Appendix B Mathematical Proofs

Appendix B.1 Proof of Lemma 1

Proof. By the classical Weyl inequalities in e.g., Horn and Johnson (2013) and Tao (2012), we have $|\lambda_j(A+\epsilon)-\lambda_j(A)| \leq K\,\|\epsilon\|$, for all $j=1,2,\ldots,d$, where $A,\epsilon \in \mathcal{M}_d^+$, and K is some constant. This establishes the Lipchitz property. ∎

Appendix B.2 Proof of Lemma 2

Proof. It is straightforward to show (by the implicit function theorem, see Magnus and Neudecker (1999) Theorem 8.7) that any simple eigenvalue and its corresponding eigenvector, written as functions of A, $\lambda_g(A)$ and $\gamma_g(A)$, are C^∞. To calculate their derivatives, note that $A\gamma_g=\lambda_g\gamma_g$, hence we have $(\partial_{jk}A)\gamma_g + A(\partial_{jk}\gamma_g) = (\partial_{jk}\lambda_g)\gamma_g + \lambda_g(\partial_{jk}\gamma_g)$. Pre-multiplying $\gamma_g^\intercal$ on both sides yields $\partial_{jk}\lambda_g = \gamma_g^\intercal(\partial_{jk}A)\gamma_g = \gamma_{gj}\gamma_{gk}$. Rewrite it into $(\lambda_g\mathbb{I}-A)\partial_{jk}\gamma_g = (\partial_{jk}A)\gamma_g - (\partial_{jk}\lambda_g)\gamma_g$, which leads to $(\lambda_g\mathbb{I}-A)^+(\lambda_g\mathbb{I}-A)\partial_{jk}\gamma_g = (\lambda_g\mathbb{I}-A)^+(\partial_{jk}A)\gamma_g$. As a result, $\partial_{jk}\gamma_g = (\lambda_g\mathbb{I}-A)^+J_{jk}\gamma_g$.

In the case when all eigenvalues are simple, by direct calculation we have

$$\partial_{jk}\gamma_{gh} = \sum_{p\neq g}\frac{1}{\lambda_g-\lambda_p}\gamma_{ph}\gamma_{pj}\gamma_{gk},$$

where we use the fact that $(\gamma_1^\intercal,\gamma_2^\intercal,\ldots,\gamma_d^\intercal)^\intercal A(\gamma_1,\gamma_2,\ldots,\gamma_d) = \mathrm{Diag}(\lambda(A))$. Further,

$$\begin{aligned}
\partial^2_{jk,lm}\gamma_{gh} =& -\sum_{p\neq g}\frac{1}{(\lambda_g-\lambda_p)^2}(\partial_{lm}\lambda_g-\partial_{lm}\lambda_p)\gamma_{ph}\gamma_{pj}\gamma_{gk} + \sum_{p\neq g}\frac{1}{\lambda_g-\lambda_p}\partial_{lm}\left(\gamma_{ph}\gamma_{pj}\gamma_{gk}\right)\\
=& -\sum_{p\neq g}\frac{1}{(\lambda_g-\lambda_p)^2}\left(\gamma_{gl}\gamma_{gm}\gamma_{ph}\gamma_{pj}\gamma_{gk} - \gamma_{pl}\gamma_{pm}\gamma_{ph}\gamma_{pj}\gamma_{gk}\right)\\
&+\sum_{p\neq g}\sum_{q\neq p}\frac{1}{(\lambda_g-\lambda_p)(\lambda_p-\lambda_q)}\gamma_{ql}\gamma_{pm}\gamma_{qh}\gamma_{pj}\gamma_{gk}\\
&+\sum_{p\neq g}\sum_{q\neq p}\frac{1}{(\lambda_g-\lambda_p)(\lambda_p-\lambda_q)}\gamma_{ql}\gamma_{pm}\gamma_{qj}\gamma_{ph}\gamma_{gk}\\
&+\sum_{p\neq g}\sum_{q\neq g}\frac{1}{(\lambda_g-\lambda_p)(\lambda_g-\lambda_q)}\gamma_{qk}\gamma_{ql}\gamma_{gm}\gamma_{ph}\gamma_{pj},
\end{aligned}$$

which concludes the proof. ∎

Appendix B.3 Proof of Lemma 3

Proof. The proof is by induction. Consider, at first the following optimization problem:

$$\max_{\gamma_s} \int_0^u \gamma_s^\intercal c_s \gamma_s ds, \text{ s.t. } \quad \gamma_s^\intercal \gamma_s = 1, \quad 0 \le s \le u \le t.$$

Using a sequence of Lagrange multipliers λ_s, the problem can be written as solving

$$c_s \gamma_s = \lambda_s \gamma_s, \quad \text{and} \quad \gamma_s^\intercal \gamma_s = 1, \text{ for any } 0 \le s \le t.$$

Hence, the original problem is translated into eigenanalysis.

Suppose the eigenvalues of c_s are ordered as in $\lambda_{1,s} \ge \lambda_{2,s} \ge \ldots \ge \lambda_{d,s}$. Note that $\gamma_s^\intercal c_s \gamma_s = \lambda_s$, so that $\lambda_s = \lambda_{1,s}$, and $\gamma_s = \gamma_{1,s}$ is one of the corresponding eigenvectors (if $\lambda_{1,s}$ is not unique), and the maximal variation is $\int_0^t \lambda_{1,s} ds$.

Suppose that we have found $\gamma_{1,s}, \ldots, \gamma_{k,s}$, for $1 \le k < d$ and $0 \le s \le t$, the $(k+1)$th principal component is defined by solving the following problem:

$$\max_{\gamma_s} \int_0^u \gamma_s^\intercal c_s \gamma_s ds, \quad \text{ s.t. } \quad \gamma_s^\intercal \gamma_s = 1, \quad \text{and} \quad \gamma_{j,s}^\intercal c_s \gamma_s = 0, \text{ for } 1 \le j \le k, \quad 0 \le u \le t.$$

Using similar technique of Lagrange multipliers, λ_s, and $\nu_{1,s}, \ldots, \nu_{k,s}$, we find

$$c_s \gamma_s = \lambda_s \gamma_s + \sum_{j=1}^k \nu_{j,s} c_s \gamma_{j,s}.$$

Multiplying on the left $\gamma_{l,s}^\intercal$, for some $1 \le l \le k$, we can show that $\nu_{l,s} c_s \gamma_{l,s} = 0$. Indeed,

$$0 = \lambda_{l,s} \gamma_{l,s}^\intercal \gamma_s = \gamma_{l,s}^\intercal c_s \gamma_s = \gamma_{l,s}^\intercal \lambda_s \gamma_s + \sum_{j=1}^k \nu_{j,s} \gamma_{l,s}^\intercal c_s \gamma_{j,s} = \nu_{l,s} \lambda_{l,s}.$$

Therefore, since l is an arbitrary number between 1 and k, we have $c_s \gamma_s = \lambda_s \gamma_s$. Hence, $\lambda_s = \lambda_{k+1,s}$, $\gamma_s = \gamma_{k+1,s}$ is one of the eigenvectors associated with the eigenvalue $\lambda_{k+1,s}$. This establishes the first part of the theorem.

