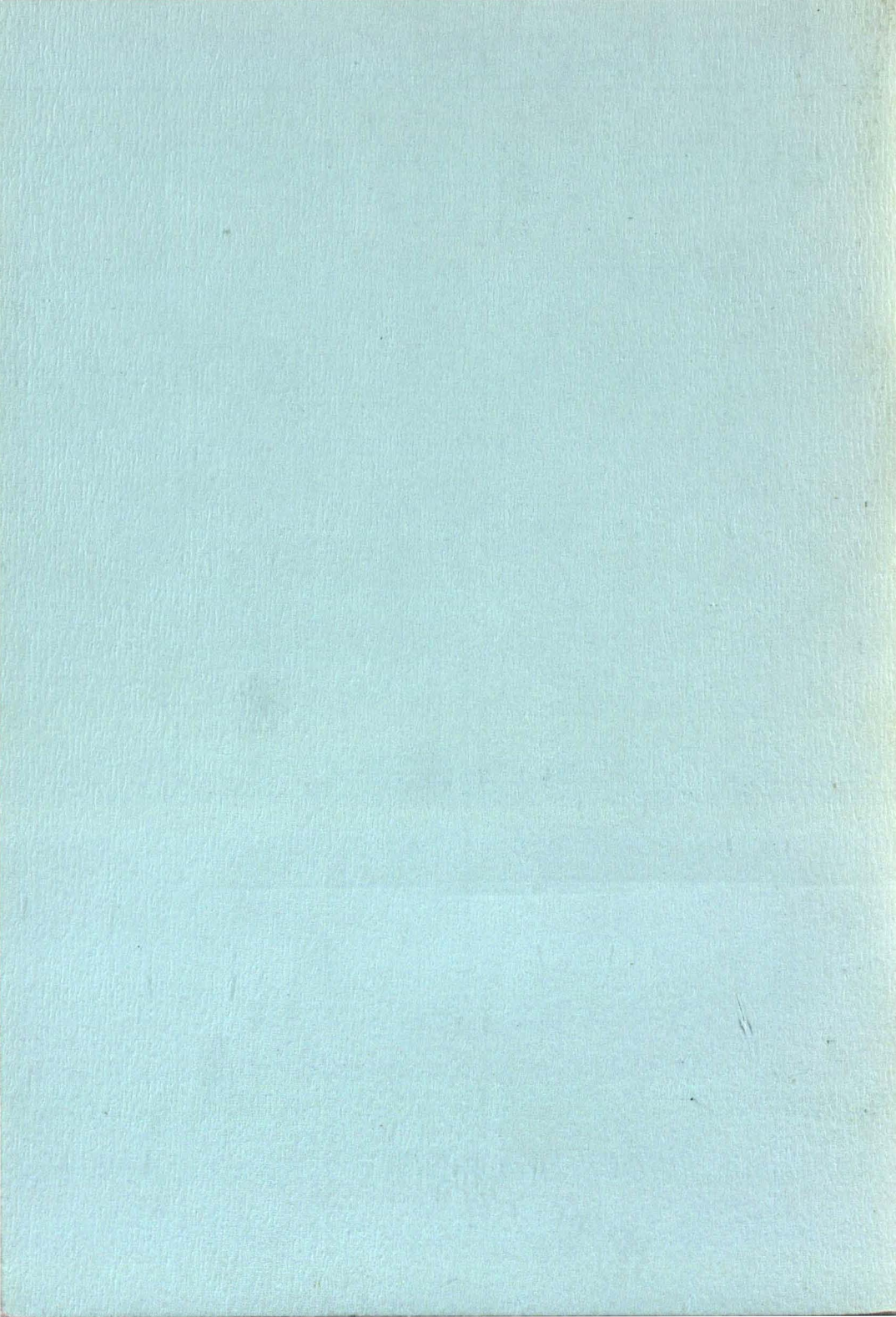# M O N A S H
U N I V E R S I T Y

DEPARTMENT OF MATHEMATICS

AUSTRALIA

Charles Felix Moppert

THE MONASH
SUNDIAL

THE MONASH SUNDIAL

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MONASH UNIVERSITY DEPARTMENT OF
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Preface

The late Carl Felix Moppert was a senior lecturer in the department of Mathematics at Monash University from 1967 until his death from leukaemia in 1984. His speciality was Geometry, a subject he could and did pursue in its abstract modern forms. Yet he also saw it, as indeed did Euclid, as being related to the world of experience.

This found expression in three remarkable projects, undertaken during the last ten or so years of his life. These were:

(a) the design and construction of the Foucault pendulum in the Science North building at Monash – the most accurate such pendulum in the world;

(b) the design and construction of the vertical analemnic sundial on the north wall north wall of Monash's Union Building (the subject of this monograph);

(c) the concept of an energy-efficient and novel pump whose full potential he did not live to develop.

Of these three projects, the first has been reported in the technical literature (*Q.H.R. Astr. Soc.* **21**, *1980*, 108-118). The

third was the subject of a brief popular article (*Function* 8(4), *1984*, 2-7), and a prototype, constructed of discarded hospital equipment, was demonstrated at a Monash Open Day in August, 1984. (The combination of ingenuity and courage implicit in this history epitomises the man.)

In terms of sheer effort, however, it was the sundial that claimed most of his attention. It stands and still works (when the sun shines) as a monument to its creator. Students, staff and visitors alike stop to read the legend and try to see if they too can use it to determine the time and date (it tells the careful reader both).

What the brief on-site plaque cannot say, of course, is how the elaborate configuration of curves and loops was designed. Carl felt that without this explanation the sundial itself was lacking – it should not be, he thought, a mere wonder to be gazed upon, but a device that should be understood, and understood in mathematical terms.

To this end, he produced a manuscript of some 70 pages: a manuscript he tried strenuously to see to publication. In this he was unsuccessful; at the time of his death a highly condensed account had appeared in the Monash journal of school mathematics, *Function* (5(5), *1981, 2-9)*, and that was all.
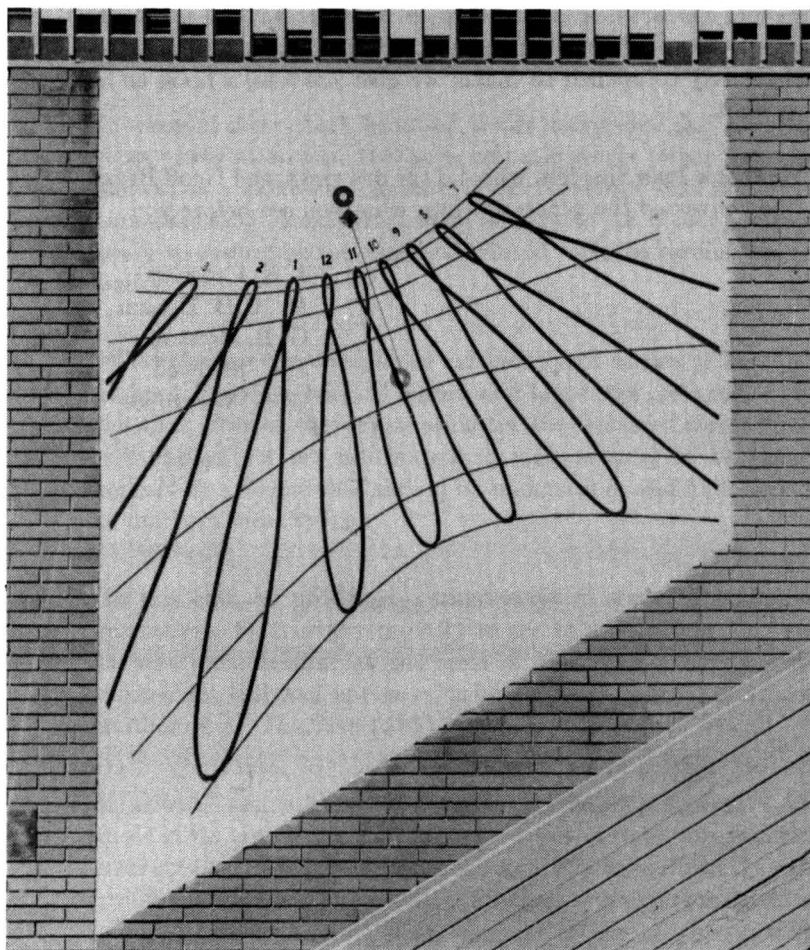
In tribute to his memory we have reprinted this work. Almost in its entirety. Omitted, as now out of date, are detailed instructions on how to programme an HP25 to perform the calculations required. Reluctantly (for there is much real elegance of technique displayed there), we came to the conclusion that these passages detracted from a modern reader's understanding of the situation, rather than enhancing it, which was what Carl would have sought.

