On the Solvability of the Pressure Driven Hydraulic Steady-State Equations Considering Feedback-Control Devices

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ABSTRACT

Water supply networks (WSNs) represent an important part of urban technical infrastructure. Recently, the resilience of water distribution networks facing different physical and cyber threats has gained increasing attention. In order to improve the operation of complex water distribution systems under both, extreme and normal hydraulic conditions, mathematical simulation models are indispensable tools for engineers, network planners, network operators and decision makers. Especially, in the case of extremely disruptive events that might be caused by natural hazards or deliberate malevolent attacks by humans, the proper operation of the system for maintaining the supply of drinking water to at least parts of the population is a very challenging task. Resilient behaviour can be reached only by adaptive system operation including isolation of parts of the network and control of pressure and flows in the system. For that purpose, different kinds of control devices are used that may be remotely controlled or which are operated in the field. Existing hydraulic simulation software often fails to calculate reliable results for systems under control and pressure insufficiency.
A mathematical framework for the simulation of the steady-state flow in reticulation water supply networks with special consideration of feedback control devices and pressure dependent demands is proposed in this paper. First, the importance of the steady-state calculation in the face of disruptive events is stressed and a brief review of synthetic and analytical methods for the hydraulic steady-state calculation is given. In this paper, the well-known content model is extended by the content of pressure dependent outflows. The nonlinear consumption functions in combination with linear inequality constraints (box constraints) replace the constant demands of demand driven analysis. It is also shown that the range of solvable problems in combination with flow control devices is enlarged by the relaxation of fixed demands.

Keywords: Pressure driven modelling, control valve, convex optimization

1 BACKGROUND

1.1 Introduction

The influence of insufficient pressure conditions ([1], [2], [3]) and the impact of control devices (has been widely studied in the context of mathematical modeling of Water Distribution Networks (WDN). However, there is still a lack of robust simulation algorithms that are capable of dealing with extreme situations where a number of pipes are in failure mode and, as a result, the system might be decomposed into several disconnected parts. The stable and robust calculation of WDN hydraulics and water quality under anomalous operational conditions as they appear under extreme events like natural disasters, electrical power blackouts or other technical failures is a basic requirement for all model-based decisions and, eventually, for improving the resilience of drinking water supply. Existing simulation techniques are not capable of dealing with these situations and often fail to calculate a feasible solution. There is a strong need for improved mathematical methods that successfully deal with ill-posed systems and other situations where existing modelling techniques reach the limits of their theoretical basis.

One important step towards robust and realistic modelling of extreme situations is the development of a robust hydraulic system solver that can deal, amongst other things, with system-wide insufficient pressure conditions. Consider a scenario with numerous failures of system devices like pumps, control valves or pipe breaks and the network is decomposed into different parts that might be connected to the sources only by pipes with insufficient diameter or not connected at all. In this case, the state-of-the-art demand driven models fail to converge or to calculate reliable results. The way to analyze such situations with existing tools is to remove the closed (or broken) pipes from the system, check connectivity and analyze the different resulting components separately with a remaining risk that flow control devices cause further decomposition. This approach is not practicable especially for online calculations since the system matrix has to be changed due to multiple incidence matrix manipulations. In contrast to pipes whose status is closed and known a priori, the state of flow control devices is unknown and dependent on the hydraulic conditions of the entire system. From a mathematical point of view, the closed pipes can be modelled by equality constraints of the flow, whereas flow control devices are modelled by inequality constraints. In this paper, after a brief background, a method is presented that extends the well-known Content Model for hydraulic steady-state calculation by introducing the content of pressure dependent demands. It will be shown that by replacing the fixed demands by a more general Pressure Outflow Relation (POR) the range of solvable systems is greatly enlarged. The method is demonstrated with a small example system.