For any càdlàg and adapted process γ_s,

$$\left[\int_0^u \gamma_{s-}^\intercal dX_s, \int_0^u \gamma_{s-}^\intercal dX_s\right]^c = \int_0^t \gamma_s^\intercal c_s \gamma_s ds.$$

Hence the statement follows from the gth-step optimization problem. Note that the validity of the integrals above is warranted by the continuity of λ given by Lemma 1. ∎

Appendix B.4 Proof of Lemma 4

Proof. The first statement of the proof follows by immediate calculations from Theorem 1.1 in Lewis (1996b) and Theorem 3.3 in Lewis and Sendov (2001). The second statement is discussed and proved in, e.g., Ball (1984), Sylvester (1985), and Silhavý (2000). Finally, the last statement on convexity is proved in Davis (1957) and Lewis (1996a). ∎

Appendix B.5 Proof of Lemma 5

Proof. Obviously, for any $1 \le g_1 < g_2 < \ldots < g_r \le d$, the set defined in (4), $\mathcal{D}(g_1, g_2, \ldots, g_r)$, is an open set in $\mathbb{R}_d^+/\{0\}$. Define $f(x) = |\bar{x}_{g_r}| + \sum_{i \neq j} |\bar{x}_{g_i} - \bar{x}_{g_j}|$, which is a continuous and convex function. It is differentiable at x if and only if $x \in \mathcal{D}(g_1, g_2, \ldots, g_r)$. Therefore, by Lemma 4, $f \circ \lambda$ is convex, and it is differentiable at A if and only if $\lambda(A) \in \mathcal{D}(g_1, g_2, \ldots, g_r)$, i.e., $A \in \mathcal{M}(g_1, g_2, \ldots, g_r)$. On the other hand, a convex function is almost everywhere differentiable, see Rockafellar (1997), which implies that $\mathcal{M}(g_1, g_2, \ldots, g_r)$ is dense in $\mathcal{M}_d^{++}$. Moreover, $\mathcal{M}(g_1, g_2, \ldots, g_r)$ is the pre-image of the open set $\mathbb{R}^+/\{0\}$ under a continuous function $h \circ \lambda$, where $h(x) = \prod_{i \neq j} |\bar{x}_{g_i} - \bar{x}_{g_j}||\bar{x}_{g_r}|$. Therefore, it is open. ∎

Appendix B.6 Proof of Theorem 1

Proof. Throughout the proof, we adopt the usual localization procedure as detailed in Jacod and Protter (2012). Note that

$$V(\Delta_n, X; F) = k_n \Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} f\left(\widehat{\lambda}_{ik_n\Delta_n}\right) = k_n \Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} (f \circ \lambda)(\widehat{c}_{ik_n\Delta_n}).$$

By Assumption 2 and Lemma 4, $f \circ \lambda$ is a continuous vector-valued function. Moreover, for $c \in \mathcal{M}_d^+$, $\|f \circ \lambda(c)\| \le K(1 + \|\lambda(c)\|^\zeta) \le K(1 + \|c\|^\zeta)$. Below we prove this theorem for any spectral function F that is bounded by $K(1 + \|c\|^\zeta)$.

We start with a function F bounded by K everywhere. We extend the definition of $\widehat{c}$ to the entire interval $[0,t]$ by letting:

$$\widehat{c}_s = \widehat{c}_{(i-1)k_n\Delta_n}, \text{ for } \quad (i-1)k_n\Delta_n \leq s < ik_n\Delta_n.$$

Note that for any $t > 0$, we have

$$\begin{aligned}
&\mathrm{E}\left\| V(\Delta_n, X; F) - \int_0^t F(c_s)ds \right\| \\
\leq & k_n\Delta_n \mathrm{E}\left\| F(\widehat{c}_{[t/(k_n\Delta_n)]k_n\Delta_n}) \right\| + \int_0^{[t/(k_n\Delta_n)]k_n\Delta_n} \mathrm{E}\left\| F(\widehat{c}_s) - F(c_s) \right\| ds + \int_{[t/(k_n\Delta_n)]k_n\Delta_n}^t \mathrm{E}\left\| F(c_s) \right\| ds \\
\leq & Kk_n\Delta_n + \int_0^{[t/(k_n\Delta_n)]k_n\Delta_n} \mathrm{E}\left\| F(\widehat{c}_s) - F(c_s) \right\| ds.
\end{aligned}$$

By the fact that $\widehat{c}_s - c_s \xrightarrow{\mathrm{p}} 0$, it follows that $\mathrm{E}\left\| F(\widehat{c}_s) - F(c_s) \right\| \to 0$, which is bounded uniformly in s and n because F is bounded. Therefore, by the dominated convergence theorem, we obtain the desired convergence.

Next we show the convergence holds under the polynomial bound on F. Denote ψ to be a C^∞ function on $\mathbb{R}^+$ such that $1_{[1,\infty)}(x) \leq \psi(x) \leq 1_{[1/2,\infty]}(x)$. Let $\psi_\varepsilon(c) = \psi(\|c\|/\varepsilon)$, and $\psi'_\varepsilon(c) = 1 - \psi_\varepsilon(c)$. Since the function $F \cdot \psi'_\varepsilon$ is continuous and bounded, the above argument implies that $V(\Delta_n, X; F \cdot \psi'_\varepsilon) \xrightarrow{\mathrm{p}} \int_0^t F \cdot \psi'_\varepsilon(c_s)ds$, for any fixed ε. When ϵ is large enough, we have $\int_0^t F \cdot \psi'_\varepsilon(c_s)ds = \int_0^t F(c_s)ds$ by localization, since c_s is locally bounded. On the other hand, $F \cdot \psi_\varepsilon(c) \leq K \|c\|^\zeta 1_{\{\|c\| \geq \varepsilon\}}$, for $\varepsilon > 1$. So it remains to show that

$$\lim_{\varepsilon \to \infty} \limsup_{n \to \infty} \mathrm{E}\left(k_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \left\| \widehat{c}_{ik_n\Delta_n} \right\|^\zeta 1_{\{\|\widehat{c}_{ik_n\Delta_n}\| > \varepsilon\}} \right) = 0.$$

By (9.4.7) of Jacod and Protter (2012), there exists some sequence a_n going to 0, such that

$$\mathrm{E}\left(\left\| \widehat{c}_{ik_n\Delta_n} \right\|^\zeta 1_{\{\|\widehat{c}_{ik_n\Delta_n}\| > \varepsilon\}} | \mathcal{F}_{ik_n\Delta_n} \right) \leq \frac{K}{\varepsilon^\zeta} + Ka_n\Delta_n^{(1-\zeta+\varpi(2\zeta-\gamma))},$$

which establishes the desired result. ∎

Appendix B.7 Proof of Proposition 1

Proof. We divide the proof into several steps. To start, we need some additional notations. Let X' and c' denote the continuous parts of the processes X and c, respectively. Also, we introduce $\widehat{c}'_{ik_n\Delta_n}$ to denote the estimator constructed similarly as in (5) with X replaced by X' and without truncation, namely

$$\widehat{c}'_{ik_n\Delta_n} = \frac{1}{k_n\Delta_n}\sum_{j=1}^{k_n}(\Delta^n_{ik_n+j}X')(\Delta^n_{ik_n+j}X')^\intercal.$$

In addition, $\widehat{\lambda}'_{ik_n\Delta_n}$ corresponds to the vector of eigenvalues of $\widehat{c}'_{ik_n\Delta_n}$, and

$$V'(\Delta_n, X; F) = k_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} f\left(\widehat{\lambda}'_{ik_n\Delta_n}\right).$$

