Supplementary Appendix

ESM (Electronic Supplementary Material)

associated with, but not printed with, the journal paper

Recursive modular modelling methodology for lumped-parameter dynamic systems

by

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Recursive modular modelling methodology applied to a 3RRR mechanism

In order to illustrate the application of the modular methodology presented in the paper "Recursive modular modelling methodology for lumped - parameter dynamic systems", highlighting its generality and the advantages of applying it to complex system, the modeling of a 3RRR according to this approach is discussed in detail in this Supplementary Appendix. The reader is also invited to check the modeling of this mechanism presented in [1] in which a computational package for planar mechanisms based on a non-recursive form of the modular methodology is applied.

For the sake of clarity, the figures already presented in the paper to introduce a possible hierarchical description for this system are repeated in this Supplementary Appendix (see Figures 1 – 3).

The 3RRR mechanism is assumed to be planar, mounted in a horizontal plane fixed with respect to an inertial reference frame. A coordinate system can be defined with axes x and y being tangent to the plane and axis z being orthogonal to it. The origin can be set so that it coincides with the center of the platform $\mathscr{L}$ in the reference configuration of the system. This coordinate system is also assumed to remain fixed with respect to an inertial reference frame.

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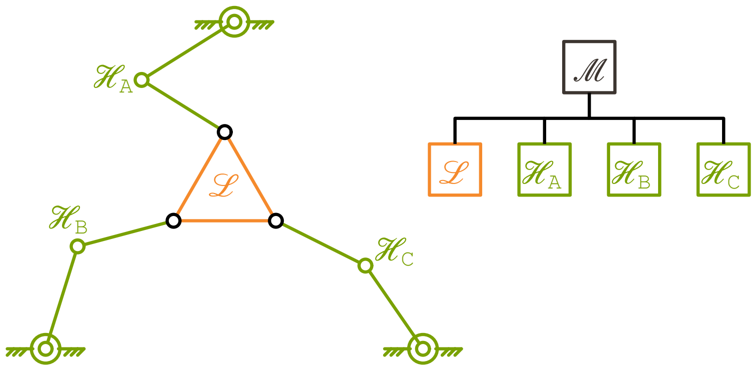

Figure 1: 3$\underline{\mathrm{R}}$RR parallel mechanism (system $\mathscr{M}$) partitioned in 4 modules.

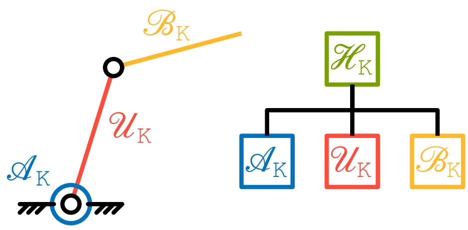

Figure 2: Generic active $\underline{\mathrm{R}}$R kinematic chain ($\mathscr{H}_{\mathrm{K}}$) partitioned in 3 modules.

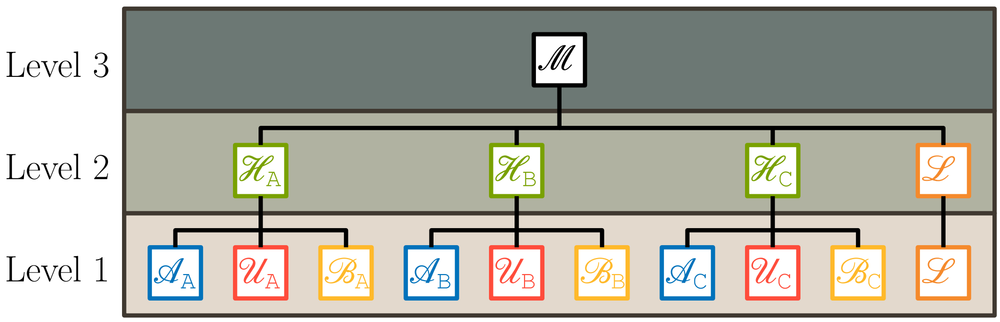

Figure 3: Hierarchical description of the 3$\underline{\mathrm{R}}$RR parallel mechanism.

Adopt the following conventions for the constant parameters of the system:

- $l_{\mathscr{U}}$ and $l_{\mathscr{B}}$ respectively denote the distances between the centres of the revolute joints in the extremities of the bars $\mathscr{U}_{\mathtt{K}}$ and $\mathscr{B}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$; $l_{\mathscr{L}}$ denotes the distance between the centre of the triangular platform $\mathscr{L}$ and the centre of any revolute joint in one of its vertices.

- $\bar{\alpha}_{\mathtt{K}}$ denotes the angle (measured counterclockwise) between the line joining the center of the triangular platform $\mathscr{L}$ to the centre of the revolute joint linked to the chain $\mathscr{H}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ and the x axis when the mechanism is in the reference configuration.

