 & 0 & P_{44}\end{pmatrix}\begin{bmatrix}s_{xx}^{(pm)}(\vec{r})\\ s_{yy}^{(pm)}(\vec{r})\\ 2s_{xy}^{(pm)}(\vec{r})\end{bmatrix} \\ &= \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}\begin{bmatrix}q_{xx}^{(pm)}(\vec{r})\\ q_{yy}^{(pm)}(\vec{r})\\ q_{xy}^{(pm)}(\vec{r})\end{bmatrix}, \end{aligned} \tag{S34}$$

where we have defined the elements of a tensor $\mathbf{q}^{(pm)}(\vec{r})$:

$$\begin{bmatrix}q_{xx}^{(pm)}(\vec{r})\\ q_{yy}^{(pm)}(\vec{r})\\ q_{xy}^{(pm)}(\vec{r})\end{bmatrix} \equiv -n^4\begin{pmatrix}P_{11} & P_{12} & 0\\ P_{12} & P_{11} & 0\\ 0 & 0 & P_{44}\end{pmatrix}\begin{bmatrix}s_{xx}^{(pm)}(\vec{r})\\ s_{yy}^{(pm)}(\vec{r})\\ 2s_{xy}^{(pm)}(\vec{r})\end{bmatrix}. \tag{S35}$$

The photo-elastic tensor elements used in the above relations are P_{11} = 0.121, P_{12} = 0.27, and $P_{44} = \frac{1}{2}(P_{11}-P_{12})$. The photo-elastic perturbation to the dielectric tensor seen by the probe is calculated through the overlap integrals between the local dielectric modification and the transverse profile of the probe wave:

$$\begin{aligned} \overline{\delta\tilde{\varepsilon}}_{kl}^{(pm)}(\Omega) &= \iint \delta\tilde{\varepsilon}_{kl}^{(pm)}(\vec{r},\Omega)\left|E_T^{(2)}(\vec{r})\right|^2 ds \\ &= \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}\iint q_{kl}^{(pm)}(\vec{r})\left|E_T^{(2)}(\vec{r})\right|^2 ds \\ &= \frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}Q_{PE,kl}^{(pm)}. \end{aligned} \tag{S36}$$

The elements of the photo-elastic overlap tensor $\mathbf{Q}_{PE}^{(pm)}$, used in Eq. (S36), are defined by:

$$Q_{PE,kl}^{(pm)} \equiv \iint q_{kl}^{(pm)}(\vec{r})\left|E_T^{(2)}(\vec{r})\right|^2 ds. \tag{S37}$$

This definition provides a generalization of the photo-elastic overlap integrals $Q_{PE}^{(m)}$ of Eqs. (S11), (S14) and (S15) in section 1.

Let us denote the eigen-values of the tensor $\mathbf{Q}_{PE}^{(pm)}$ as $Q_{PE}^{(pm,i)}$, $i=1,2$, and the corresponding eigen-vectors as $\hat{e}_i^{(pm)}$. Scattering by the guided acoustic mode TR_{pm} introduces birefringence to the propagation of the probe wave, with principal axes $\hat{e}_i^{(pm)}$ and respective index perturbations given by:

$$\delta\tilde{n}^{(pm,i)}(\Omega) = \frac{1}{2n}\frac{Q_{ES}^{(pm)}H_{pm}(\Omega)\tilde{P}(\Omega)}{2nc\rho_0\Omega_{pm}\Gamma_{pm}}Q_{PE}^{(pm,i)}. \tag{S38}$$

Note that the eigen-vectors $\hat{e}_i^{(pm)}$ may vary among modes. The XPM of a probe wave that is polarized along $\hat{e}_i^{(pm)}$ due to mode TR_{pm} is given by:

$$\delta\tilde{\phi}_{OM}^{(pm,i)}(\Omega) = k_0 \cdot \delta\tilde{n}^{(pm,i)}(\Omega)\cdot L. \tag{S39}$$