1.2 Content Model for Demand Driven Modelling (DDM)

It has been shown by different authors that in DDM analysis the calculation of the steady-state of pressurized pipe networks is equivalent to the minimization of the so-called system content [4].

$$\begin{gathered} \min_{\mathbf{q}\in R^{np}} C(\mathbf{q}) = \sum_{j=1}^{np} W_j + \sum_{i=1}^{nr} V_i \\ s.t.\ -\mathbf{A}_1^T\mathbf{q} - \mathbf{d} = \mathbf{0}_{nj} \end{gathered} \tag{1}$$

The continuity constraint includes the $n_p \times n_j$ arc-node incidence matrix $\mathbf{A}_1$ ($\mathbf{A}_{1,j,i} = 1$, if arc j leaves node i, $\mathbf{A}_{1,j,i} = -1$, if arc j enters node i and 0 otherwise), the flow vector $\mathbf{q}$ (size np: number of pipes) and demand vector $\mathbf{d}$ (size nj: number of nodes without fixed head). The content of pipe j is given by:

$$W_j = \int_0^{q_j} G_{jj}\, x_j dx_j = \int_0^{q_j} \left(r_j|x_j|^{\alpha-1} + K_j|x_j|\right) x_j dx_j. \tag{2}$$

Here, q_j is the flow for pipe j; $r_j = r_j(x_j)$ is the pipe resistance which depends on flow for the Darcy-Weisbach head loss; and α the exponent of the hydraulic equation (usually for the Darcy-Weisbach or Hazen-Williams formulations where $\alpha \geq 1$). The second term on the right-hand side in Eq. (2) refers to the local minor loss of valves and fittings. The second sum in Eq. (1) is over all fixed head nodes (number nr). The content in this case is defined by:

$$V_i = \int_0^{Q_i} h_{0,i}\, dx_i = -h_{0,i}(\mathbf{A}_0^T\mathbf{q})_i. \tag{3}$$

with the external in- or outflow at the fixed head node $(\mathbf{A}_0^T\mathbf{q})_i$ and $h_{0,i}$ is the known head at node i. The system content is strictly convex (which is guaranteed by the strict monotonicity of the head loss equation) and a norm-coercive function of q ($|W_j(q_j)| \longrightarrow +\infty$ if $|q_j| \rightarrow +\infty$). This guarantees the existence and uniqueness of the solution if the mass constraint set is non-empty.

1.3 Content Model for DDM Analysis Including Flow Control Devices

In a previous publication [5] it was shown that flow control devices such as check valves and flow control valves can be modelled by a multivalued subdifferential mapping. As a consequence, the convex and differentiable problem of minimizing the system content is replaced by either the minimization of an unconstrained, convex and subdifferentiable content function or by the minimization of a convex and differentiable Content function over a polyhedral set that is described by the continuity equation in combination with additional inequality constraints describing the restrictions imposed by flow control devices:

$$\min_{\mathbf{q} \in R^{np}} C(\mathbf{q}) = \sum_{j=1}^{np} W_j(q_j) - \mathbf{q}^T \mathbf{A}_0 \mathbf{h}_0 \qquad (4)$$

$$s.t. \; -\mathbf{A}_1^T \mathbf{q} - \mathbf{c} = \mathbf{0}_{nj} \; ; \; \mathbf{q} - \mathbf{q}_{\max} \le \mathbf{0}; -\mathbf{q} + \mathbf{q}_{\min} \le \mathbf{0}$$

The constrained minimization problem has a unique solution if and only if the polyhedral set defined by the constraints is nonempty and the Karush-Kuhn-Tucker conditions (KKT) hold. Pressure controlling devices are not included, here. It would require the formulation of additional local minimization problems that result in a game theoretical model. For solution a Nash-Equilibrium can be calculated, then.

2 EXTENDED CONTENT MODEL

2.1 Content and Multivalued Pressure Outflow Relation (POR)

In this section, the content minimization problem of Eq. (4) is extended to include pressure outflow relationships (PORs). For an analytical investigation, the concept of sub-gradients is used. The generalized mathematical framework of Subdifferential Calculus, better known as Convex Analysis, provides strong concepts including general necessary and sufficient conditions. This is especially important for the derivation of statements about existence and uniqueness of steady state solutions with interacting PDM nodes and control devices. A common POR is the so-called Wagner function [6] where the outflow c is a function of the actual pressure h: $c(h)$. Other POR forms can be found for example in [6]. The following multivalued (sub-differential) mapping describes the pressure dependent behaviour by the inverse mapping $h(c)$:

$$\mathrm{h}_i(\mathrm{c}_i) := \begin{cases} \emptyset & , \mathrm{c}_i < 0 \\ (-\infty, \mathrm{h}_{i,min}] & , \mathrm{c}_i = 0 \\ \mathrm{h}_{i,min} + \mathrm{k}_i \mathrm{c}_i |\mathrm{c}_i| & , 0 < \mathrm{c}_i < \mathrm{d}_i \\ [\mathrm{h}_{i,S}, +\infty) & , \mathrm{c}_i = \mathrm{d}_i \\ \emptyset & , \mathrm{c}_i > \mathrm{d}_i \end{cases} \qquad (5)$$