- $\bar{x}_{\mathtt{K}}$ and $\bar{y}_{\mathtt{K}}$ denote the Cartesian coordinates of the fixed centres of the active revolute joints of the kinematic chains $\mathscr{H}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$.

- $m_{\mathscr{B}}$ and $m_{\mathscr{L}}$ respectively denote the masses of the bars $\mathscr{B}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ and of the platform $\mathscr{L}$.

- $I_{\mathscr{B}}$ and $I_{\mathscr{L}}$ respectively denote the moments of inertia with respect to the centres of mass (which are supposed to coincide with the geometric centres) of the bars $\mathscr{B}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ and of the platform $\mathscr{L}$.

- $J_{\mathscr{A}}$ and $J_{\mathscr{U}}$ respectively denote the moments of inertia of the rotors of the actuators $\mathscr{A}_{\mathtt{K}}$ and of the bars $\mathscr{U}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ with respect to the centres of the active revolute joints constituted by these subsystems.

- κ_m and κ_e respectively denote the motor torque constant and the back emf constant, β denotes the viscous damping of the rotors, λ the inductance of the armature windings and ρ the associated electrical resistance of actuators $\mathscr{A}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$.

- η denotes the speed ratio in the reducers of the actuators $\mathscr{A}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$.

Define the following generalized coordinates for the system:

- x, y and θ respectively representing the Cartesian coordinates of the geometric centre (also centre of mass) of $\mathscr{L}$ and the angle of rotation of this platform with respect to the reference configuration (measured counterclockwise).

- $\phi_{\mathtt{K}}$ denoting the angle between the longitudinal direction of the bars $\mathscr{U}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ and the x axis.

- $x_{\mathtt{K}}$, $y_{\mathtt{K}}$, $\psi_{\mathtt{K}}$ respectively representing the Cartesian coordinates of the geometric centres (also centres of mass) of the bars $\mathscr{B}_{\mathtt{K}}$, and the angle between the longitudinal direction of these bars and the x axis.

Take as quasi-velocities for this model the time derivatives of the generalized coordinates along with the following extra variables:

- $\omega_{\mathtt{K}}$ representing the angular velocities of the axes of the actuators $\mathscr{A}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$.
- $i_{\mathtt{K}}$ representing the electrical current in the armature circuits of the actuators $\mathscr{A}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$.

Let the higher order generalized variables be trivially defined. This mechanism is a holonomic system in which generalized variables up to order 2 are enough to describe both dynamic and constraint equations. Ordering the variables according to the order convention of the level 0 of the hierarchy shown in Figure 3, it can be stated that:

$$q^{\langle 0 \rangle} = \big(\phi_{\mathtt{A}}, x_{\mathtt{A}}, y_{\mathtt{A}}, \psi_{\mathtt{A}}, \phi_{\mathtt{B}}, x_{\mathtt{B}}, y_{\mathtt{B}}, \psi_{\mathtt{B}}, \phi_{\mathtt{C}}, x_{\mathtt{C}}, y_{\mathtt{C}}, \psi_{\mathtt{C}}, x, y, \theta\big) \tag{1}$$

$$\begin{aligned} q^{\langle 1 \rangle} = \big(&\omega_{\mathtt{A}}, i_{\mathtt{A}}, \dot{\phi}_{\mathtt{A}}, \dot{x}_{\mathtt{A}}, \dot{y}_{\mathtt{A}}, \dot{\psi}_{\mathtt{A}}, \omega_{\mathtt{B}}, i_{\mathtt{B}}, \dot{\phi}_{\mathtt{B}}, \dot{x}_{\mathtt{B}}, \dot{y}_{\mathtt{B}}, \dot{\psi}_{\mathtt{B}}, \\ &\omega_{\mathtt{C}}, i_{\mathtt{C}}, \dot{\phi}_{\mathtt{C}}, \dot{x}_{\mathtt{C}}, \dot{y}_{\mathtt{C}}, \dot{\psi}_{\mathtt{C}}, \dot{x}, \dot{y}, \dot{\theta}\big) \end{aligned} \tag{2}$$

$$q^{\langle 2 \rangle} = \dot{q}^{\langle 1 \rangle} \tag{3}$$

The equations of motion associated to this system can be written in the following form:

$$M q^{\langle 2 \rangle} = f + \gamma_r \tag{4}$$

with:

$$M = \operatorname{diag}\big(M_{\mathscr{H}}, M_{\mathscr{H}}, M_{\mathscr{H}}, M_{\mathscr{L}}\big) \tag{5}$$