The two final chapters, which consist of alternative calculations and other peripheral material, of only marginal relevance to the main topic, have also been omitted.

Otherwise, modulo all of those small amendments editors feel absolutely compelled to make, we give you Carl's book as he wrote it.

We thank Jean Sheldon, who did the drawings, and Geoff Bryan who converted the typescript into what you see before you.

<div align="right">

J. N. Crossley
M. A. B. Deakin
G. B. Preston
J. C. Stillwell
G. A. Watterson

</div>

The Monash Sundial. *The shadow falls as shown twice a year: on October 15th at 10:25 am and on March 1st at 10:55 am (EST in both cases).*

In the background, the sundial in its final stages of construction. In the foreground, the late Dr. C. F. Moppert, designer of the sundial and author of this book.

Introduction

If I had known what is written in this little book before I decided to build a sundial, I could have calculated the lot in a day. This book should therefore save the reader from the roundabout way in which I learned the necessary facts.

It is no use pretending that the reader will find the going easy. I took great effort to present things as simply as I could; however, I don't expect anybody to understand this text without work. The knowledge I assume the reader to have is high-school trigonometry.
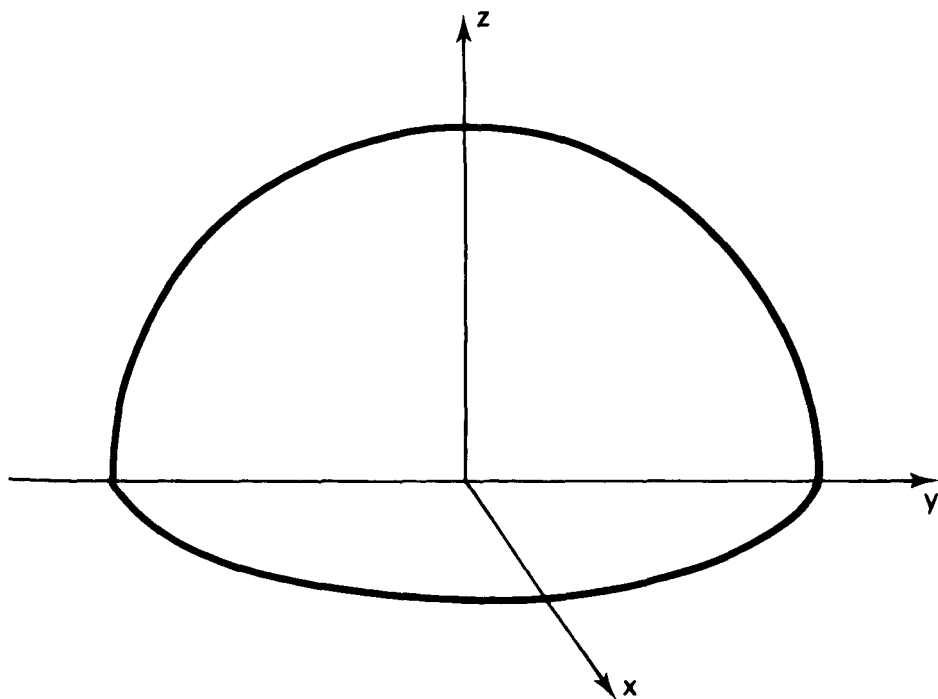
Although educated people today know about quasars, pulsars, galaxies, black-holes and UFO's, basic knowledge which was common fifty years ago has been lost. Some of my colleagues - professional astronomers - have asked me the most inane questions. Although it is hard to believe, in many excellent books the basic figure - a three-dimensional coordinate system in perspective - is hopelessly wrong (see Figure 0).

The text is divided into eight chapters.

§1 Gives the basic facts concerning the sky.

§2 Describes the coordinate system.

§3 Explains spherical trigonometry.

§4 Describes the conventional sundial.

§5 Explains complementary sundials.

§6 Calculates the length of the shadow and how the time can be calculated.

§7 Discusses the various ways in which time can be measured.

§8 Gives the calculations for the Monash sundial.

Carl Moppert
Warrandyte
August 1980

Figure 0. Nonsensical perspective of a 3-dimensional coordinate system. The curve representing a circle should be a semi-ellipse whose tangents at the end points are parallel to the x -axis.
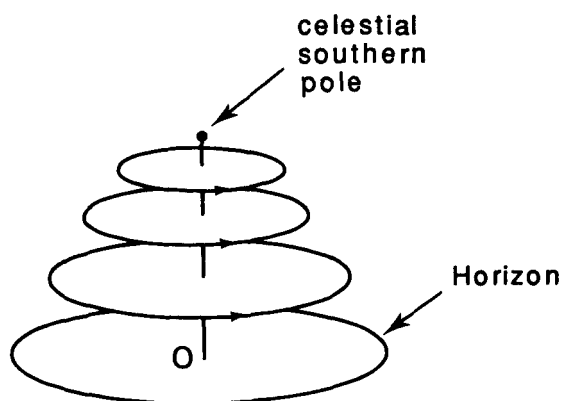
1 Looking at the Sky

For an understanding of the sundial it is easiest to adopt a
geocentric standpoint: the earth is at rest and the stars and
the sun move relative to it.

If we point a camera at night towards the sky and leave the
shutter open for at least several minutes, then we will see that
all stars move on circles about a common centre. The centre is
the *south celestial pole* (from the Latin *coelum*, the sky) and
is exactly due south. *The elevation of the south celestial pole
from the horizon in degrees is equal to our latitude* (Figure 1).
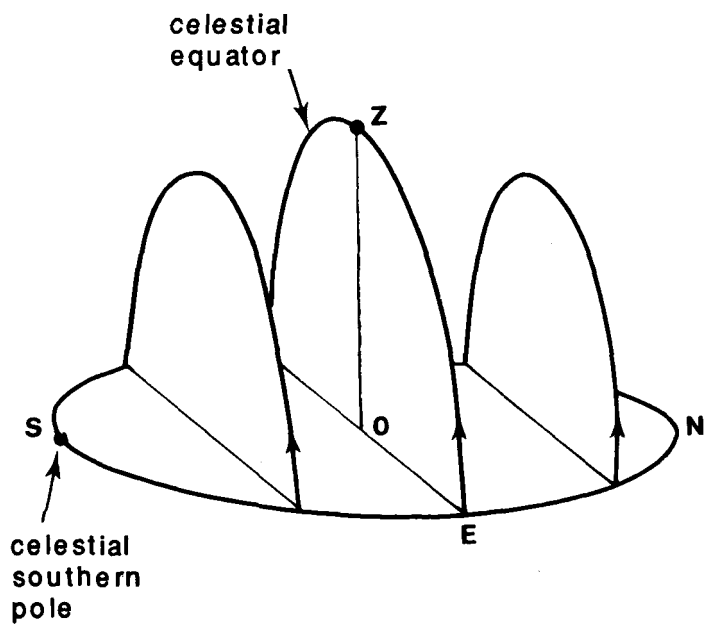
The latitude is counted positive in the northern hemisphere and
negative in the southern one. The sun and all the stars follow
these circles from east to west. All fixed stars have exactly
the same angular velocity, or : *every fixed star performs a full
circle in 23 hours 56 minutes and 4.091 seconds.* In the yearly
average, the sun takes exactly 24 hours to perform a full circle.
This time can vary from day to day by up to 30 seconds.

The semicircle in the sky about the southern pole which starts
exactly due east and ends exactly due west is called the *ce-
lestial equator.* The sun moves along the celestial equator on

Figure 1. Apparent motion of the fixed stars as seen from a point O with latitude λ.

the equinoxes, i.e. on the 20th of March and on the 23rd of September.

Every fixed star always follows exactly the same circle in the sky (hence its name !). The sun does not. During any day the sun follows near enough a circle, however this circle changes during the year. In the southern hemisphere it is nearer the south celestial pole in summer and nearer the north celestial pole in winter . As we have said before, it is along the equator of the sky at the dates which divide the summer half-year from the winter half, i.e. at either of the two equinoxes.

How does the sky appear if we are on the earth's south pole or on the earth's equator? If we are on the earth's south pole, then all fixed stars move along circles which are parallel to the horizon (Figure 2) and if we are on the earth's equator (e.g. in Singapore) then all the fixed stars move along circles which are perpendicular to the horizon (Figure 3).