Two equivalent, mode-specific opto-mechanical nonlinear coefficients may be defined in a manner similar to Eq. (S16):

$$\gamma_{OM}^{(pm,i)} \equiv \frac{k_0 Q_{ES}^{(pm)}Q_{PE}^{(pm,i)}}{4n^2c\rho_0\Omega_{pm}\Gamma_{pm}}, \tag{S40}$$

so that:

$$\delta\tilde{\phi}_{OM}^{(pm,i)}(\Omega) = \gamma_{OM}^{(pm,i)}H_{pm}(\Omega)\tilde{P}(\Omega)L. \tag{S41}$$

Unlike the inter-core XPM due to radial acoustic modes discussed in section 1, large overlap is expected between the resonant spectra of scattering by different TR_{pm} modes. Therefore, phase modulation of the probe wave at a given Ω may well be affected by multiple modes. Care must be taken in the summation of XPM due to different modes, since the respective principal axes of opto-mechanical birefringence are generally not the same. To work around this difficulty, we may define a frequency-dependent opto-mechanical tensor, which brings together the contributions of all modes at given Ω:

$$\tilde{\mathbf{H}}_{OM}(\Omega) \equiv \sum_{p,m}\frac{Q_{ES}^{(pm)}}{\Gamma_{pm}\Omega_{pm}}H_{pm}(\Omega)\mathbf{Q}_{PE}^{(pm)}. \tag{S42}$$

We note again that summation should also be carried out over the two azimuthal solutions for the material displacement vector of each TR_{pm} mode. One may then find the frequency-dependent, overall unit eigen-vectors $\hat{\tilde{h}}_{OM}^{(i)}(\Omega)$ and eigen-values $\tilde{H}_{OM}^{(i)}(\Omega)$ of $\tilde{\mathbf{H}}_{OM}(\Omega)$, $i=1,2$. Opto-mechanical XPM to a probe wave polarized along $\hat{\tilde{h}}_{OM}^{(i)}(\Omega)$ can be expressed as:

$$\delta\tilde{\phi}_{OM}^{(i)}(\Omega) = \frac{k_0 L}{4n^2c\rho_0}\tilde{H}_{OM}^{(i)}(\Omega)\tilde{P}(\Omega). \tag{S43}$$

Frequency-dependent (rather than mode-dependent) nonlinear coefficients may be defined:

$$\tilde{\gamma}_{OM}^{(i)}(\Omega) \equiv \frac{k_0}{4n^2c\rho_0}\tilde{H}_{OM}^{(i)}(\Omega). \tag{S44}$$

The frequency-dependence of the opto-mechanical eigen-vectors $\hat{\tilde{h}}_{OM}^{(i)}(\Omega)$ may lead to the depolarization of broadband probe waves. Last, the combined probe modulation due to multiple pump waves propagating in different cores may be calculated through the summation over contributions $\tilde{P}(\Omega)\tilde{\mathbf{H}}_{OM}(\Omega)$, driven from all cores. The addition of terms may be constructive or destructive, depending on the eigen-values and eigen-vectors of the opto-mechanical tensors and on the radio-frequency phase relations among the Fourier components of the multiple pump waves.

The analysis above describes inter-core Brillouin scattering through guided torsional-radial acoustic modes when the fiber is free of linear birefringence. However, photo-elastic perturbations due to these modes become more complex when linear birefringence is present as well [11,12]. The local dielectric tensor is then further modified by an additional rotational term, which is determined by the anti-symmetric strain tensor [12,13]. Let us denote, without loss of generality, the $\hat{x}$ and $\hat{y}$ directions as the principal axes of birefringence at the core in which the probe wave is propagating, and define the rotational part of the displacement gradient as [13]:

$$\tilde{W}_{xy}^{(pm)}(\vec{r},\Omega) \equiv \frac{1}{2}\left[\frac{\partial\tilde{U}_x^{(pm)}(\vec{r},\Omega)}{\partial y} - \frac{\partial\tilde{U}_y^{(pm)}(\vec{r},\Omega)}{\partial x}\right]. \tag{S45}$$