with $\mathrm{k}_i = \frac{\mathrm{h}_{i,S} - \mathrm{h}_{i,min}}{\mathrm{d}_i^2}$ $, if$ $\mathrm{d}_i \neq 0$; $\mathrm{h}_{i,min}$ the elevation plus the minimum pressure head at node *i*; and $\mathrm{h}_{i,S}$ is the service head at node i (i.e.: the elevation plus the service pressure head above which the full demand is satisfied). For this sub-differential mapping, a convex and lower semi-continuous content function exists:

$$\bar{W}_i(\mathrm{c}_i) = \begin{cases} \infty & ,\mathrm{c}_i < 0 \\ \mathrm{h}_{i,min}\mathrm{c}_i + \frac{1}{3}\mathrm{k}_i|\mathrm{c}_i|^3 & ,0 \le \mathrm{c}_i \le \mathrm{d}_i \\ \infty & ,\mathrm{c}_i > \mathrm{d}_i \end{cases} \tag{6}$$

Figure 1 shows the graph of the multivalued mapping for $\mathrm{h(c)}$ and the corresponding lower semi-continuous convex Content function. Within the interval $]0, d_i[$. $\bar{W}_i$ is strictly convex.

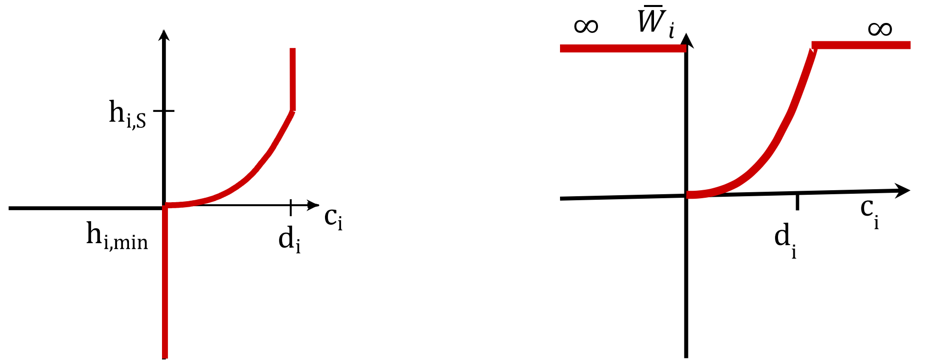

Figure 1. Multivalued mapping for the inverse POR function $\boldsymbol{h(c)}$ *and the corresponding convex, lower semi-continuous Content-Function* $\bar{W}_i$

The different variables and their physical meanings may be visualized in Figure 2. The curves a, b and c refer to the hydraulic grade lines (HGLs) corresponding to three different cases:

a) full supply with $\mathrm{c}_i = \mathrm{d}_i$ and $\mathrm{h}_i \ge \mathrm{h}_{i,S}$;

b) reduced supply with $0 < \mathrm{c}_i < \mathrm{d}_i$ and $\mathrm{h}_i = \hat{\mathrm{h}}(\mathrm{c})$;

c) no supply with $\mathrm{c}_i = 0$ and $\mathrm{h}_i \le \mathrm{h}_{i,min}$.

The multivalued POR shown in Figure 2 can be realized in Epanet by a series of control devices enabling the solution of PDM problems by existing DDM solvers ([1], [2], [3]). To achieve that, one FCV (flow control valve), one TCV (throttle valve), one check valve and one reservoir are connected to the original demand node in the Epanet model as shown in Figure 3.