Every fixed star rises in Singapore from the horizon along a perpendicular line. At the south pole, the equator of the sky is

celestial
southern
pole

Horizon

O

Figure 2. Apparent motion of the fixed stars as seen from the south pole, O.

Figure 3. Apparent motion of the fixed stars as seen from a point on the equator, e.g. from Singapore.

along the horizon. In Singapore, the equator of the sky is the perpendicular circle starting due east and ending due west.

2 Declination and Hour Angle

As we said earlier, the circle which the sun describes every day is not always the same; it shifts a little bit every day. More exactly, the sun does not follow a circle in the sky but a spiral. However, it is sufficiently accurate to say that during any day the sun moves along a circle about the south celestial pole.

In order to describe the position of this circle, we need the notion of the *declination* of the sun. Point your outstretched left arm to the south celestial pole in the sky. Stretch out your right arm in such a way that the two arms form a right angle. You observe that you can do so in many ways, in fact you can rotate your right arm in such a way that the left arm keeps pointing to the celestial south pole while the right arm always forms a right angle with the left one. Your right arm is then always in a *fixed plane* and this plane is the *celestial equatorial plane*. It is the plane through the equator of the sky.

In Figure 4, O is the observer and S_c the south celestial pole. The point P moves along the *circle* on which it lies. We see that the angle $\angle S_c OP$ remains constant and that $90°$ minus $\angle S_c OP$

is the constant angle between OP and the celestial equatorial plane. This second angle also remains constant. Its opposite, i.e. the negative angle $\angle S_c OP$ minus 90°, is called the *declination of the point P* in the sky. If P happens to move along the celestial equator then its declination is zero. The declination of the south celestial pole is $-90°$. If P is between the celestial equator and the southern pole then its declination is negative, otherwise it is positive. The declination of the sun changes over one year between $-23\frac{1}{2}°$ and $23\frac{1}{2}°$. Figure 5 gives the change of the declination of the sun during any one year. In order to fix the position of the point P (Figure 4) fully , the declination of its circle is not sufficient. From where we stand, we can imagine a vertical plane in the North-South direction. In this plane, there is the south celestial pole and also the highest point Z in the sky, the *zenith*, which is straight above us . We can also imagine a plane which contains us (the point O), the south celestial pole S_c and the point P. These two planes form an angle τ , say. There is another way of arriving at the angle τ (Figure 6). The plane through O, P and S_c meets the celestial equatorial plane EOWZ' along the line OP' . The vertical plane in the N-S direction is the plane $NOSS_cZZ'$. The angle τ is therefore $\tau = \angle Z'OP'$. In the same figure, the declination δ of P is the angle POP', taken with a negative sign.

What I have said in the preceeding paragraph is perhaps the most difficult part of the whole exercise. We have points in the sky which may be fixed or moving; e.g., the celestial south pole or the zenith are fixed, while stars are moving. A "point" in the sky is not really a point, but a direction. In the following paragraph, we shall introduce the celestial globe and there we shall again identify directions with points.

I talk about the angle between planes. Two planes intersect along a line (parallel planes do not occur here). We select a point in this line and draw in each plane a line which is perpendicular to the line of intersection. The angle between
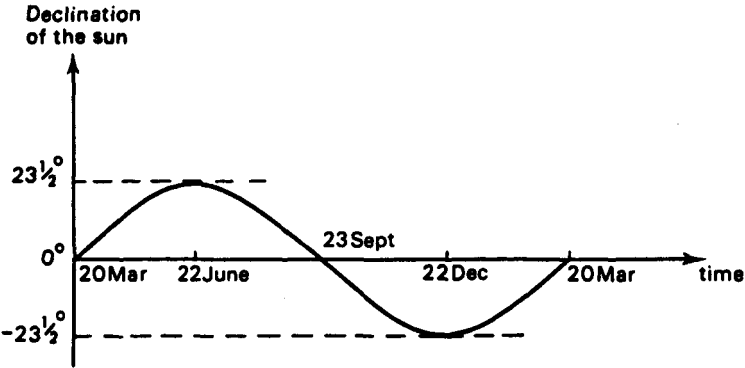
Figure 4. Declination and hour angle.

Figure 5. Change of the declination of the sun during a year.

Figure 6. The hour angle, τ, of P is the angle P'OZ'. *The declination of P is minus the angle* POP'.

these two perpendiculars is called the angle between the two planes.

There is another way of arriving at this angle. Choose any point, it does not matter where it is : on one of the planes, on the line of intersection or anywhere. Through this point, imagine two lines: one perpendicular to the first plane and the other perpendicular to the second one. The angle between these two lines is again the angle between the two planes.

You cannot understand these arguments without making a small model.

We shall call the angle τ the *hour angle* of the point P. (Traditionally not τ but $180° - \tau$ is the hour angle. Our definition is more appropriate for the southern hemisphere.) The two angles τ and δ determine the direction of the point P uniquely. For a fixed star, δ has always the same value and τ increases from $0°$ to $360°$ in 23 hours 56 minutes and 4.091 seconds. For the sun, δ is not always the same but it can be taken as constant during any one day. The value τ increases for the sun from $0°$ to $360°$ in, on the average, 24 hours.

3 Spherical Trigonometry

There is a much easier way of looking at the situation. In all the figures so far, we have looked a the sky 'from the inside', our own stand–point O was the point of reference. Now we look at the *celestial sphere* from the outside. In fact, what is the celestial sphere? There is in reality of course no such thing. However, we can imagine a glass sphere with our head as its centre. We mark on this sphere each point along which we see some star or the sun or the moon. We then step outside this sphere and look at it from the outside. We are all familiar with a globe. The reader is advised to take a globe and experiment with it during the following discussion.

Two points A and B on a sphere can be either diametrically opposite or not. The north pole and the south pole on the globe are diametrically opposite. Any point on the globe has exactly one point which is diametrically opposite to it. We arrive at this point by imagining a straight line through the first point and the centre of the globe. Where this line penetrates the globe on the other side the other point is found. Any point and its diametrically opposite one determine a diameter of the globe.

Let A and B be two points which are not diametrically opposite.
We fix a thread to A and to B and stretch the thread tightly
over the sphere. It then shows the shortest path from A to B,
the *geodesic*. If an airplane flies from A to B it always does so
(if possible) along the geodesic in order to economise on time
and fuel. The geodesic is always along a *great circle* on the
globe. This means that we can find the geodesic from A to B
in a different way: the two points A, B and the centre O of the
globe determine a plane. This plane intersects the globe along
a great circle and the geodesic from A to B is the portion of
this great circle between A and B.

We can measure the geodesic distance between A and B, i.e. the
length of the thread in kilometres. However, it is sufficient to
measure it by giving the angle AOB and we shall use this kind
of measurement on the sphere from now on. We see then that
the distance from the north pole to any point on the equator
(e.g. Singapore) is 90°. The distance between Melbourne and
Singapore is about 50°.

Let us now fix three points A, B, C on the globe (Figure 7).
Connecting them by geodesics we get a *spherical triangle*. The
lengths a, b, c of its sides are angles. Its angles α, β, γ are
of course also angles. As b and c are along great circles on
the sphere and as these are in the two planes determined by
ACO and ABO, the angle α is in fact the angle between these
two planes. It is important to understand that both lengths
and angles in spherical triangles are measured by angles. The
angle-sum in a spherical triangle is always more than 180°.
Take, for example, the north pole, Singapore and Kampala in
Uganda as the vertices of a triangle. Singapore has longitude
105° and Kampala (also roughly on the equator) has longi-
tude 31°. The spherical distance from Kampala to Singapore
is therefore 105° − 31° = 74° (Figure 8). As both these points
are on the equator, the angles Singapore-Kampala-North pole
and Kampala-Singapore-North pole are both 90°. For the same

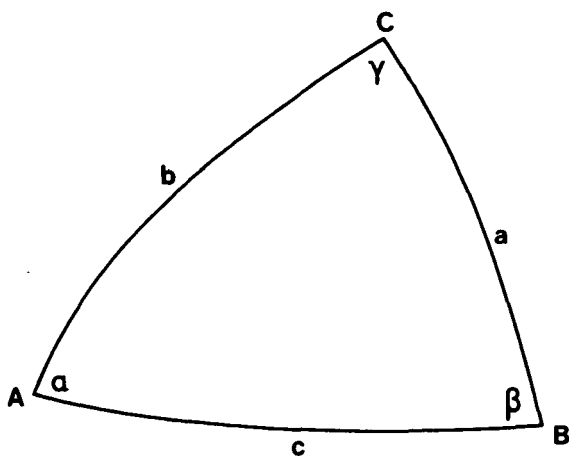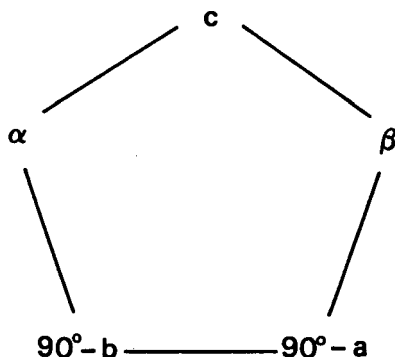*Figure 7. Spherical triangle. The sides a,b,c are measured as angles.*

reason, the angle Kampala-North pole-Singapore is equal to the spherical distance Kampala-Singapore, i.e. equal to 74° . The angle-sum in this triangle is $90° + 90° + 74° = 254°$.