We also define

$$\bar{c}_{ik_n\Delta_n} = \frac{1}{k_n\Delta_n}\int_{ik_n\Delta_n}^{(i+1)k_n\Delta_n} c_s ds, \quad \beta^n_{ik_n} = \widehat{c}'_{ik_n\Delta_n} - c_{ik_n\Delta_n}, \quad \alpha^n_l = (\Delta^n_l X')(\Delta^n_l X')^\intercal - c_{l\Delta_n}\Delta_n, \quad \text{and}$$

$$\eta^n_i = \left(\mathrm{E}\left(\sup_{i\Delta_n\le u\le i\Delta_n+k_n\Delta_n}|b_{i\Delta_n+u} - b_{i\Delta_n}|^2|\mathcal{F}_{i\Delta_n}\right)\right)^{1/2}.$$

We first collect some known estimates in the next lemma:

Lemma 6. *Under the assumptions of Proposition 1, we have*

$$\mathrm{E}\left(\sup_{0\le u\le s}\|c_{t+u} - c_t\|^q\,|\mathcal{F}_t\right) \le K s^{1\wedge q/2}, \quad \|\mathrm{E}\,(c_{t+s} - c_t|\mathcal{F}_t)\| \le Ks, \tag{B.1}$$

$$\mathbb{E}\left\|(\Delta^n_i X)(\Delta^n_i X)^\intercal 1_{\{\|\Delta^n_i X\|\le u_n\}} - (\Delta^n_i X')(\Delta^n_i X')^\intercal\right\| \le K a_n \Delta_n^{(2-\gamma)\varpi+1}, \ \text{for some } a_n \to 0, \tag{B.2}$$

$$\mathrm{E}\left(\left\|\widehat{c}_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\|^q\right) \le K a_n \Delta_n^{(2q-\gamma)\varpi+1-q}, \ \text{for some } q\ge 1, \ \text{and } a_n \to 0, \tag{B.3}$$

$$\mathrm{E}\left\|\widehat{c}'_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}\right\|^p \le K k_n^{-p/2}, \ \text{for some } p \ge 1, \tag{B.4}$$

$$\mathrm{E}\left(\|\alpha^n_i\|^q\,|\mathcal{F}_{i\Delta_n}\right) \le K\Delta_n^q, \ \text{for some } q\ge 0, \tag{B.5}$$

$$\|\mathrm{E}\,(\alpha^n_i|\mathcal{F}_{i\Delta_n})\| \le K\Delta_n^{3/2}\left(\Delta_n^{1/2} + \eta^n_i\right), \tag{B.6}$$

$$\left|\mathrm{E}\left(\alpha^{n,jk}_i\alpha^{n,lm}_i - \left(c^{jl}_{i\Delta_n}c^{km}_{i\Delta_n} + c^{jm}_{i\Delta_n}c^{kl}_{i\Delta_n}\right)\Delta_n^2|\mathcal{F}_{i\Delta_n}\right)\right| \le K\Delta_n^{5/2}, \tag{B.7}$$

$$\left\|\mathrm{E}\left(\beta^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n}\right)\right\| \le K{\Delta_n}^{1/2}\left(k_n\Delta_n^{1/2} + \eta^n_{ik_n}\right), \tag{B.8}$$

$$\left\| \mathrm{E}\left(\beta_{ik_n}^{n,jk}\beta_{ik_n}^{n,lm}|\mathcal{F}_{ik_n\Delta_n}\right) - k_n^{-1}(c_{ik_n\Delta_n}^{jl}c_{ik_n\Delta_n}^{km} + c_{ik_n\Delta_n}^{jm}c_{ik_n\Delta_n}^{kl})\right\|$$
$$\leq K{\Delta_n}^{1/2}\left(k_n^{-1/2} + k_n\Delta_n^{1/2} + \eta_{ik_n}^n\right), \tag{B.9}$$

$$\mathrm{E}\left(\left\|\beta_{ik_n}^n\right\|^q|\mathcal{F}_{ik_n\Delta_n}\right) \leq K\left(k_n^{-q/2} + k_n\Delta_n\right),\ \textit{for some } q \geq 2, \tag{B.10}$$

$$\Delta_n\mathrm{E}\sum_{i=1}^{[t/\Delta_n]}\eta_i^n \to 0. \tag{B.11}$$

Proof of Lemma 6. These estimates are given by Lemma A.2 in Li and Xiu (2016), (4.3), (4.8), (4.10), (4.11), (4.12), (4.18), Lemmas 4.2 and 4.3 of Jacod and Rosenbaum (2013), and Lemma 13.2.6 of Jacod and Protter (2012). ∎

Now we return to the proof of Proposition 1.

1) We show that we can restrict the domain of function f to some compact set, where both the estimates $\{\widehat{c}_{ik_n\Delta_n}\}_{i=0,1,2,\dots,[t/(k_n\Delta_n)]}$ and the sample path of $\{c_s\}_{s\in[0,t]}$ take values. By (B.4), we have for $p \geq 1$,

$$\mathbb{E}\left\|\widehat{c}'_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}\right\|^p \leq Kk_n^{-p/2}.$$

Therefore, by the maximal inequality, we deduce, by picking $p > 2/\varsigma - 2$,

$$\mathbb{E}\left|\sup_{0\leq i\leq[t/(k_n\Delta_n)]}\left\|\widehat{c}'_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}\right\|^p\right| \leq K\Delta_n^{-1}k_n^{-p/2-1} \to 0,$$

therefore, $\sup_{0\leq i\leq[t/(k_n\Delta_n)]}\left\|\widehat{c}'_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}\right\| = o_p(1)$. Moreover, by (B.2) we have

$$\begin{aligned}\mathbb{E}\left|\sup_{0\leq i\leq[t/(k_n\Delta_n)]}\left\|\widehat{c}_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\|\right| \leq& \frac{1}{k_n\Delta_n}\sum_{i=1}^{[t/\Delta_n]-k_n}\mathbb{E}\left\|(\Delta_i^nX)(\Delta_i^nX)^\intercal 1_{\{\|\Delta_i^nX\|\leq u_n\}} - (\Delta_i^nX')(\Delta_i^nX')^\intercal\right\| \\ \leq& Ka_n\Delta_n^{(2-\gamma)\varpi-1+\varsigma} \to 0.\end{aligned}$$

As a result, we have as $\Delta_n \to 0$,

$$\sup_{0\leq i\leq[t/(k_n\Delta_n)]}\left\|\widehat{c}_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}\right\| \xrightarrow{\mathrm{p}} 0. \tag{B.12}$$

Note that by Assumption 3, for $0 \leq s \leq t$, $c_s \in \mathcal{C} \cap \mathcal{M}^*(g_1, g_2, \dots, g_r)$, where $\mathcal{C}$ is a convex and open set. Therefore, $\{\bar{c}_{ik_n\Delta_n}\}_{i=0,1,2,\ldots,[t/(k_n\Delta_n)]} \in \mathcal{C}$ by convexity. For n large enough, $\{\widehat{c}_{ik_n\Delta_n}\}_{i=0,1,2,\ldots,[t/(k_n\Delta_n)]} \in \mathcal{C}$, with probability approaching 1, by (B.12). Since $\bar{\mathcal{C}} \subset \mathcal{M}(g_1, g_2, \ldots, g_r)$, we can restrict the domain of f to the compact set $\lambda(\bar{\mathcal{C}}) \subset \mathcal{D}(g_1, g_2, \ldots, g_r)$, in which f is C^∞ with bounded derivatives. Moreover, because $\lambda_{g_j}(\cdot), 1 \leq j \leq r$ are continuous functions, $\min_{1\leq j\leq r-1}(\lambda_{g_j}(\cdot) - \lambda_{g_{j+1}}(\cdot))$ is hence continuous, so that $\inf_{c\in\mathcal{C}}\{\min_{1\leq j\leq r-1}(\lambda_{g_j}(c) - \lambda_{g_{j+1}}(c))\} \geq \epsilon > 0$. It follows from Lemma 4 and Theorem 3.5 of Silhavý (2000) that $F(\cdot)$ is C^∞ with bounded derivatives on $\mathcal{C}$.