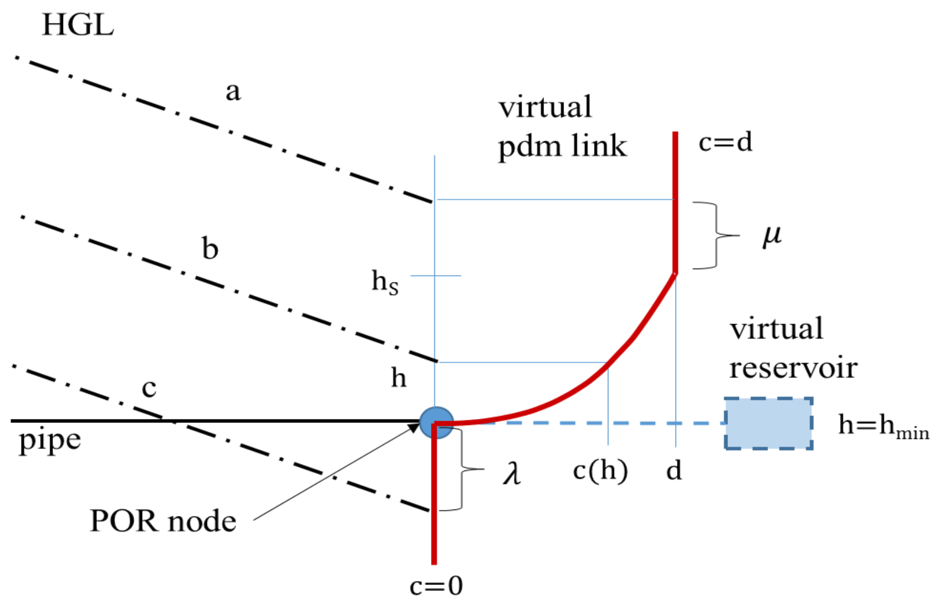

Figure 2. Multi-valued mapping for POR and Lagrangian multipliers for three HGLs a – c.

The FCV limits the maximum outflow to the required demand (upper bound $c_i \leq d_i$), the TCV includes the nonlinear flow-head relationship (inverse POR function: $h_{p,i}(c_i) = k_i c_i|c_i|$) and the check valve prevents backflow from the reservoir into the system (lower bound $c_i \geq 0$). The elevation of the reservoir is set to the elevation (minimum pressure head) h_{min} of the demand node. The setting of the FCV is the demand d and the headloss coefficient value of the TCV is chosen such that the headloss for flow $c = d$ equals the minimum service pressure h_S.

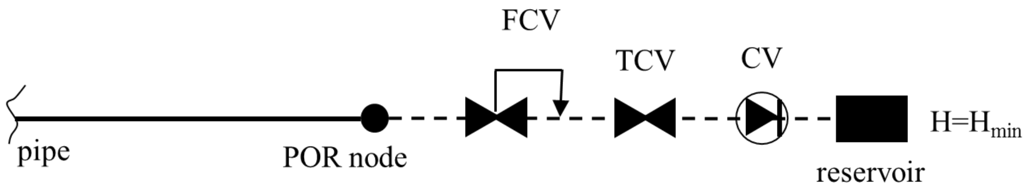

Figure 3: Modifications in EPANET needed for PDM modelling.

This approach has also been recently proposed by other researchers. However, the applicability of the approach is limited due to following reason. First, the settings of the TCVs have to be updated for every time step of an extended period simulation according to changes in d. Second, the size of the model is dramatically increased by the modifications. For every demand node, an addition of three nodes and three links is required. Third, the solvability of the system relies on the heuristics implemented in Epanet for handling flow control devices.

2.2 System Content with Pressure Dependent Demands

The steady-state is given by the minimum of the system content that is extended by the content of the non-zero demand nodes. In the continuity constraints the fixed demand **d** is replace by **c**.

$$\min_{[\mathbf{q},\mathbf{c}]\in\boldsymbol{R}^{np+nj}} C(\mathbf{q},\mathbf{c}) = \sum_{j=1}^{np} W_j(q_j) + \sum_{i=1}^{nj} \overline{W}_i(c_i) - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 \qquad (7)$$
$$s.t.\quad -\mathbf{A}_1^T\mathbf{q} - \mathbf{c} = \mathbf{0}_{nj}$$

The function C is strictly convex and norm-coercive in q and c. As explained above, the content of the PDM demand nodes $\overline{W}$ as shown in Eq. (6) is an unconstrained, convex and sub-differentiable function. Therefore, there always exists a minimum. However, in practice we are only interested in solutions where $C(\mathbf{q},\mathbf{c}) < \infty$. For numerical treatment, it is convenient to replace the unconstrained minimization problem (Eq. (7)) by a constrained one with differentiable function $\widehat{W}$ that is defined only for $0 \leq c_i \leq d_i$ and is identical with $\overline{W}$ within the compact set defined by the box constraints. The unconstrained minimization problem with subdifferentiable objective function $\overline{W}$ is replaced by the following constrained problem with differentiable function $\widehat{W}$:

$$\min_{[\mathbf{q},\mathbf{c}]\in\mathbf{R}^{np+nj}} C(\mathbf{q},\mathbf{c}) = \sum_{j=1}^{np} W_j + \sum_{i=1}^{nj} \widehat{W}_i(c_i) - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 \qquad (8)$$
$$s.t.\quad -\mathbf{A}_1^T\mathbf{q} - \mathbf{c} = \mathbf{0}_{nj};\ -\mathbf{c} \leq \mathbf{0}_{nj};\ \mathbf{c} \leq \mathbf{d}$$

An equivalent formulation was proposed in [8].