As in ordinary trigonometry, the easiest triangle to calculate is the right-angled one (Figure 9). There is a simple rule for it which we shall not prove but shall use very often.
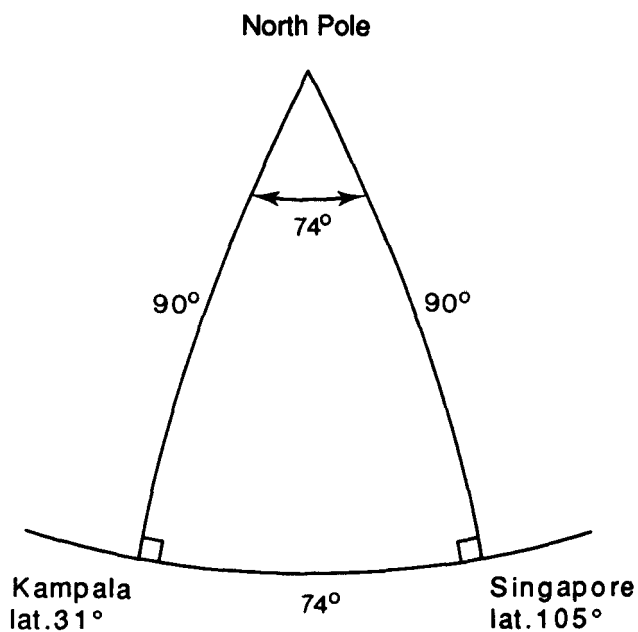
We draw the 'Napier diagram'.

It has five 'entries' : $c, \beta, 90° - a, 90° - b, \alpha$.

Take for example the entry β. The entries c and $90° - a$, are called adjacent to β and the entries α and $90° - b$ are called opposite to β. Accordingly, each entry has two adjacent and two opposite entries.

Napier's rule then says: For a right-angled triangle, the cosine of each entry in the Napier diagram is equal to the product of either:

(i) the two co-tangents of the adjacent entries,

Figure 8. The spherical triangle on the globe with the vertices at Kampala, Singapore (both on the equator) and the North Pole.

or

(ii) the two sines of opposite entries.

Remember that

$$
\begin{aligned}
\cot \alpha &= \frac{1}{\tan \alpha}, \\
\cot(90^\circ - \alpha) &= \tan \alpha \\
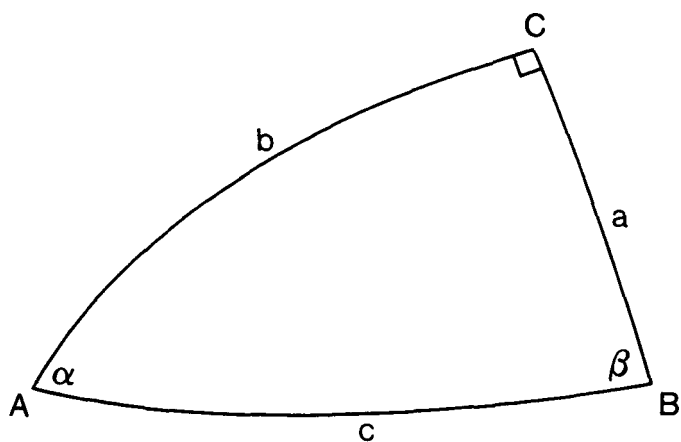\sin(90^\circ - \alpha) &= \cos \alpha,
\end{aligned}
$$

etc.

The quantities a, b, c, α, β in Napier's rule are the sides and angles of the right-angled triangle (Figure 9). For example:

$$
\begin{aligned}
\cos c &= \cot \alpha \cot \beta = \sin(90^\circ - b) \sin(90^\circ - a) = \cos b \cos a \\
\cos \alpha &= \cot \beta \cot(90^\circ - a) = \cot \alpha \cot(90^\circ - a) = \cot \beta \tan a \\
&= \sin(90^\circ - a) \sin \beta = \cos a \sin \beta \\
\cos(90^\circ - b) &= \sin b = \cot \alpha \cot(90^\circ - a) = \cot \alpha \tan \alpha = \sin c \sin \beta.
\end{aligned}
$$

Let us calculate the spherical distance from Melbourne to Singapore!

City	Longitude	Latitude
Melbourne	145°	−38°
Singapore	105°	0°

Figure 9. Right-angled triangle.

We take the point C on the equator with the same longitude as Melbourne:

Point C   longitude 145°   latitude 0°

We then have the right-angled triangle Melbourne-Singapore-Point C, the right angle being C (Figure 10). We know the spherical distances :

Singapore to Point C = 145° − 105° = 40°
Point C to Melbourne = 0° − (−38°) = 38°.

Napier's rule then gives the spherical distance c :

Melbourne to Singapore : $\cos c = \cos 40° \cdot \cos 38°$.

So c = 53°, approximately.

In which direction should a pilot fly to get from Melbourne to Singapore? As the side b in Figure 10 is in the north-south direction, the answer is the angle α. From Napier's rule we choose an equation connecting α, a, b:

$$\cos(90° − b) = \cot \alpha \cot(90° − a)$$

and thus
$$\tan \alpha = \tan a / \sin b.$$

It follows that
$$\alpha = 54° \text{west}.$$

The advent of the electronic calculator enables us to solve complicated problems of spherical trigonometry in minutes.
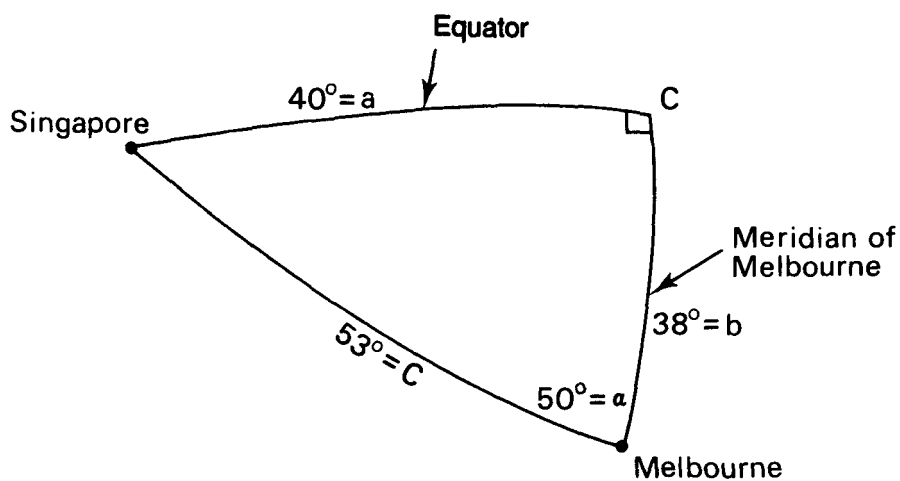
Figure 10. Calculating the distance Melbourne-Singapore

4 The Conventional Sundial

Conventional sundials may be mounted either horizontally or
vertically. The conventional sundial mounted on a vertical wall
consists of a stick OG (Figure 11) which points exactly to the
celestial southern pole. This stick is called the *gnomon* and
its shadow indicates the hour. In Figure 11 we have a vertical
wall on which the gnomon is fixed. OS is the vertical line in the
wall. The angle SOG is the angle between the vertical and OG.
Looking at Figure 11 we see that this angle is $90° - \lambda$ where
λ is the latitude of where we are. The vertical plane through
SOG is in the North-South direction. The angle between this
plane and the wall is thus the angle ϕ between the wall and the
North-South direction.

In Figure 11 the point G' is the point in which a perpendicular
to the wall, dropped from G, meets the wall. As GG' is per-
pendicular to the wall and the wall is vertical, GG' horizontal.