The rotational displacement gradient vanishes for purely radial modes, however it is nonzero for torsional-radial acoustic modes. The birefringence-related term of the local photo-elastic perturbation to the dielectric tensor is given by [13]:

$$\delta\tilde{\boldsymbol{\varepsilon}}_b^{(pm)}(\vec{r},\Omega) = n\cdot\Delta n_{xy}\tilde{W}_{xy}^{(pm)}(\vec{r},\Omega)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{S46}$$

where Δn_{xy} is the difference between the effective refractive indices of probe light polarized along the two principal axes of linear birefringence. The overall photo-elastic perturbation tensor is given by $\delta\tilde{\boldsymbol{\varepsilon}}^{(pm)}(\vec{r},\Omega)+\delta\tilde{\boldsymbol{\varepsilon}}_b^{(pm)}(\vec{r},\Omega)$. Complete calculations of the probe phase modulation due to torsional radial modes therefore mandate knowledge of the fiber linear birefringence. The analysis of this additional term is beyond the scope of the current work. Note that the birefringence-related term $\delta\tilde{\boldsymbol{\varepsilon}}_b^{(pm)}(\vec{r},\Omega)$ is typically much smaller than $\delta\tilde{\boldsymbol{\varepsilon}}^{(pm)}(\vec{r},\Omega)$ [13].

Numerical analysis of stimulated Brillouin scattering by guided TR_{pm} acoustic modes was carried out following the above formalism, for the seven-core commercial fiber studied in this work. The analysis suggests that hundreds of different torsional-radial modes contribute to inter-core phase modulation. Azimuthal orders $0 \le p \le 36$ were considered. The calculated XPM spectra become broad and quasi-continuous up to frequencies of 1 GHz, depending on the choices of cores. These spectra are markedly different from those of XPM driven by radial modes only, which consist of few discrete, sparse resonances with larger spectral separations. Equivalent nonlinear coefficients as high as 1.8 $[\mathrm{W}\times\mathrm{km}]^{-1}$ were calculated for 526 MHz frequency. Guided acoustic waves Brillouin scattering driven by pump light in outer cores is therefore qualitatively different from a corresponding process that is stimulated by light at the inner core.

Inter-core XPM due to pump light at an outer core was measured experimentally, using short pump pulses (see Main Text). A broad, quasi-continuous spectrum, consisted of a large number of modes, was observed in agreement with expectations. Opto-mechanical cross-talk was significant up to frequencies of 750 MHz, limited by the bandwidth of pump pulses. Several resonances observed in the measured XPM spectrum match the frequencies of specific acoustic mode groups. These results corroborate the qualitative predictions of the analysis, and provide a first observation of scattering involving general TR_{pm} modes in all-solid, cylindrical fiber. Multi-core fibers therefore exhibit richer opto-mechanical coupling phenomena that cannot be addressed in standard, single-mode fibers. Continuous spectra, involving a large number of complex acoustic modes, were previously observed in a single-core, photonic crystal fiber [14].

Unlike the earlier discussion of R_{0m} modes, the details of calculated and measured spectra of XPM by TR_{pm} modes do not fully agree. A possible explanation for the observed differences is due to the contribution of linear birefringence according to Eq. (S46), however this effect might not be substantial enough. Another potential cause for discrepancy is depolarization. The probe wave modulation is polarization-dependent, with principal axes that vary with frequency. Hence the visibility of interference between CW and CCW probe waves may become frequency-dependent, and distort the measured spectra. Further, torsional-radial modes of azimuthal orders $p > 36$ might also contribute to inter-core cross-talk. Lastly, the doping profiles of the cores may modify the exact resonance frequencies and transverse profiles of high-order torsional-radial modes. Acoustic guiding in the core is known, for example, in backwards stimulated Brillouin scattering. This effect was not included in the analysis.

The quantitative study of guided acoustic waves Brillouin scattering in multi-core fibers remains the subject of ongoing work. Nevertheless, broadband inter-core cross-talk that is mediated by a large number of high-order, guided acoustic modes was demonstrated in both analysis and experiment.

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