2) Next, we have

$$\begin{aligned}\left\|V(\Delta_n, X; F) - V'(\Delta_n, X; F)\right\| &\leq k_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \left\|F(\widehat{c}_{ik_n\Delta_n}) - F(\widehat{c}'_{ik_n\Delta_n})\right\| \\ &\leq Kk_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \left\|\widehat{c}_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\|.\end{aligned}$$

By (B.3), we have

$$\mathrm{E}\left(\left\|\widehat{c}_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\|\right) \leq Ka_n\Delta_n^{(2-\gamma)\varpi},$$

where a_n is some sequence going to 0, as $n \to \infty$, which implies

$$V(\Delta_n, X; F) - V'(\Delta_n, X; F) = O_p(a_n\Delta_n^{(2-\gamma)\varpi}). \tag{B.13}$$

As a result, given the conditions on ϖ, we have

$$k_n\left(V(\Delta_n, X; F) - V'(\Delta_n, X; F)\right) = o_p(1),$$

hence we can proceed with V' in the sequel.

3) Then we show for each $1 \leq h \leq d$, we have

$$\begin{aligned}k_n\Bigg(V'(\Delta_n, X; F_h) - k_n\Delta_n &\sum_{i=0}^{[t/(k_n\Delta_n)]}\Bigg(F_h(c_{ik_n\Delta_n}) \\ &- \frac{1}{2k_n}\sum_{j,k,l,m=1}^{d}\partial^2_{jk,lm}F_h(c_{ik_n\Delta_n})\left(c_{jl,ik_n\Delta_n}c_{km,ik_n\Delta_n} + c_{jm,ik_n\Delta_n}c_{kl,ik_n\Delta_n}\right)\Bigg)\Bigg) = o_p(1).\end{aligned}$$

where F_h is the hth entry of the vector-valued function F.

To prove it, we decompose the left hand side into 4 terms:

$$R^n_{1,h} = k_n^2\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \Big(F_h(c_{ik_n\Delta_n} + \beta^n_{ik_n}) - F_h(c_{ik_n\Delta_n}) - \sum_{l,m=1}^{d} \partial_{lm} F_h(c_{ik_n\Delta_n})\beta^{n,lm}_{ik_n}$$

$$- \frac{1}{2}\sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm} F_h(c_{ik_n\Delta_n})\beta^{n,lm}_{ik_n}\beta^{n,jk}_{ik_n}\Big), \tag{B.14}$$

$$R^n_{2,h} = k_n^2\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \frac{1}{2}\sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm} F_h(c_{ik_n\Delta_n}) \left(\beta^{n,lm}_{ik_n}\beta^{n,jk}_{ik_n} - \frac{1}{k_n}\left(c_{jl,ik_n\Delta_n}c_{km,ik_n\Delta_n} + c_{jm,ik_n\Delta_n}c_{kl,ik_n\Delta_n}\right)\right), \tag{B.15}$$

$$R^n_{3,h} = k_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \sum_{l,m=1}^{d} \partial_{lm} F_h(c_{ik_n\Delta_n}) \sum_{u=1}^{k_n} (c_{lm,(ik_n+u)\Delta_n} - c_{lm,ik_n\Delta_n}), \tag{B.16}$$

$$R^n_{4,h} = k_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \sum_{l,m=1}^{d} \partial_{lm} F_h(c_{ik_n\Delta_n}) \sum_{u=1}^{k_n} \alpha^{n,lm}_{ik_n+u}, \tag{B.17}$$

We first consider $R^n_{1,h}$. By (B.10), we have

$$\mathrm{E}(|R^n_{1,h}|) \le K k_n^2 \Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \mathrm{E}\left\|\beta^n_{ik_n}\right\|^3 \le K k_n^2\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} (k_n^{-3/2} + k_n\Delta_n)$$
$$\le K k_n^2\Delta_n + K k_n^{-1/2} \to 0.$$

As to $R^n_{2,h}$, we denote the term inside the summation of $R^n_{2,h}$ as $\nu^n_{ik_n}$. So we have

$$R^n_{2,h} = k_n^2\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \left(\nu^n_{ik_n} - \mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n}) + \mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})\right).$$

By (B.9) we have

$$|\mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})| \le K{\Delta_n}^{1/2}\left(k_n^{-1/2} + k_n\Delta_n^{1/2} + \eta^n_{ik_n}\right).$$

On the other hand, by (B.10), we can derive

$$\mathrm{E}\left(\nu^n_{ik_n} - \mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})\right)^2 \le K k_n\Delta_n.$$

Then Doob's inequality implies that

$$\mathrm{E}\left(\sup_{s\leq t}\left|\sum_{i=0}^{[s/(k_n\Delta_n)]}\left(\nu^n_{ik_n}-\mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})\right)\right|\right)\leq Kt.$$

As a result, by (B.11)

$$\begin{aligned}\mathrm{E}\left(|R^n_{2,h}|\right)\leq & k_n^2\Delta_n\mathrm{E}\left(\sup_{s\leq t}\left|\sum_{i=0}^{[s/(k_n\Delta_n)]}\left(\nu^n_{ik_n}-\mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})\right)\right|\right)+k_n^2\Delta_n\sum_{i=0}^{[t/(k_n\Delta_n)]}\left|\mathrm{E}(\nu^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})\right|\\ \leq & Kk_n^2\Delta_n+Kk_n\sqrt{\Delta_n}\left(k_n\Delta_n\sum_{i=0}^{t/(k_n\Delta_n)}\eta_{ik_n}\right)\rightarrow 0.\end{aligned}$$

The proof for $\mathrm{E}(|R^n_{3,h}|)\rightarrow 0$ is similar. Denote the term inside the summand as $\xi^n_{ik_n}$. By (B.1) and the Cauchy-Schwarz inequality, we have

$$|\mathrm{E}(\xi^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n})|\leq Kk_n^2\Delta_n,\quad \mathrm{E}\left(|\xi^n_{ik_n}|^2|\mathcal{F}_{ik_n\Delta_n}\right)\leq Kk_n^3\Delta_n.$$

By Doob's inequality again,

$$\begin{aligned}\mathrm{E}\left(|R^n_{3,h}|\right)\leq & \,k_n\Delta_n\sum_{i=0}^{[t/(k_n\Delta_n)]}\mathrm{E}\left(\left|\mathrm{E}\left(\xi^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n}\right)\right|\right)+k_n\Delta_n\left(\sum_{i=0}^{[t/(k_n\Delta_n)]}\mathrm{E}\left(|\xi^n_{ik_n}|^2\right)\right)^{1/2}\\ \leq & \,Kk_n^2\Delta_n\rightarrow 0.\end{aligned}$$

For $R^n_{4,h}$, it can be shown in the proof of Theorem 2 below that $R^n_{4,h}=O_p(k_n\sqrt{\Delta_n})=o_p(1)$.