We assume that the length of the gnomon, i.e. *the length of
OG, is unity.* Furthermore, we make the length of OS equal to
unity and then we mark T on the line OG' produced so that

33

*Figure 11. Conventional sundial. OG is the gnomon. GG' is
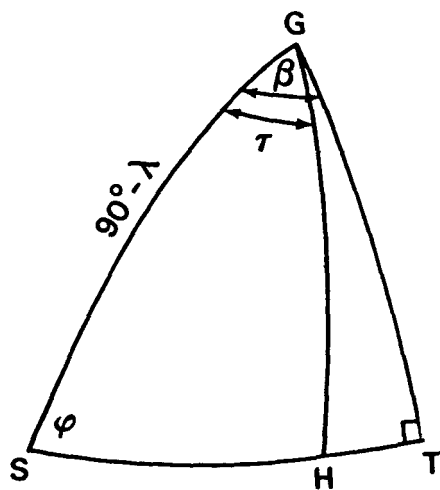horizontal and perpendicular to the wall.*

Figure 12. Spherical triangle for the sundial.

OT is unity. As G, T, and S have the same distance from O, we can place a sphere with unit radius about O. On this sphere we have then the geodesics SG, GT, and TS. As the plane through $OG'TG$ is perpendicular to the wall, the angle at T in the spherical triangle STG is $90°$ (Figure 12). Its side SG is, as we have seen, $90° - \lambda$. The angle at S of the spherical triangle is the angle ϕ between the north-south direction and the wall.

The spherical triangle STG has a right angle at T. Its hypotenuse is $SG = 90° - \lambda$. In order to apply Napier's rule we put $90° - \lambda = c, \phi = \alpha$. Then we must put $ST = b$. In the Napier diagram, c and $90° - b$ are adjacent to α, hence

$$\cos \alpha = \cot c \cot(90° - b)$$

or

$$\tan b = \cos \alpha \tan c$$

i.e.
$$\tan ST = \cos \phi \tan c.$$

Thus
$$ST = \operatorname{artan}\left(\frac{\cos \phi}{\tan \lambda}\right). \qquad (4.1)$$

The side GT of our triangle must be named a. In the Napier diagram, α and c are opposite $90° - a$.

Thus
$$\cos(90° - a) = \sin \alpha \sin c$$

or
$$\sin a = \sin \alpha \sin c,$$

i.e.
$$\sin GT = \sin \phi \cos \lambda$$

Thus
$$GT = \operatorname{arsin}(\sin \phi \cos \lambda). \qquad (4.2)$$

Finally, we want to find the angle $\beta = \angle SGT$. In Napier's diagram, β and α are adjacent to c, thus

$$\cos c = \cot \alpha \cot \beta$$

or

$$\tan \beta = \frac{\cot \alpha}{\cos c},$$

i.e.

$$\tan \beta = \frac{\cot \phi}{\sin \lambda}.$$

Thus

$$\beta = \text{artan} \left(\frac{1}{\sin \lambda \tan \phi} \right). \tag{4.3}$$

If $\phi = 90°$, i.e. if the wall is in the East–West direction, then $\cos \phi = 0$; hence $ST = \text{artan } 0 = 0°$. As $\sin \phi = 1$ we have then $TG = \text{arsin} \cos \lambda = 90° - \lambda$.

Let τ represent the hour angle of the sun. This is the angle between the rays of the sun and the vertical plane in the North–South direction. The rays of the sun along the gnomon OG (Figure 11) determine a plane $OHPG$ where P is the shadow of the point G on the wall. As the plane OSG is vertical, the angle between the former plane and the latter is the angle SGH in Figure 12.

In Figure 12. we now look at the spherical triangle GHT. It has a right angle at T. We know its side $a = GT$ and the angle $\beta - \tau$ at G. We must call the other side b, i.e. $b = HT$. The angle $\beta - \tau$ must be considered as β in Napier's diagram. There β and $90° - b$ are adjacent to $90° - a$, thus

$$\cos(90° - a) \quad = \quad \cot \beta \cot(90° - b)$$

or

$$\tan b \quad = \quad \sin a \tan \beta,$$

i.e.

$$\tan HT = \sin GT \tan(\beta - \tau) = \sin\phi\cos\lambda\tan(\beta - \tau)$$

$$HT = \text{artan}(\sin\phi\cos\lambda\tan(\beta - \tau)). \qquad (4.4)$$

In the same triangle GHT we calculate HG. As it is its hypotenuse, we call it c. We know its side $GT = a$ and its angle $\beta - \tau$, playing the role of β. In the Napier diagram, c and $90° - a$ adjacent to β. Thus

$$\cos\beta = \cot c \cot(90° - a)$$

or

$$\tan c = \frac{\tan a}{\cos\beta},$$

i.e.

$$\tan HG = \frac{\tan GT}{\cos(\beta - \tau)}.$$

Using Equation (4.2), we find

$$HG = \text{artan}\left(\frac{\tan\text{arsin}(\sin\phi\cos\lambda)}{\cos(\beta - \tau)}\right). \qquad (4.5)$$

Let us check these formulae for particular values of τ! First we put $\tau = \beta$. Then $\beta - \tau = 0°$, $\cos(\beta - \tau) = 1$ and $\tan(\beta - \tau) = 0$. Thus $HT = \text{artan}\, 0 = 0°$; $HG = \text{artan}(\tan\text{arsin}(\cos\lambda\sin\phi)) = \text{arsin}(\cos\lambda\sin\phi) = GT$ and this is correct as we see from Figure 12. Next we put $\tau = 0°$. Then $HT = \text{artan}(\cos\lambda\sin\phi\tan\beta)$ and from Equation (4.3) we see that $\tan\beta = \dfrac{1}{\sin\lambda\tan\phi}$. It follows then $HT = \text{artan}\left(\dfrac{\cos\lambda\sin\phi}{\sin\lambda\tan\phi}\right) = \text{artan}\left(\dfrac{\cos\phi}{\tan\lambda}\right) = ST$. This is correct as for $\tau = 0°$, $H = S$ and thus $HT = ST$. For $\tau = 0°$, the sun is in the vertical plane in which the gnomon

lies. The shadow of the gnomon is then a vertical line on the wall, i.e. it lies along the line OS.

We are now able to design a conventional sundial. For the wall in question, we determine ϕ. The angle λ is the latitude of our position. We calculate GT from Equation (4.2) and β from Equation (4.3). As the hour angle of the sun increases by $360°$ in every 24 hours it increases by $15°$ in every hour. $\tau = 0°$ corresponds to the local noon. The vertical line OS on the wall is thus marked 12. The markings for 11 and 1 we get by putting $\tau = \pm 15°$ and calculating $HT = \angle HOT$ from (4.4). The markings for 10 and 2 we get by putting $\tau = \pm 30°$, etc.

5 The Orientation of the Wall

We have been somewhat careless in the determination of the angle ϕ. Two opposite walls of a rectangular building form the same angle with the North–South direction; however, sundials on these walls will be different. We have not yet taken into account which side of the wall is exposed to the sun. There is in fact no wall standing in an open place which is never exposed to the sun.