2.3 System Content with POR and Flow Control

In a previous publication [5] it was shown that flow control devices can be treated with a very similar approach using PORs as described in the previous section. The hydraulic behaviour can be described by a multivalued monotone (subdifferential) mapping q->h and a convex lower semi-continuous content function. The combined constrained minimization of the continuous content function is then:

$$\begin{aligned} &\min_{[\mathbf{q},\mathbf{c}] \in \mathbf{R}^{np+nj}} C(\mathbf{q},\mathbf{c}) = \mathbf{q}^T\overline{\mathbf{\Delta h}} - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 + \tfrac{1}{3}\mathbf{c}^T\mathbf{N}\mathbf{c} + \mathbf{c}^T\mathbf{h}_{\min} \\ &s.t. \quad -\mathbf{A}_1^T\mathbf{q} - \mathbf{c} = \mathbf{0}_{nj};\ -\mathbf{c} \le \mathbf{0}_{nj};\ \mathbf{c} \le \mathbf{d};\ \ \mathbf{q} - \mathbf{q}_{\max} \le \mathbf{0};\ \ \mathbf{q} + \mathbf{q}_{\min} \le \mathbf{0} \end{aligned} \tag{9}$$

where $\mathbf{N}$ is the $(nj \times nj)$ diagonal matrix with diagonal elements $N_{ii} = k_i|c_i|$. The Lagrangian of the minimization problem Eq. (9) is defined as:

$$\begin{aligned} L(\mathbf{q},\mathbf{c},\mathbf{h},\boldsymbol{\lambda},\boldsymbol{\mu}) = \mathbf{q}^T\overline{\mathbf{\Delta h}} - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 + \tfrac{1}{3}\mathbf{c}^T\mathbf{N}\mathbf{c} + \mathbf{c}^T\mathbf{h}_{\min} - \mathbf{h}^T(\mathbf{A}_1^T\mathbf{q} + \mathbf{c}) + \boldsymbol{\mu}^T(\mathbf{c} - \mathbf{d}) - \boldsymbol{\lambda}^T\mathbf{c} + \\ \boldsymbol{\kappa}^T(-\mathbf{q} + \mathbf{q}_{\min}) + \boldsymbol{\nu}^T(\mathbf{q} - \mathbf{q}_{\max}) \\ \boldsymbol{\lambda} \ge \mathbf{0}_{nj},\ \boldsymbol{\mu} \ge \mathbf{0}_{nj},\ \boldsymbol{\kappa} \ge \mathbf{0}_{np},\ \boldsymbol{\nu} \ge \mathbf{0}_{np} \end{aligned} \tag{10}$$

with the Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ which correspond to the constraints of the POR and $\boldsymbol{\kappa}$ and $\boldsymbol{\nu}$ which refer to the link flow constraints. Necessary conditions for a solution of Eq. (9) are provided by the KKT conditions:

$$\begin{aligned} \mathbf{G}(\mathbf{q})\mathbf{q} - \mathbf{A}_1\mathbf{h} - \mathbf{A}_0\mathbf{h}_0 - \mathbf{V}_{\mathrm{L}}\boldsymbol{\kappa}^* + \mathbf{V}_{\mathrm{U}}\boldsymbol{\nu}^* = \mathbf{0}_{np} \\ \mathbf{N}(\mathbf{c})\mathbf{c} + \mathbf{h}_{\min} - \mathbf{U}_{\mathrm{L}}{}^T\boldsymbol{\lambda}^* + \mathbf{U}_{\mathrm{U}}{}^T\boldsymbol{\mu}^* = \mathbf{h} \\ -\boldsymbol{A}_1^T\mathbf{q} - \mathbf{c} = \mathbf{0} \\ (\boldsymbol{\lambda}^*)^T\mathbf{c} = \mathbf{0};\ (\boldsymbol{\mu}^*)^T(\mathbf{c}^* - \mathbf{d}) = \mathbf{0};\ (\boldsymbol{\kappa}^*)^T(\mathbf{q}^* - \mathbf{q}_{\min}) = \mathbf{0};\ (\boldsymbol{\nu}^*)^T(\mathbf{q}^* - \mathbf{q}_{\max}) = \mathbf{0} \end{aligned} \tag{11}$$