4) Finally, it is sufficient to show that

$$k_n\left(\sum_{i=0}^{[t/(k_n\Delta_n)]}\left(\int_{ik_n\Delta_n}^{(i+1)k_n\Delta_n}F_h(c_{ik_n\Delta_n})ds-\int_{ik_n\Delta_n}^{(i+1)k_n\Delta_n}F_h(c_s)ds\right)-\int_{[t/(k_n\Delta_n)]k_n\Delta_n}^{t}F_h(c_s)ds\right)\xrightarrow{\mathrm{p}}0,$$

as the similar result holds if we replace $F_h(c_{ik_n\Delta_n})$ by $\partial^2_{jk,lm}F_h(c_{ik_n\Delta_n})\left(c_{jl,ik_n\Delta_n}c_{km,ik_n\Delta_n}+c_{jm,ik_n\Delta_n}c_{kl,ik_n\Delta_n}\right)$. Since F_h is bounded, the second term is bounded by $Kk_n^2\Delta_n\rightarrow 0$. As to the first term, we notice that

$$\zeta^n_{ik_n} = \int_{ik_n\Delta_n}^{(i+1)k_n\Delta_n} F_h(c_{ik_n\Delta_n})ds - \int_{ik_n\Delta_n}^{(i+1)k_n\Delta_n} F_h(c_s)ds$$

is measurable with respect to $\mathcal{F}_{(i+1)k_n\Delta_n}$, and that

$$|\mathrm{E}\left(\zeta^n_{ik_n}|\mathcal{F}_{ik_n\Delta_n}\right)| \leq K(k_n\Delta_n)^2, \quad \mathrm{E}\left(|\zeta^n_{ik_n}|^2|\mathcal{F}_{ik_n\Delta_n}\right) \leq K(k_n\Delta_n)^3,$$

so the same steps as in (2) and (3) yield the desired results. ∎

Appendix B.8 Proof of Theorem 2

Proof. To start, we decompose

$$\frac{1}{\sqrt{\Delta_n}}\left(\widetilde{V}(\Delta_n, X; F) - k_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} F(c_{ik_n\Delta_n})\right) = \frac{1}{k_n\sqrt{\Delta_n}}\left(R^n_1 + R^n_2 + R^n_3 + R^n_4 + R^n_5 + R^n_6\right), \tag{B.18}$$

where $R^n_i = \left(R^n_{i,1}, R^n_{i,2}, \ldots, R^n_{i,d}\right)^\intercal$, for $i = 1, 2, 3, 4$, and 5, with $R^n_{1,h}$, $R^n_{2,h}$, $R^n_{3,h}$ and $R^n_{4,h}$ given by equations (B.14) - (B.17). In addition, $R^n_{5,h}$ and $R^n_{6,h}$ are given by

$$\begin{aligned}
R^n_{5,h} =& \frac{k_n\Delta_n}{2}\sum_{i=0}^{[t/(k_n\Delta_n)]}\sum_{j,k,l,m=1}^{d}\left(\partial^2_{jk,lm}F_h(c_{ik_n\Delta_n})\left(c_{jl,ik_n\Delta_n}c_{km,ik_n\Delta_n} + c_{jm,ik_n\Delta_n}c_{kl,ik_n\Delta_n}\right)\right.\\
&\left.-\partial^2_{jk,lm}F_h(\widehat{c}'_{ik_n\Delta_n})\left(\widehat{c}'_{jl,ik_n\Delta_n}\widehat{c}'_{km,ik_n\Delta_n} + \widehat{c}'_{jm,ik_n\Delta_n}\widehat{c}'_{kl,ik_n\Delta_n}\right)\right).\\
R^n_{6,h} =& \frac{k_n\Delta_n}{2}\sum_{i=0}^{[t/(k_n\Delta_n)]}\sum_{j,k,l,m=1}^{d}\left(\partial^2_{jk,lm}F_h(\widehat{c}'_{ik_n\Delta_n})\left(\widehat{c}'_{jl,ik_n\Delta_n}\widehat{c}'_{km,ik_n\Delta_n} + \widehat{c}'_{jm,ik_n\Delta_n}\widehat{c}'_{kl,ik_n\Delta_n}\right)\right.\\
&\left.-\partial^2_{jk,lm}F_h(\widehat{c}_{ik_n\Delta_n})\left(\widehat{c}_{jl,ik_n\Delta_n}\widehat{c}_{km,ik_n\Delta_n} + \widehat{c}_{jm,ik_n\Delta_n}\widehat{c}_{kl,ik_n\Delta_n}\right)\right) + k_n\left(V(\Delta_n, X; F) - V'(\Delta_n, X; F)\right).
\end{aligned}$$

We have shown in the proof of Proposition 1 that $R^n_i = o_p(k_n\sqrt{\Delta_n})$, for $i = 1, 2, 3$. Therefore, these terms do not contribute to the asymptotic variance of $\widetilde{V}'(\Delta_n, X; F)$.

Next, we show that $R^n_{5,h}$ is also $o_p(k_n\sqrt{\Delta_n})$. By (B.10) and the mean-value theorem, we have

$$\mathrm{E}|R^n_{5,h}| \leq Kk_n\Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \mathrm{E}\left\|c_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\| \leq K(k_n^{-1/2} + k_n\Delta_n) = o_p(k_n\sqrt{\Delta_n}).$$

As to $R^n_{6,h}$, by (B.13) and the mean-value theorem, we have

$$\mathrm{E}|R^n_{6,h}| \leq K k_n \Delta_n \sum_{i=0}^{[t/(k_n\Delta_n)]} \mathrm{E}\left\|\widehat{c}_{ik_n\Delta_n} - \widehat{c}'_{ik_n\Delta_n}\right\| = O_p(a_n \Delta_n^{(2-\gamma)\varpi}) = o_p(k_n\sqrt{\Delta_n}).$$

Hence, $\varpi \geq \frac{1-\varsigma}{2-\gamma} > \frac{1}{4-2\gamma}$ is sufficient to warrant the desired rate.

As a result, except for the term that is related to R^n_4, all the remainder terms on the right-hand side of (B.18) vanish. We write R^n_4 as

$$R^n_4 = k_n \sum_{i=1}^{[t/(k_n\Delta_n)]k_n} \sum_{l,m=1}^{d} \omega_i^{n,lm}\alpha_i^{n,lm}, \quad \text{where} \quad \omega_i^{n,lm} = \partial_{lm}F(c_{[(i-1)/k_n]k_n\Delta_n}).$$

where $\omega_i^{n,lm}$ is a vector measurable with respect to $\mathcal{F}_{(i-1)\Delta_n}$, and $\|\omega_i^n\| \leq K$. To prove the stable convergence result, we start with

$$\frac{1}{\sqrt{\Delta_n}}\mathrm{E}\left\|\sum_{i=0}^{[t/(k_n\Delta_n)]k_n} \omega_i^{n,lm}\mathrm{E}\left(\alpha_i^{n,lm}|\mathcal{F}_{i\Delta_n}\right)\right\| \leq \frac{1}{\sqrt{\Delta_n}} \sum_{i=0}^{[t/(k_n\Delta_n)]k_n} K\Delta_n^{3/2}(\sqrt{\Delta_n} + \mathrm{E}\,(\eta_i^n)) \to 0,$$

where we use (B.6) and (B.11). Moreover, by (B.5), we have

$$\frac{1}{\Delta_n^2}\mathrm{E}\left(\sum_{i=0}^{[t/(k_n\Delta_n)]k_n} \|\omega_i^n\|^4\,\mathrm{E}\left(\|\alpha_i^n\|^4\,|\mathcal{F}_{i\Delta_n}\right)\right) \leq K\Delta_n \to 0.$$