In Figures 13a and 13b we illustrate how to measure the angle ϕ for the two sides of the same wall as seen from above, looking down on the wall. We see that specifying the wall and the side of it on which we desire to put the sundial determines ϕ uniquely. We also see that ϕ ranges from 0° to 360°. In Figure 14 we have illustrated a wall in the East–West direction and the position of the gnomon on its southern and on its northern side (the figure applies to the wall being in the southern hemisphere). For the southern side of the wall we have $\phi = 270°$ and for its northern side, $\phi = 90°$. No matter which side of the wall we choose, the angle β from Equation (4.3) will be the same, and $\tan \phi = \tan(180° + \phi)$. As $\cos \phi = -\cos(180° + \phi)$, the angle ST
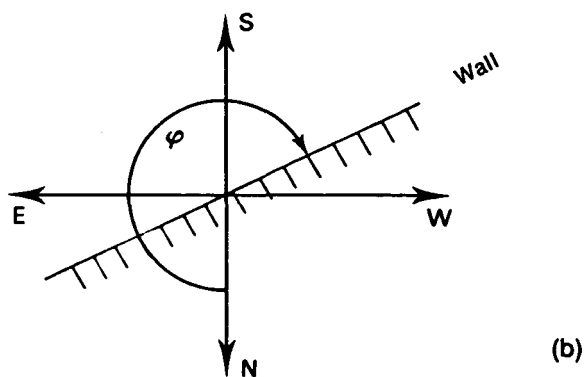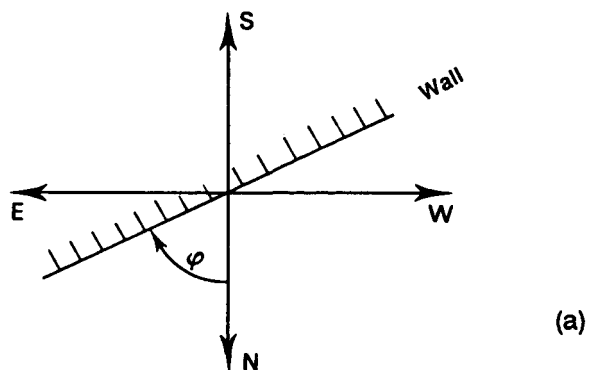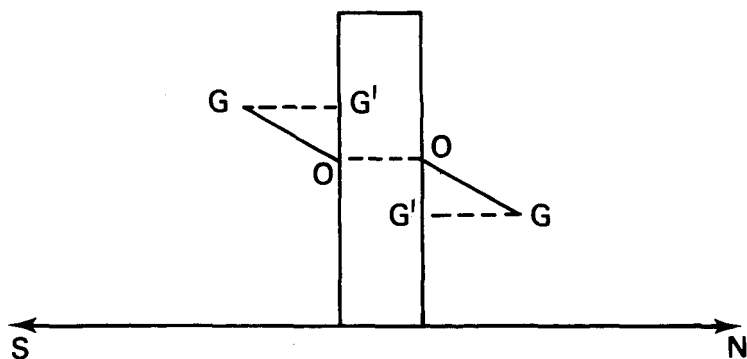
Figure 13. Orientation of the wall

will be positive for one side and negative for the other and the same applies to the angle GT. Two sundials on the two sides of the same wall (or of parallel walls) are called *complementary*. Of course the sun does not shine on them both simultaneously. In Chapter 6 we shall calculate when the sun shines on a wall.

Figure 14. Complementary sundials

6 The Length of the Shadow

In Figure11 the shadow of the tip G of the gnomon is the point P on the wall. We shall calculate the length OP of the shadow of the gnomon. In the triangle OPG we take the length of the side OG as the unit (Figure 15). The angle $\lambda = \angle POG$ is equal to the angle HG (Figures 11, 12). We can therefore calculate it from Equation (4.5). The angle OGP is equal to $90° - \delta$ where δ is the declination of the sun (remember that the gnomon points to the south celestial pole!). The angle OPG is thus $90° + \gamma - \delta$. The sine-theorem then gives

$$\frac{1}{\sin(90° + \gamma - \delta)} = \frac{OP}{\sin(90° - \delta)} .$$

$$OP = \frac{\cos \delta}{\cos(\gamma - \delta)} = \frac{\cos \delta}{\cos(HG - \delta)}. \qquad (6.1)$$

Formula (6.1) provides a lot of information. During any day, δ is constant or can at least be taken as such (it varies at the most by 24' per day). In Figures 11 and 15, $\angle OGP = 90° + \delta$. As this angle is constant, the direction from G towards the sun
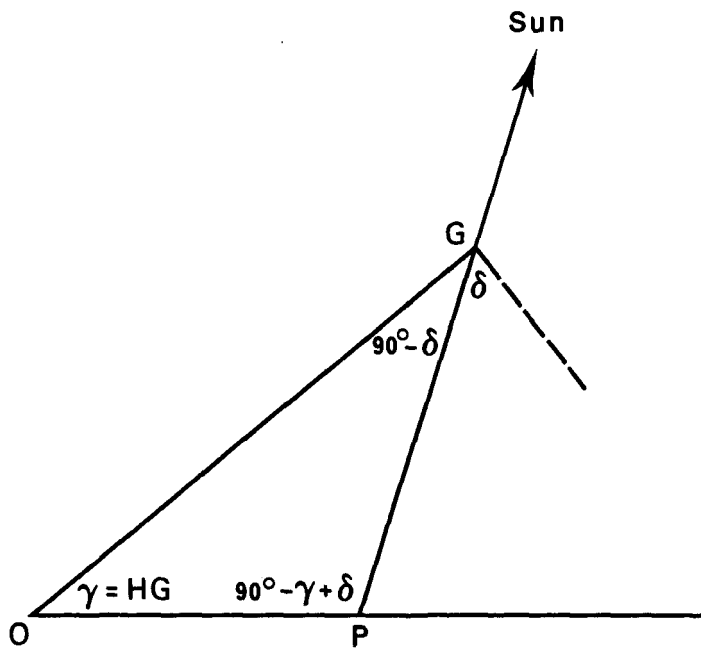
Figure 15. The length OP of the shadow.

in Figure 11 moves along a circular cone with axis OG and vertex G. The same can of course be said about the produced segment GP. It follows that the path of the shadow P is the intersection of this circular cone with the wall, i.e. *this path is a conic section*.

In Formula (6.1), δ remains constant during any one day and HG changes. Now $\cos(HG + \delta)$ decreases as HG increases and as $\cos(HG + \delta)$ is in the denominator, OP increases as HG increases. By the way, this behaviour is evident from Figure 15. The minimum of OP occurs for the minimum of HG and this takes place for $H = T$ (Figure 11), i.e. for $\tau = \beta$.

Outside the tropics, the conic section in question is always a hyperbola. It follows that *the axis of this hyperbola lies along the line OT in Figure 11.* The asymptotes of the hyperbola correspond to the values of HG for which OP becomes infinite, i.e. for $\cos(HG + \delta) = 0$. For this we have $HG + \delta = 90°$. Let $\tau = \tau_A$ in this case. Then

$$\tan HG = \cot \delta = \frac{\tan \text{arsin}(\sin \phi \cos \lambda)}{\cos(\beta - \tau_A)}$$

or

$$\cos(\beta - \tau_A) = \tan \delta \tan \text{arsin} (\sin \phi \cos \lambda). \qquad (6.2)$$

The angle β is determined by ϕ and λ (Formula (4.3)). Accordingly, Formula (6.2) allows us to calculate τ from ϕ, λ and δ.

However, $|\cos(\beta - \tau)| \leq 1$ and therefore Formula (6.2) makes sense only if

$$|\tan \delta \tan \text{arsin}(\sin \phi \cos \lambda)| < 1. \qquad (6.3)$$

Formula (6.3) can be changed to

$$|\tan \delta| \leq |\cot \text{arsin}(\sin \phi \cos \lambda)|$$

and clearly $\cot(\mathrm{arsin}(\sin\phi\cos\lambda)) = \tan(90° - \mathrm{arsin}(\sin\phi\cos\lambda))$.

Now for arbitrary angles x, y we have

$|\tan x| \leq |\tan y|$ *is equivalent to* $|x| \leq |y|$ *if we restrict x and y to the interval from* $-90°$ *to* $+90°$.

Accordingly, Formula (6.3) is equivalent to

$$|\delta| \leq |90° - \mathrm{arsin}(\sin\phi\cos\lambda)|. \qquad (6.4)$$

Again, for any x and y, $|x| \leq |y|$ is equivalent to $|\cos x| \leq |\cos y|$ (under the same restriction). It follows that Formula (6.4) can be replaced by

$$|\cos\delta| \geq |\cos(90° - \mathrm{arsin}(\sin\phi\cos\lambda))|$$

and $\cos(90° - \mathrm{arsin}(\sin\phi\cos\lambda))| = \sin\phi\cos\lambda$. The condition (6.3) appears thus finally in the form:

$$|\cos\delta| \geq |\sin\phi\cos\lambda|. \qquad (6.5)$$

The declination δ of the sun ranges between $\pm 23°26'$ and these angles are of course the boundary latitudes of the tropics. Outside the tropics we have thus $|\delta| < |\lambda|$, hence $\cos\delta > \cos\lambda \geq |\cos\lambda\sin\phi|$. The above inequality is thus always satisfied there.

In order to understand why in the tropics the shadow P need not trace a hyperbola it is best to consider the simplest case : a sundial on the equator placed on the wall in an East–West direction.The gnomon OG is then horizontal and we place such a gnomon on either side of this wall: on the northern side the gnomon points to the celestial north pole and on the southern side to the celestial southern pole. The sun shines on the northern side of the wall from the 21st of March to the 21st of September and on its southern side from the 21st of September to the 21st of March. During any one day, the line GP moves

along a circular cone with axis OG and P thus describes a circle. Of course this is in fact only a semi–circle from sunrise to sunset. The radius of this circle approaches infinity as the time of the year approaches the equinox.