$$M_{\mathscr{H}} = \operatorname{diag}\big(J_{\mathscr{A}}, \lambda, J_{\mathscr{U}}, m_{\mathscr{B}}, m_{\mathscr{B}}, I_{\mathscr{B}}\big) \tag{6}$$

$$M_{\mathscr{L}} = \operatorname{diag}\big(m_{\mathscr{L}}, m_{\mathscr{L}}, I_{\mathscr{L}}\big) \tag{7}$$

$$f = \big(f_{\mathscr{H}_{\mathtt{A}}}, f_{\mathscr{H}_{\mathtt{B}}}, f_{\mathscr{H}_{\mathtt{C}}}, f_{\mathscr{L}}\big) \tag{8}$$

$$f_{\mathscr{H}_{\mathtt{K}}} = \big(-\beta\omega_{\mathtt{K}} - \kappa_m i_{\mathtt{K}}, -\kappa_e \omega_{\mathtt{K}} - \rho i_{\mathtt{K}} + v_{\mathtt{K}}, 0, 0, 0, 0\big) \tag{9}$$

$$f_{\mathscr{L}} = \big(0, 0, 0\big) \tag{10}$$

In these equations, $v_{\mathtt{K}}$ represent the voltage sources of the actuators $\mathscr{A}_{\mathtt{K}}$ $(\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C})$ which should be treated as control inputs. All frictional effects but the ones in the actuators were neglected. Also, once the mechanism is mounted in a horizontal plane, there are no terms in the equations associated to gravitational forces.

The constraint equations associated specifically to the Level 1 of the hierarchy can be expressed as follows (for $\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C}$):

$$\dot{\phi}_{\mathtt{K}} - \frac{\omega_{\mathtt{K}}}{\eta} = 0 \tag{11}$$

$$x_{\mathtt{K}} - \bar{x}_{\mathtt{K}} - l_{\mathscr{U}} \cos\phi_{\mathtt{K}} - \frac{1}{2} l_{\mathscr{B}} \cos\psi_{\mathtt{K}} = 0 \tag{12}$$

$$y_{\mathtt{K}} - \bar{y}_{\mathtt{K}} - l_{\mathscr{U}} \sin\phi_{\mathtt{K}} - \frac{1}{2} l_{\mathscr{B}} \sin\psi_{\mathtt{K}} = 0 \tag{13}$$

These constraint equations can also be expressed in the following form:

$$\tilde{A}_1 q^{\langle 2\rangle} = \tilde{b}_1 \tag{14}$$

with:

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{1,\mathscr{H}_\mathtt{A}} & 0 & 0 & 0 \\ 0 & \tilde{A}_{1,\mathscr{H}_\mathtt{B}} & 0 & 0 \\ 0 & 0 & \tilde{A}_{1,\mathscr{H}_\mathtt{C}} & 0 \end{bmatrix} \qquad \tilde{b}_1 = \begin{bmatrix} \tilde{b}_{1,\mathscr{H}_\mathtt{A}} \\ \tilde{b}_{1,\mathscr{H}_\mathtt{B}} \\ \tilde{b}_{1,\mathscr{H}_\mathtt{C}} \end{bmatrix} \tag{15}$$

$$\tilde{A}_{1,\mathscr{H}_\mathtt{K}} = \begin{bmatrix} -\frac{1}{\eta} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & l_\mathscr{U} \sin\phi_\mathtt{K} & 1 & 0 & \frac{1}{2} l_\mathscr{B} \sin\psi_\mathtt{K} \\ 0 & 0 & -l_\mathscr{U} \cos\phi_\mathtt{K} & 0 & 1 & -\frac{1}{2} l_\mathscr{B} \cos\psi_\mathtt{K} \end{bmatrix} \tag{16}$$

$$\tilde{b}_{1,\mathscr{H}_\mathtt{K}} = \begin{bmatrix} 0 \\ -l_\mathscr{U} \dot{\phi}_\mathtt{K}^2 \cos\phi_\mathtt{K} - \frac{1}{2} l_\mathscr{B} \dot{\psi}_\mathtt{K}^2 \cos\psi_\mathtt{K} \\ -l_\mathscr{U} \dot{\phi}_\mathtt{K}^2 \sin\phi_\mathtt{K} - \frac{1}{2} l_\mathscr{B} \dot{\psi}_\mathtt{K}^2 \sin\psi_\mathtt{K} \end{bmatrix} \tag{17}$$

The constraint equations associated specifically to the Level 2 can be described by the following expressions (for $\mathtt{K} = \mathtt{A}, \mathtt{B}, \mathtt{C}$):

$$x + l_\mathscr{L} \cos(\theta + \bar{\alpha}_\mathtt{K}) - x_\mathtt{K} - \frac{1}{2} l_\mathscr{B} \cos\psi_\mathtt{K} = 0 \tag{18}$$

$$y + l_\mathscr{L} \sin(\theta + \bar{\alpha}_\mathtt{K}) - y_\mathtt{K} - \frac{1}{2} l_\mathscr{B} \sin\psi_\mathtt{K} = 0 \tag{19}$$