Also, similar to (4.18) in Jacod and Rosenbaum (2013), we have $\mathrm{E}\left(\alpha_i^{n,lm}\Delta_i^n N|\mathcal{F}_{i\Delta_n}\right) = 0$, for $N = W$ or any N that is an arbitrary bounded martingale orthogonal to W, which readily implies

$$\frac{1}{\sqrt{\Delta_n}} \sum_{i=0}^{[t/(k_n\Delta_n)]k_n} \omega_i^{n,lm}\mathrm{E}\left(\alpha_i^{n,lm}\Delta_i^n N|\mathcal{F}_{i\Delta_n}\right) \xrightarrow{\mathrm{p}} 0.$$

Finally, note that for any $1 \leq p, q \leq d$, by (B.7),

$$\frac{1}{\Delta_n} \sum_{i=0}^{[t/(k_n\Delta_n)]k_n} |\omega_{i,p}^{n,jk}\omega_{i,q}^{n,lm}| \left\|\mathrm{E}\left(\alpha_i^{n,jk}\alpha_i^{n,lm}|\mathcal{F}_{i\Delta_n}\right) - (c_{i\Delta_n,jl}c_{i\Delta_n,km} + c_{i\Delta_n,jm}c_{i\Delta_n,kl})\,\Delta_n^2\right\| \leq K\Delta_n^{1/2},$$

which implies

$$\begin{aligned}
&\frac{1}{\Delta_n}\sum_{i=0}^{[t/(k_n\Delta_n)]k_n}\omega_{i,p}^{n,jk}\omega_{i,q}^{n,lm}\mathrm{E}\left(\alpha_i^{n,jk}\alpha_i^{n,lm}|\mathcal{F}_{i\Delta_n}\right)\\
=&\frac{1}{\Delta_n}\sum_{i=0}^{[t/(k_n\Delta_n)]k_n}\omega_{i,p}^{n,jk}\omega_{i,q}^{n,lm}\left(c_{i\Delta_n,jl}c_{i\Delta_n,km}+c_{i\Delta_n,jm}c_{i\Delta_n,kl}\right)\Delta_n^2+o_p(1)\\
\xrightarrow{\mathrm{p}}&\int_0^t\partial_{jk}F_p(c_s)\partial_{lm}F_q(c_s)\left(c_{s,jl}c_{s,km}+c_{s,jm}c_{s,kl}\right).
\end{aligned}$$

Finally, by Theorem IX.7.28 of Jacod and Shiryaev (2003), we establish

$$\frac{1}{k_n\sqrt{\Delta_n}}R_n^4\xrightarrow{\mathcal{L}-\mathrm{s}}\mathcal{W}_t,$$

where $\mathcal{W}_t$ is conditional Gaussian on an extension of the probably space, with a covariance matrix

$$\mathrm{E}\left(\mathcal{W}_{p,t}\mathcal{W}_{q,t}|\mathcal{F}\right)=\sum_{j,k,l,m=1}^{d}\int_0^t\partial_{jk}F_p(c_s)\partial_{lm}F_q(c_s)\left(c_{s,jl}c_{s,km}+c_{s,jm}c_{s,kl}\right).$$

∎

Appendix B.9 Proof of Proposition 2

Proof. As we have seen from the above proof, we have for any $c\in\mathcal{C}$,

$$\|\partial_{jk}F_p(c)\partial_{lm}F_q(c)\left(c_{jl}c_{km}+c_{jm}c_{kl}\right)\|\leq K(1+\|c\|^2),$$

which, combined with the same argument in the proof of Theorem 1, establishes the desired result.

∎

Appendix B.10 Proof of Corollary 1

Proof. The first statement on consistency follows immediately from Theorem 1, as Assumption 2 holds with $\zeta=1$. Next, we prove the central limit result. For any $1\leq p\leq d$, we define f_p^λ as,

$$f_p^\lambda(\bar{x})=\frac{1}{g_p-g_{p-1}}\sum_{j=g_{p-1}+1}^{g_p}\bar{x}_j,$$

hence we have

$$\partial f_p^\lambda(\bar{x}) = \frac{1}{g_p - g_{p-1}} \sum_{v=g_{p-1}+1}^{g_p} e^v, \quad \partial^2 f_p^\lambda(\bar{x}) = 0,$$

and f^λ is C^∞ and Lipchitz. By Lemma 4 we can derive

$$\partial_{jk} F_p^\lambda(c_s) = \sum_{u=1}^{d} O_{uj} \partial_u f_p^\lambda(\lambda(c_s)) O_{uk} = \frac{1}{g_p - g_{p-1}} \sum_{u=1}^{d} \sum_{v=g_{p-1}+1}^{g_p} O_{uj} e_u^v O_{uk} = \frac{1}{g_p - g_{p-1}} \sum_{v=g_{p-1}+1}^{g_p} O_{vj} O_{vk}.$$

Therefore, the asymptotic covariance matrix is given by

$$\begin{aligned}
&\int_0^t \sum_{j,k,l,m=1}^{d} \partial_{jk} F_p^\lambda(c_s) \partial_{lm} F_q^\lambda(c_s) \left(c_{jl,s} c_{km,s} + c_{jm,s} c_{kl,s}\right) ds \\
=&\frac{1}{g_p - g_{p-1}} \frac{1}{g_q - g_{q-1}} \int_0^t \sum_{j,k,l,m=1}^{d} \sum_{v=g_{p-1}+1}^{g_p} \sum_{u=g_{q-1}+1}^{g_q} O_{vj} O_{vk} O_{ul} O_{um} \left(c_{jl,s} c_{km,s} + c_{jm,s} c_{kl,s}\right) ds \\
=&\frac{2}{(g_p - g_{p-1})(g_q - g_{q-1})} \int_0^t \sum_{l,m=1}^{d} \sum_{v=g_{p-1}+1}^{g_p} \sum_{u=g_{q-1}+1}^{g_q} O_{vl} O_{vm} O_{ul} O_{um} \lambda_{v,s}^2 ds \\
=&\frac{2}{(g_p - g_{p-1})(g_q - g_{q-1})} \int_0^t \sum_{v=g_{p-1}+1}^{g_p} \sum_{u=g_{q-1}+1}^{g_q} \delta_{u,v} \lambda_{v,s}^2 ds \\
=&\frac{2\delta_{p,q}}{(g_p - g_{p-1})} \int_0^t \lambda_{g_p,s}^2 ds. && \text{(B.19)}
\end{aligned}$$