Formula (6.2) enables us to calculate when the sun shines on a given wall. As an example, we take the wall on which the Monash sundial is mounted. The latitude of Monash is $\lambda = -37°55' = -37.92°$ and the angle of the wall is $\phi = 106°51' = 106.85°$. Formula (4.3) gives $\beta = 26.24°$. We find that

$$\tan \arcsin (\sin \phi \cos \lambda) = \pm 1.1514.$$

Formula (6.2) is then

$$\cos(26.24° - \tau) = \pm 1.1514 \tan \delta$$

or

$$\tau = 26.24° \pm \arccos (\pm 1.1514 \tan \delta).$$

At the equinoxes, the declination of the sun is $\delta = 0$ and thus $\tau = 26.24° \pm 90°$ or $\tau = 116.24°$, or $\tau = -63.76°$. The first one of these values is useless as the sun is above the horizon for only $|\tau| \leq 90°$. This means that the sun hits the wall from the moment of sunrise and goes behind the wall at the moment at which τ takes the value $-63.76°$. In the next paragraph we shall find out how to find the corresponding time of the day.

The argument I am now putting forward is not scientific; it is based on trial and error. It would of course be quite easy to put it in scientific clothing. I shall explain as I go on. In our formula

$$\tau = 26.24° \pm \arccos (\pm 1.1514 \tan \delta)$$

we have four values of τ corresponding to a given value of δ. Of course this is wrong; we cannot have more than two such values by the nature of the problem. The question is: should we

choose arcos($+1.1514 \tan \delta$) or arcos($-1.1514 \tan \delta$)? In mid–summer we have $\delta = -23.44°$ and in mid–winter $\delta = +23.44°$. It follows that $|1.1514 \tan \delta| = 0.4992$ in both mid–summer and in mid–winter.

Now
$$\text{arcos} 0.4992 = \pm 60.05°$$

and arcos(-0.4992) $= \pm 119.95°$.

Furthermore, $\tau = 26.24° \pm 60.05° = 86.29°$ or $-33.81°$ while $\tau = 26.24° \pm 119.95° = 146.19°$ or $-93.71°$.

My experimental argument goes as follows: In mid-summer the sun is on the wall for a long period and in mid–winter for a short one. The correct formula is thus

$$\tau = 26.24° \pm \text{arcos}(-1.1514 \tan \delta).$$

This means then that in mid-winter, the sun hits the wall from sunrise to sunset and in mid-summer between the times corresponding to $\tau = -33.81°$ and $\tau = 86.29°$.

I have not yet finished with Formula (6.2). We have agreed that in mid-summer the sun hits the wall between the time corresponding to $\tau = -33.81°$ and $86.29°$. What happens for $86.29° \leq \tau \leq 90°$ and $-90° \leq \tau \leq 33.81°$? Obviously, the sun *then hits the back of the wall.* (In the case of the Monash sundial, this is the south wall of the Monash Union building.) This means that the sun hits the south wall in the morning and in the evening but not in the middle of the day. In mid-winter the situation is different: the sun then hits the north wall during the whole day and the south wall never.

If δ is such that

$$\text{arcos}(-1.1514 \tan \delta) = -(90° - 26.64°) = -63.76°$$

i.e. $\delta = -21.01°$, then the values of τ are $26.24° \mp 63.76°$ and these values are $90°$ and $-37.52°$. At the corresponding date the sun hits the southern wall in the evening only and no more in the morning.

If δ is such that

$$\text{arcos}(-1.1514 \tan \delta) = -(90° + 26.24°) = -116.24°$$

i.e. $\delta = 21.00°$ when the values of τ are $26.24° \mp 116.24°$ and these values are $-90°$ and $142.48°$. At the corresponding date the sun hits the north wall only.

Now $\delta = -21°$ on the 16 January and 27 November and $\delta = 21°$ on 26 May and 19 July. The south wall of the Union building has thus:

- Morning sun from 27 November till 16 January,

- Evening sun from 19 July till 27 November

- No sun at all from 26 May till 27 November .

Our calculations prove to be useful even without any sundial !

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7 Three Kinds of Time

The time to which we set our watches is called *standard time (ST)*. In Australia there are three standard times, one for Victoria, New South Wales, Tasmania, the A.C.T. and Queensland; one for South Australia and the Northern Territory and one for Western Australia. If it is 12 noon in Melbourne then it is 11.30 am in Darwin and 10 am in Perth (these three standard times are not the kinds of time referred to in the title of this paragraph!). The map of the earth is divided into *"time zones"* which you may find in any atlas. When it is 12 noon in Melbourne it is 2 am in England. The standard time in England is called Greenwich Standard Time.

The division of the world into time zones (about one for every hour of the day) is clearly a compromise. Ideally, every place should have its own standard time, the *local standard time (LST)*. It would depend solely on the longitude of the place. Longitudes are counted East and West from Greenwich, 180° corresponding to 12 hours' time difference. We have thus

$$LST = \text{Greenwich } ST + \frac{l}{15} \text{ (East)}$$

$$LST = \text{Greenwich } ST - \frac{l}{15} \text{ (West)}, \qquad (7.1)$$

where l is the longitude of the place.

The longitude for Monash is $l = 145°8'$ East. Its local standard time is thus

$$LST(\text{Monash}) = \text{Greenwich } ST + 9\text{h}\,40\text{min}\,32\text{sec}. \qquad (7.2)$$

The time zones are chosen in such a way that they differ from each other (and thus from Greenwich ST) by full hours or half hours. They thus give LST for places the longitudes of which are multiples of $15°$ or $7\frac{1}{2}°$. In particular, the standard time for Victoria, is the LST for places with longitude $150°$ East, i.e. for places on the meridian through Eden in NSW.

(By the way, we now understand the word *meridian*: the Latin word *meridies* means noon; places on the same meridian have noon at the same time.)

We have calculated the LST at Monash from the Greenwich Standard Time. This is not very useful as we do not set our watches to Greenwich time but to the standard time ST of Victoria. As this is the LST for places with the longitude $150°$, we must take into account the difference between the longitude of Monash and $150°$, i.e. $4°52'$. As we are this amount *west* of Eden, we then have

$$LST(\text{Monash}) = ST(\text{Victoria}) - 19\,\text{min}\,28\,\text{sec}. \qquad (7.3)$$

as $(4°52')/15 = 19'28''$. This means that when our watch shows 12h 19m 28s then the LST at Monash is 12 noon.[1] The two coincide for Greenwich, Eden in N.S.W. and most places the longitudes of which are multiples of $15°$. A third kind is called *apparent sun time (AST)* and this is a bit more difficult to understand. It is the difference between AST and LST which makes the loops to be seen in our sundial necessary. (Notice by

---

[1] We now have two kinds of time for each place.

the way that the *S* in *AST* and *LST* stand for different words: sun in the first, standard in the second).

The fact is that even in Greenwich or in Eden the sun is not always highest at noon. (This is due to the fact that the earth's orbit is not perfectly circular, and also to the earth's axis being tilted.) The time at which the sun is highest is called the *ephemeris transit time* of the sun and it can be found in the handbook called " *Astronomical Ephemerides* ". This book is issued every year. In order to find the ephemeris transit times of the sun, it does not matter which volume of the handbook we take, the one for 1900 is just as good as the one for 1980. The declination of the sun given there corresponds always to the same ephermeris transit time of the sun, at least for hundreds of years to come. We find, for example, in the handbook for 1978 that on December 31, 1977 the declination is $-23°07'15''$ and the ephemeris transit takes place at 12h 03min 01.71s *LST*. On the other hand, we find

Date	Declination	Ephemeris transit
$31 \cdot 12 \cdot 1978$	$-23° \, 08' \, 14.5''$	12h 02m 54.93s
$1 \cdot 1 \cdot 1979$	$23° \, 03' \, 53.8''$	12h 03m 23.61s.

Linear interpolation from these two data gives, for the declination $23° \, 07'15''$, the ephemeris transit time 12h 03m 1.47s. We thus have in one year for a given declination a change in ephemeris transit time of 0.24 seconds. In fact it is much less than this, as linear interpolation is not good enough.