Here, $\mathbf{G}(\mathbf{q})$ is the $(np \times np)$ diagonal matrix whose diagonal elements are such that $\mathbf{G}(\mathbf{q})\mathbf{q} = \mathbf{h}$ is the vector of headlosses. $\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\kappa}^*$ and $\boldsymbol{\nu}^*$ are the Lagrangian Multipliers of the active PORs and flow bounds at the solution point. $\mathbf{U}_{\mathrm{L}}, \mathbf{U}_{\mathrm{U}}, \mathbf{V}_{\mathrm{L}}, \mathbf{V}_{\mathrm{U}}$ are matrices representing the corresponding index sets for which the value is 1 for active bounds and 0 otherwise. The last four equations represent the complementary slackness condition.

2.4 Existence and Uniqueness

To prove the existence and uniqueness of a hydraulic steady state, the content formulation is advantageous. If all the content functions are strictly convex and norm-coercive ($|C_j(x_j)| \longrightarrow +\infty$ if $|x_j| \rightarrow +\infty$) then the total system content is a strictly convex and norm-coercive function of $(\mathbf{q},\mathbf{c})$. The strict convexity of the content can be proven by using the monotonicity of the particular mappings $q_i \mapsto h_i$ of network elements. From the norm-coercivity it follows that the system content has a minimum. Therefore, in order to prove the existence of a solution it is sufficient to show that the polyhedron that is described by the constraints is nonempty.

$$\boldsymbol{P} = \{\mathbf{x} \in \mathbf{R}^{n_p+n_j} \mid [\mathbf{A}_1^T\ \ \mathbf{I}]\mathbf{x} = \mathbf{0};\ \ \mathbf{x} \le \mathbf{x}_{min};\mathbf{x} \ge \mathbf{x}_{max}\} \neq \emptyset \tag{12}$$

with $\mathbf{x}^T = [\boldsymbol{q}^T\ \boldsymbol{c}^T]$, $\mathbf{x}_{min}^T = [\mathbf{q}_{min}^T\ \mathbf{0}]$, $\mathbf{x}_{max}^T = [\mathbf{q}_{max}^T\ \mathbf{d}]$. If $\mathbf{q}_{min} \le \mathbf{0}$ the pipe flow rates and nodal outflows $\mathbf{q} = \mathbf{0}_{np}$ and $\mathbf{c} = \mathbf{0}_{nj}$ are trivially feasible solutions for the set of constraints (Eq. (12)). It is proven that there always exists a solution to the problem. The more theoretical case where $\boldsymbol{q}_{i,min} > \mathbf{0}$ for some $\boldsymbol{i}$ refers to flow constraints that would require pumping is not considered here. Together with the strict convexity of the system content, the existence of a unique solution is proved.

2.5 Problems with Non-Uniqueness in Practical Applications

As was shown above the content minimization problem always has a unique solution in terms of network flows. However, in practice problems with singular system matrices and resulting non-uniqueness still occur. How is this consistent with the statement of uniqueness? The answer is that in practice it is often difficult to guarantee that the Linear Independency Constraint Qualification (LICQ) is always fulfilled during the iterative solution process. If the set of active flow constraints is changed it must be checked that the linear system that includes the continuity equation and all active flow and outflow constraints has no linearly dependent rows (has full row rank). Moreover, in practical applications it is sometimes difficult to decide which of the redundant constraints should be activated. Non-uniqueness of heads (the Lagrange multipliers) is always a consequence of linearly dependent equations of active flow constraints and continuity equations. Such situations can be avoided by careful pre-analysis that detects infeasible configurations or a check when constraints are activated. The problem of non-uniqueness of nodal pressures is discussed also in [8].