Next, we calculate the bias-correction term. Recall that the estimator is given by

$$F_p^\lambda(\widehat{c}_{ik_n\Delta_n}) = \frac{1}{g_p - g_{p-1}} \sum_{v=g_{p-1}+1}^{g_p} \widehat{\lambda}_{v,ik_n\Delta_n},$$

where $\widehat{\lambda}_{v,ik_n\Delta_n}$ is the corresponding eigenvalue of the sample covariance matrix $\widehat{c}_{ik_n\Delta_n}$. Although $\widehat{c}_{ik_n\Delta_n}$ and $c_{ik_n\Delta_n}$ may have different eigenstructure, it is easy to verify that the functional forms of the second order derivative of F_p^λ evaluated at both points turn out to be the same, so here we only provide the calculations based on $\widehat{c}_{ik_n\Delta_n}$. Since almost surely, sample eigenvalues are simple, it implies from Lemma 4 that

$$\partial^2_{jk,lm} F_p^\lambda(\widehat{c}_{ik_n\Delta_n}) = \sum_{u,v=1}^{d} \mathcal{A}_{uv}^{f_p^\lambda}(\lambda(\widehat{c}_{ik_n\Delta_n})) \widehat{O}_{ul} \widehat{O}_{uj} \widehat{O}_{vk} \widehat{O}_{vm}$$

$$= \frac{1}{g_p - g_{p-1}} \sum_{h=g_{p-1}+1}^{g_p} \sum_{u,v=1,u\neq v}^{d} \frac{e_u^h - e_v^h}{\widehat{\lambda}_{u,i\Delta_n} - \widehat{\lambda}_{v,i\Delta_n}} \widehat{O}_{ul}\widehat{O}_{uj}\widehat{O}_{vk}\widehat{O}_{vm}$$

$$= \frac{1}{g_p - g_{p-1}} \sum_{h=g_{p-1}+1}^{g_p} \sum_{u=1,u\neq h}^{d} \frac{1}{\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{u,ik_n\Delta_n}} \left(\widehat{O}_{ul}\widehat{O}_{uj}\widehat{O}_{hk}\widehat{O}_{hm} + \widehat{O}_{hl}\widehat{O}_{hj}\widehat{O}_{uk}\widehat{O}_{um}\right),$$

where $\widehat{O}$ is the orthogonal matrix such that $\widehat{O}\widehat{c}_{ik_n\Delta_n}\widehat{O}^\intercal = \mathrm{Diag}(\lambda(\widehat{c}_{ik_n\Delta_n}))$. The dependence of $\widehat{O}$ on $ik_n\Delta_n$ is omitted for brevity.

To facilitate the implementation, we consider the matrix $\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}$. Note that

$$\mathrm{Diag}(\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{1,ik_n\Delta_n}, \widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{2,ik_n\Delta_n}, \ldots, \widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{d,ik_n\Delta_n}) = \widehat{O}(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n})\widehat{O}^\intercal,$$

hence we have

$$\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+ = \widehat{O}^\intercal \mathrm{Diag}\left(\frac{1}{\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{1,ik_n\Delta_n}}, \frac{1}{\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{2,ik_n\Delta_n}}, \ldots, 0, \ldots \frac{1}{\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{d,ik_n\Delta_n}}\right)\widehat{O}.$$

As a result, we obtain

$$\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+_{km} = \sum_{u=1,u\neq p}^{d} \frac{1}{\widehat{\lambda}_{h,ik_n\Delta_n} - \widehat{\lambda}_{u,ik_n\Delta_n}} \widehat{O}_{uk}\widehat{O}_{um},$$

Therefore, we have

$$\partial^2_{jk,lm} F_p^\lambda(\widehat{c}_{ik_n\Delta_n}) = \frac{1}{g_p - g_{p-1}} \sum_{h=g_{p-1}+1}^{g_p} \widehat{O}_{hk}\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+_{jl} \widehat{O}_{hm} + \widehat{O}_{hj}\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+_{km} \widehat{O}_{hl}.$$

Now we can calculate the following term, which is used for bias-correction:

$$\begin{aligned}
&\sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm} F_p^\lambda(\widehat{c}_{i\Delta_n})\left(\widehat{c}_{jl,i\Delta_n}\widehat{c}_{km,i\Delta_n} + \widehat{c}_{jm,i\Delta_n}\widehat{c}_{kl,i\Delta_n}\right) \\
=&\frac{1}{g_p - g_{p-1}} \sum_{h=g_{p-1}+1}^{g_p} \sum_{j,k,l,m=1}^{d} \left(\widehat{O}_{hk}\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+_{jl} \widehat{O}_{hm} + \widehat{O}_{hj}\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+_{km} \widehat{O}_{hl}\right) \\
&\cdot \left(\widehat{c}_{jl,ik_n\Delta_n}\widehat{c}_{km,ik_n\Delta_n} + \widehat{c}_{jm,ik_n\Delta_n}\widehat{c}_{kl,ik_n\Delta_n}\right) \\
=&\frac{2}{g_p - g_{p-1}} \sum_{h=g_{p-1}+1}^{g_p} \widehat{\lambda}_{h,ik_n\Delta_n} \mathrm{Tr}\left(\left(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n}\right)^+ \widehat{c}_{ik_n\Delta_n}\right).
\end{aligned} \tag{B.20}$$

The last equality uses the following observation:[5]

$$(\widehat{\lambda}_{h,ik_n\Delta_n}\mathbb{I} - \widehat{c}_{ik_n\Delta_n})^+ \widehat{O}_{h,\cdot}^\intercal = 0,$$

which concludes the proof of (ii). The proof of (iii) uses the same calculations as above. Note that we can apply Theorem 2 with only Assumption 4, because the spectral function here only depends on λ_g. ∎

Appendix B.11 Proof of Proposition 3

Proof. By Lemma 5 and the uniform convergence of $\widehat{c}_{ik_n\Delta_n} - \bar{c}_{ik_n\Delta_n}$ to 0 established above, we can restrict the domain of $\gamma_g(\cdot)$ to the set $\mathcal{C}$, in which it is C^∞ with bounded derivatives. By Theorem 21 of Protter (2004), we have

$$\sum_{i=1}^{[t/(k_n\Delta_n)]-1} \gamma_{g,ik_n\Delta_n}^\intercal (X_{(i+1)k_n\Delta_n} - X_{ik_n\Delta_n}) \overset{\text{u.c.p}}{\Longrightarrow} \int_0^t \gamma_{g,s-} dX_s.$$

Therefore, it remains to show that

$$\sum_{i=1}^{[t/(k_n\Delta_n)]-1} \left(\widehat{\gamma}_{g,(i-1)k_n\Delta_n}^\intercal - \gamma_{g,ik_n\Delta_n}^\intercal\right)(X_{(i+1)k_n\Delta_n} - X_{ik_n\Delta_n}) \overset{\text{u.c.p}}{\Longrightarrow} 0.$$

Define a $\mathcal{F}_{(i+1)k_n\Delta_n}$-measurable function:

$$\xi_{ik_n} = \left(\widehat{\gamma}_{g,(i-1)k_n\Delta_n}^\intercal - \gamma_{g,ik_n\Delta_n}^\intercal\right)(X_{(i+1)k_n\Delta_n} - X_{ik_n\Delta_n}).$$

By standard estimates in (B.1) with c replaced by X, (B.3), and (B.10),

$$\begin{aligned}
\mathrm{E}|\mathrm{E}\left(\xi_{ik_n}|\,\mathcal{F}_{ik_n\Delta_n}\right)| =& \mathrm{E}|\widehat{\gamma}_{g,(i-1)k_n\Delta_n} - \gamma_{g,ik_n\Delta_n}||\mathrm{E}\left((X_{(i+1)k_n\Delta_n} - X_{ik_n\Delta_n})|\mathcal{F}_{ik_n\Delta_n}\right)| \\
\leq & K\mathrm{E}|\widehat{c}_{(i-1)k_n\Delta_n} - c_{ik_n\Delta_n}|(k_n\Delta_n) \\
\leq & K\left((k_n\Delta_n)^{1/2} + a_n\Delta_n^{(2-\gamma)\varpi} + \sqrt{k_n^{-1} + k_n\Delta_n}\right)(k_n\Delta_n)
\end{aligned}$$

[5] See page 160 of Magnus and Neudecker (1999).