All places on the same meridian have the same AST. At any place, AST noon occurs when the sun is highest in the sky, i.e. at the moment when the shadow of the gnomon in a conventional sundial is vertical. To put it differently: AST noon occurs when the shadow of any vertical object is shortest. LST and AST vary by a changing amount during the year. This difference,

$$\Delta E = LST - AST \qquad (7.4)$$

as a function of the data is called the *equation of time* (Figure 16). (This name is a little unfortunate as ΔE is not an "equation"in the usual sense. Rather it is a correction term to be entered into the equations. However, the name is standard and so is used here.) *This difference, ΔE is the same for all places on the earth* (not only for places on the same meridian).

The "ephemeris transit time " of the sun as found in the "Astronomical Ephemerides " is the LST at which the sun is highest in the sky.

Combining the relations found above we have

$$
\begin{aligned}
ST\,(\text{Victoria}) \;&=\; LST(\,\text{Monash}) + 19\,\text{m}\,28\text{s} \\
&=\; AST\,(\text{Monash}) + \Delta E + 19\,\text{m}\,28\text{s}. \qquad (7.5)
\end{aligned}
$$

At noon AST the sun is highest in the sky. Between AST noon of one day and the next the sun describes a full circle; the hour angle of the sun increases by 360°. We have then the relation between the hour angle τ of the sun and AST.
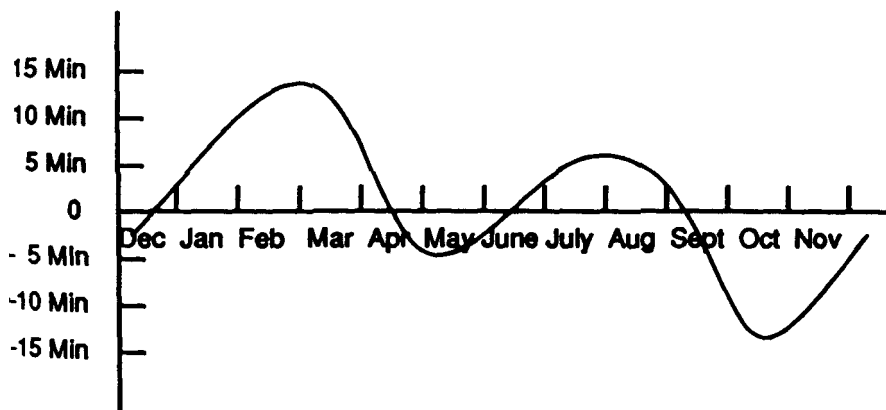
Figure 16. The equation of time

Hour angle	AST
$0°$	12 noon
$+90°$	6 am
$-90°$	6 pm
τ	$12 - \frac{\tau}{15}$.

In this last relation, namely $12 - \frac{\tau}{15}$, we count the hours of the day from 0 to 24, 0 and 24 standing for midnight. We substitute this in our relation above and find

$$\tau = -15\,ST\,(\text{Vic}) + 184°52' + 15\Delta E$$
$$= -15\,ST\,(\text{Vic}) + 184.87° + 15\,\Delta E. \qquad (7.6)$$

For 10 am ST (Vic) we have thus $\tau = 34°52' + 15\Delta E$ and for 2 pm $= 14\,ST$ we have $\tau = -25°8' + 15\Delta E$.

The handbook *"Astronomical Ephemerides"* does not give ΔE but ET, the ephemerides transit time of the sun.

Then $\Delta E = ET - 12$ and this is ΔE in minutes and seconds. For the calculator, we have to convert this into decimal points. We have

$$\beta - \tau = \beta - 184.87° + 15(ST - ET + 12) \qquad (7.7)$$

$$\beta - \tau = \beta - 4.87° + 15(ST - ET). \qquad (7.8)$$

8 Calculations for the Monash Sundial

The longitude of Monash is

$$l = 145°08' = 145.13° \quad \text{East}$$

and the latitude is

$$\lambda = -37°55' = -37.92°.$$

The wall on which the sundial is mounted has the angle (Figure 13):

$$\phi = 106.85°.$$

From Formulae (4.1), (4.2), (4.3), (4.4), (4.5) we calculate

$$
\begin{aligned}
ST &= 20.41° \\
GT &= 49.03° \\
\beta &= 26.24°.
\end{aligned}
$$

Then

$$HT = \text{artan}(0.7550)\tan(\beta - \tau)) \qquad (8.1)$$

$$HG = \text{artan}\left(\frac{1.1514}{\cos(\beta - \tau)}\right). \qquad (8.2)$$

For the meaning of these quantities see Figure 11a

It follows that

$$SH = ST - HT = 20.41° - HT. \qquad (8.3)$$

$$OG' = \cos GT = 0.6557$$
$$G'G = \sin GT = 0.7550,$$

where the length OG of the gnomon is unity. The x, y coordinates of the point G' (Figures 11 and 17) are then

$$x = 0.6145, y = 0.2287.$$

For the quantity $\beta - \tau$ we have from Equation (7.8)

$$\beta - \tau = 21.37° + 15(ST - ET). \qquad (8.4)$$

The quantities OP and SH give the position of the shadow in polar coordinates. OP is given by Formula (6.1) (here repeated)

$$OP = \frac{\cos \delta}{\cos(HG - \delta)}. \qquad (6.1)$$

Formulae (8.1-8.4), together with the Formula (6.1) and the values for l, λ, ϕ, ST, GT and β, allow computation of the position of the point P.

We have

$$\beta = 26.24°$$

$$\sin\phi\cos\lambda = 0.7550$$

$$\tan\operatorname{arsin}(\sin\phi\cos\lambda) = 1.1515$$

$$ST = \operatorname{artan}\left(\frac{\cos\phi}{\tan\lambda}\right) = 20.41°.$$

These four quantities remain unchanged for all calculations.

In order to calculate the position of the shadow for any given day (remember the year does not affect matters greatly), we need also the declination δ and the ephemeris transit time ET. These quantities I took from the *'Astronomical Ephemerides'*. It is then possible to derive the values of OP (from Equation (6.1)) and SH (from Equation (8.3)). I did this using an HP25 calculator, applying the calculations to all the values of ST when the sun shone on the north wall of the Union building, and to all the days of the year. Finally, I used the facilities provided by the calculator to convert these values (which are the polar coordinates of the point P) into cartesian coordinates x, y.

Consider a particular example; for 21.1.1978, we have

$$ET = 12\text{h } 11\text{m } 18\text{s}$$

and

$$\delta = -20.0244°.$$

Our results are then as follows.

ST	x	y
7	3.4244	8.9778
8	2.4421	3.3749
9	2.3145	2.0120
10	2.3574	1.3125
11	2.5093	0.8023
12	2.9063	0.3047
13	3.4153	-0.3699
14	5.1462	-1.8844

By making many such calculations, it is possible to derive an x and a y for each ST on any particular day. Interpolation, for example, gives (still on 21.1.1978) $x = 2.3165, y = 1.7220$ for an ST of 9h21m33s. In this way, the values of x, y were calculated for each time on each date of the year. (Fortunately, as noted on pg. 55 above, the variation from year to year is extremely small and can be neglected.)

This is how I made the calculations for the Monash sundial[2].

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[2] **Editor's Note:** The one complication not allowed for in that calculation is daylight saving. The sundial gives very accurate readings of Eastern Standard Time. During the summer months, the viewer must supply the one hour adjustment for Summer Time.
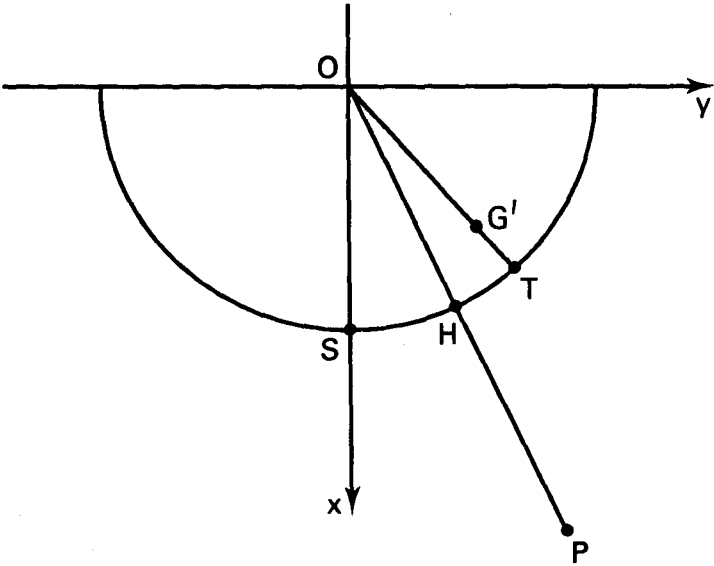
Figure 17. x, y coordinates for the design of the sundial.

M 8259/MON