3 EXAMPLE

In this section the following question is addressed: under what conditions does the PDM model still give feasible solutions by reduction of the outflow at the demand nodes? A simple example is given in Fig. 1 where a supply area, represented by the single node S (elevation 100 m) where the demand is d, is connected to two reservoirs (R1 and R2, both at elevation 200 m) by two identical pipes (L= 1000 m, D = 200 mm CHW=100). In the case of sufficient pressures and without flow control, the demand $d = q_1 + q_2$. The flows q_1 and q_2 depend on the pipe characteristics and the water levels in the reservoirs. Now, consider the case where the flow out of the reservoirs is restricted, e.g. because of a water shortage, by FCVs in pipe 1, 2. Then, the demand cannot be satisfied anymore if $d > q_1 + q_2$. In DDM this results in infeasible solutions whereas in PDM the actual delivery is reduced to less than d by the nonlinear consumption function. The following cases are distinguished:

- unique solution exists and is calculated by DDM and PDM
- unique solution exists in PDM but not in DDM
- solution exists but is unique neither in DDM nor PDM

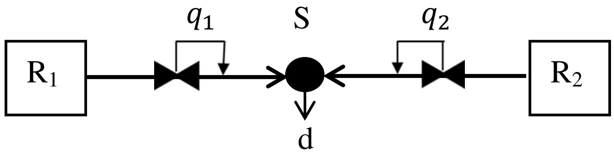

Figure 1. Example system with FCVs in Pipes 1 and 2

Following Eq. (11) the full KKT conditions are:

$$G_{1,1}\, q_1 + h - h_{R,1} - v_{L,1}\, \kappa_1 + v_{U,1}\, \nu_1 = 0$$
$$G_{2,2}\, q_2 + h - h_{R,2} - v_{L,2}\kappa_2 + v_{U,2}\, \nu_2 = 0$$
$$Nc + h_{min} - u_L\lambda + u_U\mu = 0$$
$$q_1 + q_2 = \mathrm{c};\;\; u_L \mathrm{c} = 0;\;\; u_U(\mathrm{c} - \mathrm{d}) = 0;\;\; v_{L,1}(q_1 - q_{1,min}) = 0;\;\; v_{U,1}\left(q_1 - q_{1,max}\right) = 0;$$
$$v_{L,2}(q_2 - q_{2,min}) = 0;\;\; v_{U,2}\left(q_2 - q_{2,max}\right) = 0$$

Table 1 shows the results for three different demand values (0, 200, 500 in m3/h). The settings of the FCVs are the same for all three cases: $q_{1,max} = 200$ m^3/h and $q_{2,max} = 250$ m^3/h. The units for the heads and the Lagrange multipliers are meters [m]. In the first case (d=200) both FCVs are inactive and the full demand is supplied. This situation causes no problems even in DDM analysis. In the second case, the sum of allowable flows is less than the desired demand. Both control devices are active with positive multipliers for the upper flow bounds (ν_1 and ν_2). Epanet fails to calculate a realistic solution (head at node S: -7,474,824.00 m, q1 = 225.0 m^3/h, q2 = 275 m^3/h). The third case happens, when for example both pipes are closed. Then the demand is reduced to zero but as shown in Figures 1 and 2 the head at the node can theoretically be any value below the minimum pressure. To get a unique solution it is important that the POR constraint is not active even when c = 0. Otherwise, the flow constraints are linearly dependent which contradicts the LICQ of the Kuhn-Tucker conditions. In spite of this Epanet finds a solution for the third case.

Table 1. Results for three demand values (d= 200 m^3/h, d=450 m^3/h, d=0 m^3/h)

d	$G_{1,1}$	$G_{2,2}$	N	q_1	q_2	c	h	λ	μ	κ_1	κ_2	ν_1	ν_2
200	0.07	0.07	0.15	100	100	200	193	0	63	0	0	0	0
500	0.127	0.153	0.06	200	250	450	127	0	0	0	0	47.65	34.68
0	1e10	1e10	1e10	0	0	0	100	0	0	0	0	100	100

4 CONCLUSIONS

It has been shown that the consideration of pressure dependent outflows is not only useful for more realistic modelling of pressure deficient conditions and pressure dependent leakage: it also enlarges the range of solvable problems by its ability to reduce the outflow in cases of disconnected subnets. The same technique is used for both POR functions and flow control devices. As an important result, it has been shown that there always exists a unique solution in terms of networks flows. Violation of the LICQ by the set of active flow and outflow constraints in combination with the homogeneous continuity equation results in non-uniqueness of the Lagrangian multipliers and therefore also in non-uniqueness of nodal heads.

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