Moreover, we have by the same estimates above,

$$\mathrm{E}\left(|\xi_{ik_n}|^2|\mathcal{F}_{ik_n\Delta_n}\right) \leq (k_n\Delta_n + a_n\Delta_n^{(4-\gamma)\varpi-1} + k_n^{-1} + k_n\Delta_n)k_n\Delta_n.$$

Finally, using Doob's inequality, and measurability of ξ_{ik_n}, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t}|\sum_{i=1}^{[s/(k_n\Delta_n)]-1}\xi_{ik_n}|\right)\\
\leq&\sum_{i=1}^{[t/(k_n\Delta_n)]-1}\mathrm{E}|\mathrm{E}\left(\xi_{ik_n}|\,\mathcal{F}_{ik_n\Delta_n}\right)| + \left(\sum_{i=1}^{[t/(k_n\Delta_n)]-1}\mathrm{E}\left(|\xi_{ik_n}|^2|\mathcal{F}_{ik_n\Delta_n}\right)\right)^{1/2}\\
\leq&K\left((k_n\Delta_n)^{1/2} + a_n\Delta_n^{(2-\gamma)\varpi} + \sqrt{k_n^{-1}+k_n\Delta_n}\right) + K(k_n\Delta_n + a_n\Delta_n^{(4-\gamma)\varpi-1} + k_n^{-1} + k_n\Delta_n)^{1/2}\\
\to&0,
\end{aligned}$$

because $(4-\gamma)\varpi \geq 1$ under our assumptions on ϖ and ς, which establishes the proof. ∎

Appendix B.12 Proof of Corollary 2

Proof. The (p,q) entry of the asymptotic covariance matrix is given by

$$\begin{aligned}
&\int_0^t \sum_{j,k,l,m=1}^{d}\partial_{jk}\gamma_{gp,s}\partial_{lm}\gamma_{gq,s}\left(c_{jl,s}c_{km,s}+c_{jm,s}c_{kl,s}\right)ds\\
=&\int_0^t \sum_{j,k,l,m=1}^{d}(\lambda_{g,s}\mathbb{I}-c_s)^+_{pj}(\lambda_{g,s}\mathbb{I}-c_s)^+_{ql}\gamma_{gk,s}\gamma_{gm,s}\left(c_{jl,s}c_{km,s}+c_{jm,s}c_{kl,s}\right)ds\\
=&\int_0^t \sum_{j,l=1}^{d}(\lambda_{g,s}\mathbb{I}-c_s)^+_{pj}(\lambda_{g,s}\mathbb{I}-c_s)^+_{ql}\left(\lambda_{g,s}c_{jl}+\lambda_{g,s}^2\gamma_{gl,s}\gamma_{gj,s}\right)ds\\
=&\int_0^t \lambda_{g,s}\left((\lambda_{g,s}\mathbb{I}-c_s)^+c_s(\lambda_{g,s}\mathbb{I}-c_s)^+\right)_{p,q}ds,
\end{aligned}$$

where we use $(\lambda_{g,s}\mathbb{I}-c_s)^+\gamma_{g,s}=0$, and $\sum_{k=1}^d\gamma_{gk,s}c_{km,s}=\lambda_{g,s}\gamma_{gm,s}$. To calculate the asymptotic bias, we note that the $\widehat{c}_s$ has only simple eigenvalues almost surely. Denote $\widehat{\lambda}_h$ and $\widehat{\gamma}_h$ as the corresponding eigenvalue and eigenvector. We omit the dependence on time s to simplify the notations. By Lemma

2, we obtain

$$
\begin{aligned}
&\sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm}\widehat{\gamma}_{gh}(\widehat{c}_{jl}\widehat{c}_{km}+\widehat{c}_{jm}\widehat{c}_{kl})\\
=&-2\sum_{p\neq g}\frac{\widehat{\lambda}_p\widehat{\lambda}_g}{(\widehat{\lambda}_g-\widehat{\lambda}_p)^2}\sum_{l,m=1}^{d}\widehat{\gamma}_{gl}^2\widehat{\gamma}_{pm}\widehat{\gamma}_{ph}\widehat{\gamma}_{gm}+\sum_{p\neq g}\frac{\widehat{\lambda}_p\widehat{\lambda}_g}{(\widehat{\lambda}_g-\widehat{\lambda}_p)^2}\left(\sum_{l,m=1}^{d}\widehat{\gamma}_{pl}^2\widehat{\gamma}_{gm}\widehat{\gamma}_{ph}\widehat{\gamma}_{pm}+\sum_{l,m=1}^{d}\widehat{\gamma}_{pm}^2\widehat{\gamma}_{gl}\widehat{\gamma}_{pl}\widehat{\gamma}_{ph}\right)\\
&+\sum_{p\neq g}\sum_{q\neq p}\frac{\widehat{\lambda}_p\widehat{\lambda}_g}{(\widehat{\lambda}_g-\widehat{\lambda}_p)(\widehat{\lambda}_p-\widehat{\lambda}_q)}\left(\sum_{l,m=1}^{d}\widehat{\gamma}_{ql}\widehat{\gamma}_{pl}\widehat{\gamma}_{pm}\widehat{\gamma}_{gm}\widehat{\gamma}_{qh}+\sum_{l,m=1}^{d}\widehat{\gamma}_{pm}^2\widehat{\gamma}_{gl}\widehat{\gamma}_{ql}\widehat{\gamma}_{qh}\right)\\
&+\sum_{p\neq g}\sum_{q\neq p}\frac{\widehat{\lambda}_q\widehat{\lambda}_g}{(\widehat{\lambda}_g-\widehat{\lambda}_p)(\widehat{\lambda}_p-\widehat{\lambda}_q)}\left(\sum_{l,m=1}^{d}\widehat{\gamma}_{ql}^2\widehat{\gamma}_{gm}\widehat{\gamma}_{ph}\widehat{\gamma}_{pm}+\sum_{l,m=1}^{d}\widehat{\gamma}_{qm}\widehat{\gamma}_{gl}\widehat{\gamma}_{ql}\widehat{\gamma}_{pm}\widehat{\gamma}_{ph}\right)\\
&+\sum_{p\neq g}\sum_{q\neq g}\frac{\widehat{\lambda}_p\widehat{\lambda}_q}{(\widehat{\lambda}_g-\widehat{\lambda}_p)(\widehat{\lambda}_g-\widehat{\lambda}_q)}\left(\sum_{l,m=1}^{d}\widehat{\gamma}_{pm}\widehat{\gamma}_{ql}^2\widehat{\gamma}_{gm}\widehat{\gamma}_{ph}+\sum_{l,m=1}^{d}\widehat{\gamma}_{qm}\widehat{\gamma}_{pl}\widehat{\gamma}_{ql}\widehat{\gamma}_{gm}\widehat{\gamma}_{ph}\right)\\
=&-\sum_{p\neq g}\frac{\widehat{\lambda}_p\widehat{\lambda}_g}{(\widehat{\lambda}_g-\widehat{\lambda}_p)^2}\widehat{\gamma}_{gh}.
\end{aligned}
$$

Since $\gamma_g(\cdot)$ is a C^∞ function, it is straightforward using the proof of Theorem 2 that the desired CLT holds, even though $\gamma_g(\cdot)$ is not a spectral function. ∎