These constraint equations can also be expressed in the following form:

$$\tilde{A}_2 q^{\langle 2\rangle} = \tilde{b}_2 \tag{20}$$

with:

$$\tilde{A}_2 = \begin{bmatrix} \tilde{A}_{2,\mathscr{H}_\mathtt{A}} & 0 & 0 & \tilde{A}_{2,\mathtt{A}} \\ 0 & \tilde{A}_{2,\mathscr{H}_\mathtt{B}} & 0 & \tilde{A}_{2,\mathtt{B}} \\ 0 & 0 & \tilde{A}_{2,\mathscr{H}_\mathtt{C}} & \tilde{A}_{2,\mathtt{C}} \end{bmatrix} \qquad \tilde{b}_2 = \begin{bmatrix} \tilde{b}_{2,\mathtt{A}} \\ \tilde{b}_{2,\mathtt{B}} \\ \tilde{b}_{2,\mathtt{C}} \end{bmatrix} \tag{21}$$

$$\tilde{A}_{2,\mathscr{H}_\mathtt{K}} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & \frac{1}{2} l_\mathscr{B} \sin\psi_\mathtt{K} \\ 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} l_\mathscr{B} \cos\psi_\mathtt{K} \end{bmatrix} \tag{22}$$

$$\tilde{A}_{2,\mathtt{K}} = \begin{bmatrix} 1 & 0 & -l_\mathscr{L} \sin(\theta + \bar{\alpha}_\mathtt{K}) \\ 0 & 1 & l_\mathscr{L} \cos(\theta + \bar{\alpha}_\mathtt{K}) \end{bmatrix} \tag{23}$$

$$\tilde{b}_{2,\mathtt{K}} = \begin{bmatrix} 0 \\ l_\mathscr{L} \dot{\theta}^2 \cos(\theta + \bar{\alpha}_\mathtt{K}) - \frac{1}{2} l_\mathscr{B} \dot{\psi}_\mathtt{K}^2 \cos\psi_\mathtt{K} \\ l_\mathscr{L} \dot{\theta}^2 \sin(\theta + \bar{\alpha}_\mathtt{K}) - \frac{1}{2} l_\mathscr{B} \dot{\psi}_\mathtt{K}^2 \sin\psi_\mathtt{K} \end{bmatrix} \tag{24}$$

Choosing one strategy among the four presented in Proposition 3.1, one can compute a matrix S_1 describing an operator onto the kernel of $\tilde{A}_1$ and then compute a matrix C_2 describing an operator onto the kernel of $B_2 = \tilde{A}_2 S_1$. According to the statement of Theorem 3.1, $S_2 = S_1 C_2$. Therefore, the dynamic equations of motion for the 3$\underline{\text{R}}$RR mechanism are the following, with $\dot{q}^{\langle 1 \rangle} = \alpha_2$:

$$\begin{bmatrix} S_2^* M \\ \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} \alpha_2 = \begin{bmatrix} S_2^* f \\ \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix} \tag{25}$$

Alternatively, one could opt to use the algorithm based on Udwadia-Kalaba equation presented in Section 4. In this case, let $\tilde{H}_1 = \tilde{A}_1 M^{-1/2}$ and $\tilde{H}_2 = \tilde{A}_2 M^{-1/2}$. Assume that $(\cdot)^g$ denotes $\{1, 4\}$-inverses. Let $P_0 = I$ and compute $K_1 = (\tilde{H}_1 P_0)^g = \tilde{H}_1^g$, $P_1 = I - \tilde{H}_1^g \tilde{H}_1$ and $K_2 = (\tilde{H}_2 P_1)^g$. The dynamic equations of motion for the 3$\underline{\text{R}}$RR mechanism can alternatively be expressed in the following explicit form, with $\dot{q}^{\langle 1 \rangle} = M^{-1/2} a_2$:

$$a_0 = M^{-1/2} f \tag{26}$$

$$a_1 = a_0 + K_1(\tilde{b}_1 - \tilde{H}_1 a_0) \tag{27}$$

$$a_2 = a_1 + K_2(\tilde{b}_2 - \tilde{H}_2 a_1) \tag{28}$$

References

[1] Orsino RMM. 2016 *A contribution on modelling methodologies for multibody systems*. PhD thesis. São Paulo, Brazil: University of São Paulo. See http://www.teses.usp.br/teses/disponiveis/3/3151/tde-22062016-160724/en.php.